• **Hints to solutions for HW#12**

• **Exercise 34.5-1:**
  o Prove it is in NP.
  o Prove that CLIQUE ≤p SUBGRAPH-ISOMORPHISM.

• **Exercise 34.5-2:**
  o Prove it is in NP.
  o Prove that 3-CNF-SAT ≤p 0-1 INTEGER-PROGRAMMING

• **Exercise 34.5-3:**
  o Prove it is in NP.
  o Prove that 0-1 INTEGER-PROGRAMMING ≤p INTEGER LINEAR-PROGRAMMING

• **Problem 34-1:**
  o A) The decision problem of the independent set problem (IS) can be formulated as:
    Given a graph $G=(V,E)$ and an integer number $k$, the question is: is there an
    independent set $S$ where $S \subseteq V$ such that the size of this independent set is at
    least $k$.
    In order to prove that the independent set problem is NP-complete:
    ▪ Firstly, if we are given a yes-certificate, which contains a vertex set $SOL$ as
      solution for this independent set decision problem, we can check the size of
      this vertex set $SOL$ (is at least $k$ or not) in polynomial time. Also we can
      examine whether $SOL$ is an independent set of $G$ or not in a polynomial time
      too, this can be done by checking whether each pair in $SOL$ is not connected by
      an edge in $E$.
    ▪ Secondly, to show that the independent set problem is NP-hard, we can reduce
      the clique problem, which is already known to be NP-hard, into this problem. If
      we assume that $CLIQUE(G,k)$ is an instance of the clique problem of the graph
      $G=(V,E)$ with a size $k$ of the clique, we can construct a problem instance of the
      independent set problem $IS(G', k)$ of size $k$, where $G'$ is the complement of $G$.
      $CLIQUE(G,k)$ is true iff $IS(G', k)$ is true. The mapping from the $CLIQUE$ problem
      to the $IS$ can be done in polynomial time
    ▪ By summing up the previous two points above, we show that the independent
      set problem (IS) is NP-complete.

  o B) First of all, we have to find the maximum $k_0$ such that $IS(G, k_0)$ is true. This can be
    done in polynomial time because the values of $k_0 \in [1, 2, 3 \ldots \ldots |V|]$. After that,
    for each vertex $v \in V$, we check whether it belongs to a solution set by the following
procedure: We check if $IS(G-\{v\}, k_0)$ is true or not. If it is NOT (false), then $v$ belongs to the solution set, and we continue working on the sub-problem $IS(G-\{v\} \cup Nei(v)), k_0-1)$. Where $G-S$ is the induced graph with $S$ removed from $G$ and $Nei(v)$ represents the neighbors of vertex $v$. Otherwise, if it is YES (true), we simply work on the sub-problem $IS(G-\{v\}, k_0)$. The recurrence of the running time is $T(n) = T(n-1)+O(1)$ when we assume that the running time of the black-box is constant. So, we can conclude that the running time is polynomial.

- **C)** The degree constraint of this graph $G$ (each vertex in $G$ has degree 2) makes it as a cycle. For a cycle of size $n$, the size of the independent set for it is at most $\lceil n/2 \rceil$. This independent set of a cycle can be obtained by starting from an arbitrary node and sequentially assign the values of $(1, 0, 1, 0, \ldots)$ to the vertices along the cycle. At the end, we choose the vertices which were assigned 1 and these vertices are the maximum independent set. This algorithm takes $O(|V|)$ and its correctness can be simply proved by arguing that we cannot find better result (maximum independent set) for the cycle.

- **D)** In order to find the maximum independent set of a graph $G$ when it is a bipartite (in bipartite graph there are two sides of vertices: Left and Right), we have to select the side with the larger number of vertices. The running time for this (if we assume that the given graph is a bipartite one and we already know the nodes on the left and right sides of the graph), then the running time is $O(|V|)$, $V$ is the number of vertices in the graph. This algorithm is correct because in the bipartite graph all the nodes in one side (either left or right) are not connecting to each other, but they are just connecting to the other side’s nodes. So, by selecting the side with the larger number of vertices we guarantee to find the maximum independent set of the bipartite graph $G$. 