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OPTIMAL INTERLEAVING ON TORI*

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Abstract. This paper studies t-interleaving on two-dimensional tori. Interleaving has applications in distributed data storage and burst error correction, and is closely related to Lee metric codes. A *t-interleaving* of a graph is defined as a vertex coloring in which any connected subgraph of t or fewer vertices has a distinct color at every vertex. We say that a torus can be *perfectly t-interleaved* if its *t-interleaving number* (the minimum number of colors needed for a t-interleaving) meets the sphere-packing lower bound, $\lceil t^2/2 \rceil$. We show that a torus is perfectly t-interleavable if and only if its dimensions are both multiples of $\frac{t^2+1}{2}$ (if t is odd) or t (if t is even). The next natural question is how much bigger the t-interleaving number is for those tori that are not perfectly t-interleavable, and the most important contribution of this paper is to find an optimal interleaving for all sufficiently large tori, proving that when a torus is large enough in both dimensions, its t-interleaving number is at most just one more than the sphere-packing lower bound. We also obtain bounds on t-interleaving numbers for the cases where one or both dimensions are not large, thus completing a general characterization of t-interleaving numbers for two-dimensional tori. Each of our upper bounds is accompanied by an efficient t-interleaving scheme that constructively achieves the bound.

Key words. bursts, chromatic number, cluster, error-correcting code, Lee distance, multidimensional interleaving, t-interleaving, torus

AMS subject classifications. 05C15, 05C70, 94B20

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1. Introduction.

Interleaving is an important technique used for error burst correction and network data storage. In communications, interleaving the bits of consecutive codewords guarantees that error bursts will get distributed over many codewords, thus allowing the use of conventional error-correcting codes to correct bursts of errors [16]. The concept of a one-dimensional error burst was generalized to higher dimensions by Blaum, Bruck, and Vardy in [8], where an error burst of size t is defined as a set of errors confined to a connected subgraph of t vertices in a multidimensional array. It is there that the notion of t-interleaving was introduced, the purpose being to color the vertices of a multidimensional array so that every connected subgraph of t vertices receives t distinct colors, and two- and three-dimensional t-interleaving schemes were presented. Such schemes have applications in combatting error bursts in two-dimensional magnetic media and in three-dimensional holographic storage systems and optical recording systems.

Subsequent work on t-interleaving includes [21], where t-interleaving on circulant graphs with two offsets was studied, and [24], where a dual problem of t-interleaving on two-dimensional arrays was explored. The problem of two-dimensional interleaving

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with repetitions was introduced in [7] by Blaum, Bruck, and Farrell, and was extensively studied in [10] by Etzion and Vardy. That problem is to interleave colors on a two-dimensional mesh (array or its variation) in such a way that in every connected subgraph of t vertices, each color appears at most r times. Here t and r are given parameters, and the concept of interleaving with repetitions is a generalization of t-interleaving. More work on interleaving with repetitions includes [17] and [19]. Interleaving schemes on two-dimensional arrays achieving the Reiger bound were studied by Abdel-Ghaffar in [1], where error bursts of both rectangular shapes and arbitrary connected shapes were considered. More examples of interleaving for coping with shaped error bursts include [3] and [6], where the error bursts considered are respectively circular and rectangular.

Interleaving schemes have also been used for network data storage. In [12], an algorithm was presented to interleave N colors on a tree whose edges have lengths, in such a way that for every point of the tree (including a vertex or a point part way along an edge), the smallest ball centered at the point that contains at least N vertices will contain all N colors. That algorithm is useful for minimizing data retrieval delay in distributed data storage systems in hierarchical or tree-like networks. A related interleaving algorithm aimed at the graceful degradation of data-storage performance in faulty environments was presented in [14]. In [13], a scheme called *multicluster interleaving* was studied, which is a scheme to interleave colors on a path or a cycle such that every m disjoint intervals of length L in the path or cycle together contain at least K distinct colors, where $K > L$. Multicluster interleaving can be used for data storage on array-networks, ring-networks, or disks where data gets accessed through multiple access points.

This paper is the first to study t-interleaving on two-dimensional tori. Tori provide an important network structure for parallel and distributed systems [9], [18], [20], [22]. The use of t-interleaving on tori has applications in both burst error correction and distributed data storage, similar to [8], [21], [24], [12] and [14]. Specifically, for distributed data storage, a t-interleaving on a two-dimensional torus ensures that for every vertex, the colors assigned within $\lfloor \frac{t-1}{2} \rfloor$ hops are all distinct. The topic of t-interleaving on tori is closely related to a research topic in coding theory called *Lee metric codes* [2], [4], [5], [11], [15]. In a t-interleaved n-dimensional torus, the set of vertices having any given color is a Lee metric code of length n whose minimum distance is t, and the set of Lee metric codes corresponding to different colors partitions the whole code space.

