Problem 1: It is a TAUTLOGY. It can be shown by the following truth table

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>r</th>
<th>r → p ≡ X</th>
<th>q ∨ r ≡ Y</th>
<th>q → p ≡ Z</th>
<th>X ∧ Y ∧ Z ≡ A</th>
<th>A → p</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Problem 2: It is a TAUTLOGY. Can be proved by the truth table:

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>¬(p ↔ q)</th>
<th>p ↔ q</th>
<th>¬(p ↔ q) ↔ p ↔ ¬q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Problem 3: It is EQUIVALENT. ∀x(P(x) → Q(x)) = ∀x(¬P(x) ∨ Q(x)) = ¬P(x) ∨ ∀xQ(x) = P(x) → ∀xQ(x)

Problem 4: It is NOT EQUIVALENT. For ∃x(P(x) ∨ Q(x)) to be true, there can exist just one x which satisfies either of P(x) or Q(x). For ∃x(P(x)) ∧ ∃xQ(x) to be true, there should be an x which satisfies P(x) and the same or another x which satisfies Q(x).

Problem 5: It is FALSE. Eg: for x = 1, there is no y which lies in the domain of integers which satisfies the equation, because y = ±√2

Problem 6: It is FALSE, because 1³ = 1⁴ = 1

Problem 7: It is FALSE. No value of x and y can satisfy both equations. If there was such a value, y divides both x and x + 1, which is not possible.

Problem 8: One way of proving this is:
1. ∀x((P(x) ∧ ¬Q(x)) → ¬R(x)) (Premise)
2. ∀x(R(x) → ¬(P(x) ∧ ¬Q(x))) (Contrapositive)
3. ∀x(R(x) → ¬P(x) ∨ Q(x)) (De Morgan’s law)
4. ∀x¬(P(x) ∨ Q(x)) (Premise)
5. ∀x¬(P(x) ∨ ¬Q(x)) (De Morgan’s law)
6. ∀x¬(R(x) ∨ ¬P(x) ∨ Q(x)) (Equivalent to 3)
7. ∀x¬(R(x) ∨ ¬P(x) ∨ Q(x) ∨ ¬Q(x)) (Combining 5,6)
8. ∀x¬(R(x) ∨ ¬P(x)) (Since, p ∨ ¬p = T)
9. ∀x(R(x) → ¬P(x)) (Equivalent to 8) Can also be proved by truth table.

Problem 9: DIRECT PROOF. 4=2+2,5=2+3,6=3+3,7=2+5,8=3+5,9=2+7,10=5+5 or 7+3

Problem 10: PROOF BY CONTRADICTION: Let us assume p² is a rational number and hence it can be expressed in the form a/b where a and b are both integers and have no common factors but 1. Hence we have

\[ p^2 = \frac{a}{b} \]
\[ p = \frac{a^q}{b^q} \]
\[ a^q = pb^q \]

Thus, \( p \) is a factor of \( a^q \). Thus \( p \) is a factor of \( a \). Let \( a = pk \) for some \( k \). Now \( p = \frac{a^q}{b^q} \) becomes,
\[ p = \frac{p^q k^q}{b^q} \]

which gives
\[ b^q = p^{q-1} k^q \]

Thus, \( p \) is a factor of \( b^q \) if \( q > 1 \). But, we assumed that they had no common factors but 1. Hence there is a contradiction.

**Problem 11:** NOT TRUE. Let us give a counterexample. Let, \( A = \{a, b\}, B = \{a\}, C = \{1, 2, 3\}, D = \{4, 5, 6, 7\} \).
Clearly \(|A| < |C|\) and \(|B| < |D|\) but \( A \cap B = \{a\} \) and \( C \cap D = \Phi \). Thus, \(|A \cap B| = 1 > 0 = |C \cap D|\).

**Problem 12:** Maximum cardinality is \(|A| + |B| + |C|\) when they are disjoint. Minimum cardinality is \( \max(|A|, |B|, |C|) \), in case one set contains all elements present in other two sets too.

**Problem 13:** There are \(|B|^{|A|}\) functions.
If \( |A| \leq |B| \), then there are \( \frac{|B|!}{(|B|-|A|)!} \) one to one functions.

**Problem 14:** Because there is one to one correspondence (bijection) between \( B \) and \( C \), we know that
\[ |B| = |C| \]

Also, because the function from \( A \) to \( B \) is onto,
\[ |A| \geq |B| \]

(We can prove it by contradiction. Let, \( |A| < |B| \), then by definition of function, there will be at least one element in \( B \) which will not have a corresponding element in \( A \). Hence, it is not onto, which contradicts our assumption).

From the above two results, we have \(|A| \geq |C|\)

**Problem 15:**
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} (i + j)^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} (i^2 + j^2 + 2ij) = \sum_{i=1}^{m} \sum_{j=1}^{n} (i^2) + \sum_{i=1}^{m} \sum_{j=1}^{n} (j^2) + 2 \sum_{i=1}^{m} \sum_{j=1}^{n} (ij)
\]

Now, let us evaluate all three sums
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} (i^2) = \sum_{i=1}^{m} ni^2 = n \sum_{i=1}^{m} i^2 = \frac{nm(m+1)(2m+1)}{6}
\]

Similarly,
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} (j^2) = \sum_{j=1}^{n} \sum_{i=1}^{m} (j^2) = \frac{mn(n+1)(2n+1)}{6}
\]
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} (ij) = \sum_{i=1}^{m} i \sum_{j=1}^{n} j = \frac{m(m+1)n(n+1)}{2}
\]

Hence the sum is
\[
\frac{nm(m+1)(2m+1)}{6} + \frac{mn(n+1)(2n+1)}{6} + \frac{mn(m+1)(n+1)}{4}
\]

**Problem 16** The set of \( S \) is countable, because it is a subset of \( Z^3 \), which is countable. (Can be mapped one by one to a set of integers). To show \( Z^3 \) is countable, let \( r = |a| + |b| + |c| \forall (a, b, c) \in Z^3 \). Now \( \forall r \in Z \) there is a finite list of \((a, b, c) \in Z^3 \). Example, for \( r = 0 \), we have \( \{(0, 0, 0)\} \), for \( r = 1 \), we have \( \{(-1, 0, 0), (1, 0, 0), (0, -1, 0), (0, 1, 0), (0, 0, 1)\} \) and so on.