Problem 1.

1) We will prove that this algorithm correctly solves the problem. We would need at least \((B - A) \cdot F\) liters of a gas to drive \(A\) to \(B\). Thus, the minimum total time to be spent to fill the tank is \((B - A) \cdot F/r\) minutes, which is an optimal cost. According to the algorithm given in this part, we would spend \((x_2 - x_1) \cdot F/r\) minutes at the 1st gas station, \((x_3 - x_2) \cdot F/r\) minutes at the 2nd gas station, ..., \((x_n - x_{n-1}) \cdot F/r\) minutes at the \((n-1)\)-th gas station. So the total time to be spent is \((x_n - x_1) \cdot F/r\) minutes, which is the same as \((B - A) \cdot F/r\) minutes.

2) We will disprove it by counterexample. Suppose that the tank has the capacity \(C\) liters, which is much greater than \((B - A) \cdot F\) liters. This implies that we would spend much more time to fill the tank up until it is full at \(A\) than we actually need to drive to \(B\).

Problem 2.

We use the dynamic programming. Suppose that we have \(i\) boxes \(\{1, 2, ..., i\}\) and we can select a subset of boxes such that the sum of their weights is \(j\) pounds. We can consider two cases, whether the \(i\)-th box is included in that subset. If that box is picked for \(j\) pounds, then we could find a subset of \(\{1, 2, ..., i - 1\}\) such that the sum of their weights is \(j - w_i\) pounds. If not, there would be a subset whose weight sum is just \(j\) pounds.

Define a function \(f(i, j)\) such that

\[
f(i, j) = \begin{cases} 
1, & \text{if there is a subset of } \{1, 2, ..., i - 1\} \text{ such that their weight sum is } j \\
0, & \text{otherwise.}
\end{cases}
\]

Then, the recurrence is

\[
f(i, j) = \max\{f(i-1, j - w_i), f(i-1, j)\}
\]

In order to find an optimal weight sum, we just check whether \(f(n, M) = 1\). If it is 0, then check \(f(n, M - 1)\) and so on until we get 1.

The running time of this algorithm is \(O(Mn)\).
Problem 3.

When two ants meet each other, we can think that they just pass through each other instead of turning back. In fact, those are equivalent in terms of the distances made by ants.

1) The longest distance to be taken is 9 inches. Thus, the last ant will fall off the rod at 90 seconds.

2) Since the 1st and the 3rd ants walk toward the right, and the 2nd and the 4th ants walk toward the left, the distances each ant will make would be $10 - x_1$, $x_2$, $10 - x_3$, and $x_4$ inches, respectively. Therefore, $\sum_{i=1}^{4} T_i = 10(20 - x_1 + x_2 - x_3 + x_4)$ seconds.