Here we present some definitions so that we can state our claims precisely. These definitions are straightforward generalizations of the definition of t-interleaving originally given in [8] for arrays.

DEFINITION 1.1. *Let G be a graph. By an* interleaving, *we will mean a vertex coloring, as follows. We say that G is* interleaved (*or there is an* interleaving *on G*) *if each vertex of G is assigned one of a finite number of distinct colors. We say that G is t*-interleaved (*or there is a t*-interleaving *on G*) *if every set of t vertices, forming a connected subgraph of G, is colored by t distinct colors.*

The classic vertex coloring problem is clearly also a t-interleaving problem, where $t = 2$. On the other hand, t-interleaving a graph G is the same as vertex-coloring the power graph G^t, when the power graph G^t is defined as adding an edge to G between each pair of vertices connected by a path of t or fewer vertices. Determining the chromatic number of this kind of power graph is difficult in general. To the best of our knowledge, no result on the type of graphs we are interested in has appeared in the literature.

DEFINITION 1.2. *A two-dimensional $l_1 \times l_2$ torus is a graph containing $l_1 l_2$ vertices and $2l_1 l_2$ edges. We denote its vertices by (i,j) for $0 \le i \le l_1 - 1$ and $0 \le j \le l_2 - 1$.*

$(0,0)$	$(0,1)$	$\cdots$	$(0,l_2-1)$
$(1,0)$	$(1,1)$	$\cdots$	$(1,l_2-1)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(l_1-1,0)$	$(l_1-1,1)$	$\cdots$	(l_1-1,l_2-1)

Each vertex (i,j) is incident to four edges, which connect it to its four neighbors according to the arrangement shown, wrapping around at the boundaries: $((i-1) \bmod l_1, j)$, $((i+1) \bmod l_1, j)$, $(i, (j-1) \bmod l_2)$, and $(i, (j+1) \bmod l_2)$.

Now we can define the problem of t-interleaving on tori.

DEFINITION 1.3. *The minimum number of colors used by any t-interleaving for G is called the* t-interleaving number *of G. A t-interleaving on a torus whose number of colors equals the torus' t-interleaving number is called an* optimal t-interleaving*, as it uses as few colors as possible.*

Example 1.1. The following 5×5 torus is 3-interleaved with 6 colors. The colors are shown as integers from 0 to 5. Each vertex is shown as a square cell in the grid, which is understood to have its left and right edges identified, and its top and bottom edges identified, thus forming a torus.

0	3	1	4	2
1	4	2	0	3
2	0	3	1	5
3	1	5	2	0
4	2	0	3	1

However, the 3-interleaving number of this torus is not 6, since a 3-interleaving does not require 6 colors: If we replace the two instances of color 5 with color 4, we can achieve a 3-interleaving with 5 colors. Thus the 3-interleaving number of this torus is at most 5.

To see that we need 5 colors, consider the vertex $(1,1)$ and its four neighbors $(0,1)$, $(2,1)$, $(1,0)$, and $(1,2)$, and notice that any two of them are contained in a connected subgraph of order 3. Therefore, any 3-interleaving has to assign those 5 vertices 5 distinct colors. Thus the 3-interleaving number of this torus is 5.

Note that a torus that does not have at least t rows and t columns will have the property that there is a path of length less than t which wraps around the torus, going from a vertex to itself. While the definitions can still be understood for such small tori, often the practical application of interleaving results breaks down when this happens, and we will not consider such small tori in this paper.

Assumption 1.1. When discussing t-interleaving for a torus, we will assume that the torus has at least t rows and t columns when t is odd, and at least $t-1$ rows and t columns when t is even.

