Optimal t-Interleaving on Tori[1]

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Abstract — **The number of integers needed to t-interleave a 2-dimensional torus has a sphere-packing lower bound. We present the necessary and sufficient conditions for tori to meet that lower bound. We prove that for tori sufficiently large in both dimensions, their t-interleaving numbers exceed the lower bound by at most 1. We then show upper bounds on t-interleaving numbers for other cases, completing a general picture for the problem of t-interleaving on 2-dimensional tori. Efficient t-interleaving algorithms are also presented.**

I. Introduction

An $l_1 \times l_2$ torus is a graph with the vertex set $\{v_{i,j} | 0 \le i < l_1, 0 \le j < l_2\}$, where each vertex $v_{i,j}$ has 4 neighbors: $v_{(i-1) \bmod l_1, (j-1) \bmod l_2}$, $v_{(i-1) \bmod l_1, (j+1) \bmod l_2}$, $v_{(i+1) \bmod l_1, (j-1) \bmod l_2}$ and $v_{(i+1) \bmod l_1, (j+1) \bmod l_2}$. It is 2-dimensional. By "*t-interleaving a torus*", we mean to label every vertex of the torus with an integer, such that for any two vertices labelled with the same integer, the shortest path between them contains at least t edges. (An example is shown in Fig. 1 (a).) Given a torus, our objective is to find the minimum number of *distinct* integers needed to t-interleave it — which is called the torus' *t-interleaving number* — as well as the corresponding interleaving method.

t-interleaving generalizes the traditional 1-dimensional interleaving used often in telecommunications, and was originally defined in [1]. The t-interleaving problem for tori has natural applications in distributed data storage and burst error correction, and is closely related to Lee metric codes [2].

II. Main Results

We consider those 2-dimensional tori that have at least t rows and t columns. For those tori, our results include:

- Let $|S_t| = \frac{t^2+1}{2}$ if t is odd, and let $|S_t| = \frac{t^2}{2}$ if t is even. $|S_t|$ is a lower bound for the t-interleaving numbers of tori, which we call the *sphere-packing lower bound*. We prove that an $l_1 \times l_2$ torus' t-interleaving number meets that lower bound if and only if the following condition is satisfied: $|S_t|$ divides both l_1 and l_2 if t is odd, and t divides both l_1 and l_2 if t is even. We present a set of efficient t-interleaving algorithms for such tori, which includes the lattice interleaver (a classic method used previously) as a special case.
- Define a *post-threshold size* (for a given parameter t) to be a pair (θ_1, θ_2) such that whenever $l_1 \ge \theta_1$ and $l_2 \ge \theta_2$, the t-interleaving number of an $l_1 \times l_2$ torus is either $|S_t| + 1$ or $|S_t|$. We prove that such post-threshold sizes exist for every t. We present optimal t-interleaving constructions for tori whose sizes exceed the post-threshold sizes that we found.
- We study upper bounds for t-interleaving numbers. Every $l_1 \times l_2$ torus' t-interleaving number is $|S_t| + O(t^2)$. And that upper bound is tight, even if $l_1 \to +\infty$ or $l_2 \to +\infty$. When both l_1 and l_2 are of the order $\Omega(t^2)$, the t-interleaving number of an $l_1 \times l_2$ torus is $|S_t| + O(t)$.

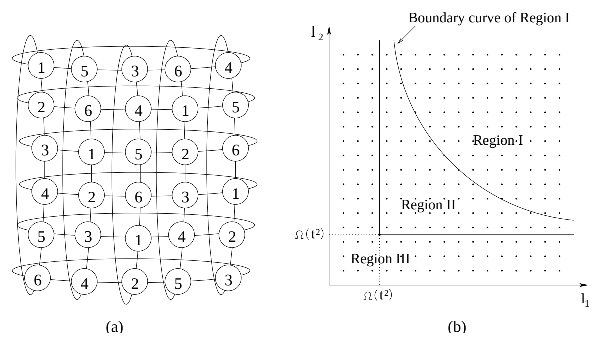

Figure 1: (a) 3-interleaving a 6×5 torus; (b) A qualitative illustration of the t-interleaving numbers.

The results can be illustrated qualitatively (not quantitatively, though) as Fig. 1 (b), which shows for any given t, how the $l_1 \times l_2$ tori can be divided into different classes based on their t-interleaving numbers. The uniform lattice of dots in Fig. 1 (b) are the sizes of the tori whose t-interleaving numbers equal $|S_t|$. The region labelled as '*Region I* ' consists of all the *post-threshold sizes*. The boundary curve of Region I is non-increasing, and symmetric with respect to the line $l_2 = l_1$. Region II is the region where $l_1 = \Omega(t^2)$ and $l_2 = \Omega(t^2)$, in which the tori's t-interleaving numbers are upper-bounded by $|S_t| + O(t)$. Region III includes every torus, where the t-interleaving number is upper-bounded by $|S_t| + O(t^2)$.

For rigorous analysis and detailed results of this paper, please refer to [3].

References

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