Our objective in this paper is to find optimal t-interleavings. The t-interleaving number of a torus is by definition the number of colors of an optimal t-interleaving, one which uses the smallest number of colors. A lower bound, which we call the *sphere-packing lower bound*, can be obtained as follows. Figure 1.1 shows six graphs (subgraphs of a torus, assuming they fit on the torus) which we call *spheres* S_1, $S_2, \dots, S_6$, respectively. In general, for any $t \ge 3$, the sphere S_t is obtained by attaching to the sphere S_{t-2} all the vertices adjacent to it. Any two vertices in S_t are connected by a path of at most $t-1$ edges, so a t-interleaving needs to color

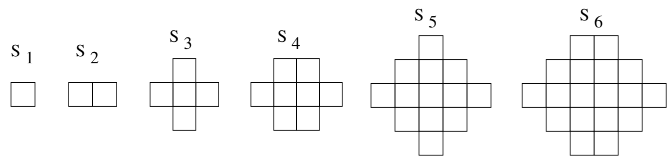

Fig. 1.1. *Six examples of spheres.*

them with different colors. So the number of vertices in S_t, which we shall denote by $|S_t|$, sets a universal lower bound for the t-interleaving number. This argument was originally proposed in [8] for studying t-interleaving on arrays. A direct calculation tells us that $|S_t| = \frac{t^2+1}{2}$ when t is odd, and $|S_t| = \frac{t^2}{2}$ when t is even. We refer to this as the *sphere-packing lower bound.*

We define *perfect t-interleaving* to be a t-interleaving using just $|S_t|$ colors, thus achieving the sphere-packing lower bound, on a torus that has at least t rows and t columns. Clearly any perfect t-interleaving is an optimal t-interleaving.

We will show that a torus can be perfectly interleaved if and only if its sizes in both dimensions are multiples of a certain function of t. Then what about tori of other sizes? Our main result will show that when a torus is sufficiently large in both dimensions, its t-interleaving number exceeds the lower bound $|S_t|$ by at most one.

A more detailed description of our results is as follows:

- We prove that an $l_1 \times l_2$ torus can be perfectly t-interleaved if and only if the following condition is satisfied: when t is odd (respectively, even), both l_1 and l_2 are multiples of $\frac{t^2+1}{2}$ (respectively, t). We reveal the close relationship between perfect t-interleaving and perfect sphere-packing, and present the *complete* set of perfect sphere-packing constructions. Based on that, we obtain a set of efficient perfect t-interleaving constructions, which includes the lattice interleaver scheme presented in [8] as a special case.
- We prove that for any torus that is sufficiently large in both dimensions, its t-interleaving number is either $|S_t|$ or $|S_t|+1$. In other words, any large torus needs at most one more color than a perfect t-interleaving would use if it were possible. More specifically, there exist integer pairs (θ_1, θ_2) such that whenever $l_1 \geq \theta_1$ and $l_2 \geq \theta_2$, the t-interleaving number of an $l_1 \times l_2$ torus is at most $|S_t|+1$. Here θ_1 and θ_2 depend on t, and naturally there is a tradeoff between them: If θ_1 takes a greater value, then the minimum value that θ_2 can take decreases or remains the same, and vice versa. We find a sequence of valid values for θ_1 and θ_2, which are shown in Theorems 4.7 and 4.8. We present optimal t-interleaving constructions for tori whose sizes exceed the found pairs (θ_1, θ_2), and we comment that those constructions, as a general interleaving method, can also be used to optimally t-interleave tori of many other sizes.
- We study upper bounds for t-interleaving numbers, and show that every $l_1 \times l_2$ torus' t-interleaving number is $|S_t| + O(t^2)$. That upper bound is tight, even if $l_1 \to +\infty$ or $l_2 \to +\infty$, meaning that having just one large dimension is not enough to guarantee any significant reduction in the t-interleaving number. When both l_1 and l_2 are of the order $\Omega(t^2)$, the t-interleaving number of an $l_1 \times l_2$ torus is $|S_t| + O(t)$.

The results can be illustrated qualitatively as Figure 1.2, but the figure is not quantitative: The coordinates of points and the shape of the curve are not exact.

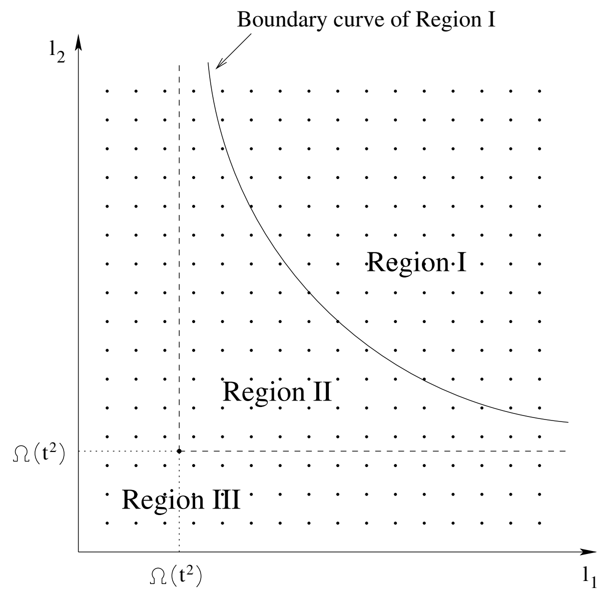

FIG. 1.2. *A qualitative illustration of the t-interleaving numbers.*

Figure 1.2 shows for any given t how the $l_1 \times l_2$ tori can be divided into different classes based on their t-interleaving numbers.

The uniform lattice of dots in Figure 1.2 represents the sizes of all the tori that can be perfectly t-interleaved. The region labeled as Region I consists of all the integer pairs (θ_1, θ_2). The boundary curve of Region I is nonincreasing and symmetric with respect to the line $l_2 = l_1$. We know the exact t-interleaving number of every torus in this region: $|S_t|$ if it is one of the lattice dots, and $|S_t| + 1$ otherwise. The most important contribution of this paper is to prove the existence of Region I and present the corresponding optimal interleaving constructions. Region II is the region where $l_1 = \Omega(t^2)$ and $l_2 = \Omega(t^2)$, in which the tori's t-interleaving numbers are upper-bounded by $|S_t| + O(t)$. Region III includes every torus, where the t-interleaving number is upper-bounded by $|S_t| + O(t^2)$. That upper bound for Region III is tight, even if l_1 or l_2 approaches $+\infty$. Thus, increasing a torus' size in only one dimension does not help reduce the t-interleaving number very effectively in general.

The rest of the paper is organized as follows. In section 2, we show the necessary and sufficient conditions for tori that can be perfectly t-interleaved, and present perfect t-interleaving constructions based on perfect sphere packing. In section 3, we present a t-interleaving method, with which we can t-interleave large tori using just one more than the optimal number of colors. In section 4, we improve upon the t-interleaving method shown in section 3, and present optimal t-interleaving constructions for tori whose sizes are large in both dimensions. As a parallel result, the existence of Region I is proved. In section 5, we prove some general bounds for the t-interleaving numbers. In section 6, we conclude this paper.

2. Perfect t-interleaving. In this section, we show the close relationship between *perfect t-interleaving* and *perfect sphere-packing*, and use it to prove the necessary and sufficient condition for tori to have perfect t-interleaving. We present the complete set of perfect sphere-packing constructions. Based on them, we derive

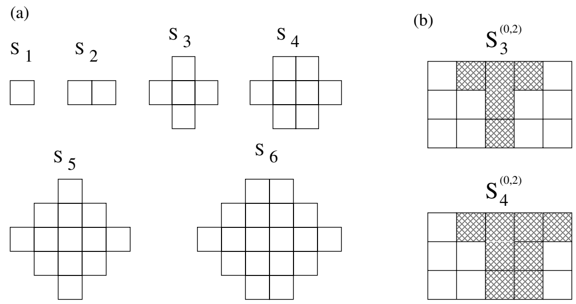

FIG. 2.1. *Examples of the sphere S_t.*

efficient perfect t-interleaving constructions.

2.1. Perfect t-interleaving and sphere-packing. The following is the definition of Lee distance in tori.

DEFINITION 2.1. *The* Lee distance *between two vertices in a torus is the number of edges in the shortest path connecting those two vertices. For two vertices in an $l_1 \times l_2$ torus G, (a_1, b_1) and (a_2, b_2), the* Lee distance *between them is denoted by $d((a_1, b_1), (a_2, b_2))$. Note that therefore, $d((a_1, b_1), (a_2, b_2)) = \min\{(a_1 - a_2) \bmod l_1, (a_2 - a_1) \bmod l_1\} + \min\{(b_1 - b_2) \bmod l_2, (b_2 - b_1) \bmod l_2\}$. Occasionally, in order to emphasize that the two vertices are in G, we also denote it by $d_G((a_1, b_1), (a_2, b_2))$.*

Clearly, an interleaving on a torus is a t-interleaving if and only if the Lee distance between any two vertices of the same color is at least t.

The following is a more detailed definition of spheres than that in the Introduction.

DEFINITION 2.2. *Let G be an $l_1 \times l_2$ torus, where $l_1 \geq 2\lfloor \frac{t-1}{2} \rfloor + 1$ and $l_2 \geq t$, and let (a, b) be a vertex in G. When t is odd, the* sphere centered at (a, b), $S_t^{(a,b)}$ *is defined to be the subgraph induced by all those vertices whose Lee distance to (a, b) is less than or equal to $\frac{t-1}{2}$. When t is even, the* sphere left-centered at (a, b), $S_t^{(a,b)}$ *is defined to be the subgraph induced by all those vertices whose Lee distance to either (a, b) or $(a, (b + 1) \bmod l_2)$ is less than or equal to $\frac{t}{2} - 1$. (a, b) is called the* center *of $S_t^{(a,b)}$ if t is odd, or the* left-center *of $S_t^{(a,b)}$ if t is even. If we do not care where the sphere is centered or left-centered, then the sphere is simply denoted by S_t. The number of vertices in the sphere is denoted by $|S_t|$.*

Example 2.1. Figure 2.1(a) shows the spheres S_1 to S_6. Figure 2.1(b) shows two spheres, $S_3^{(0,2)}$ and $S_4^{(0,2)}$, in a 3×5 torus.

For any $l_1 \times l_2$ torus, where $l_1 \geq 2\lfloor \frac{t-1}{2} \rfloor + 1$ and $l_2 \geq t$, its t-interleaving number is at least $|S_t|$. We call $|S_t|$ the *sphere-packing lower bound.* The relationship between this bound and sphere-packing will become clearer soon.

DEFINITION 2.3. *A torus G is said to have a* perfect packing of spheres S_t *if spheres S_t are packed in G in such a way that every vertex of G lies in exactly one of the spheres.*

LEMMA 2.4. (1) *Let t be odd. An interleaving on an $l_1 \times l_2$ torus (where $l_1 \geq t$ and $l_2 \geq t$) is a t-interleaving if and only if for any two vertices (a_1, b_1) and (a_2, b_2) of the same color, the two spheres centered at them, $S_t^{(a_1,b_1)}$ and $S_t^{(a_2,b_2)}$, do not share any common vertex.*

(2) *Let t be even. An interleaving on an $l_1 \times l_2$ torus (where $l_1 \geq t - 1$ and $l_2 \geq t$) is a t-interleaving if and only if for any two vertices (a_1, b_1) and (a_2, b_2) of the*

same color, the two spheres left-centered there, $S_t^{(a_1,b_1)}$ *and* $S_t^{(a_2,b_2)}$*, do not share any common vertex and, what is more,* $b_1 \neq b_2$ *or* $(a_1 - a_2) \neq \pm(t-1) \bmod l_1$*.*

Proof. (1) Let t be odd. Both $S_t^{(a_1,b_1)}$ and $S_t^{(a_2,b_2)}$ are classic spheres with radius $\frac{t-1}{2}$. If the interleaving is a t-interleaving, then the Lee distance between (a_1, b_1) and (a_2, b_2) is at least $t = 2 \cdot \frac{t-1}{2} + 1$, so $S_t^{(a_1,b_1)}$ and $S_t^{(a_2,b_2)}$ must have no intersection. The converse is also true.

(2) Let t be even. We consider two cases: $b_1 = b_2$ and $b_1 \neq b_2$.

First consider the case $b_1 = b_2$. In this case, $S_t^{(a_1,b_1)}$ and $S_t^{(a_2,b_2)}$ have no intersection if and only if $d((a_1, b_1), (a_2, b_2)) \geq 2 \cdot (\frac{t}{2} - 1) + 1 = t - 1$. Further, $d((a_1, b_1), (a_2, b_2)) = t - 1$ if and only if $(a_1 - a_2) \equiv \pm(t - 1) \bmod l_1$. So the Lee distance between (a_1, b_1) and (a_2, b_2) is at least t if and only if $S_t^{(a_1,b_1)}$ and $S_t^{(a_2,b_2)}$ have no intersection and $(a_1 - a_2) \neq \pm(t-1) \bmod l_1$, which is the conclusion we want.

Now consider the case $b_1 \neq b_2$. In this case, the Lee distance between (a_1, b_1) and (a_2, b_2) is at least $t \Leftrightarrow$ both the Lee distance between $(a_1, (b_1 + 1) \bmod l_2)$ and (a_2, b_2) and the Lee distance between $(a_2, (b_2 + 1) \bmod l_2)$ and (a_1, b_1) are at least $t - 1 \Leftrightarrow S_{t-1}^{(a_1,(b_1+1) \bmod l_2)}$ does not intersect $S_{t-1}^{(a_2,b_2)}$ and $S_{t-1}^{(a_2,(b_2+1) \bmod l_2)}$ does not intersect $S_{t-1}^{(a_1,b_1)} \Leftrightarrow S_t^{(a_1,b_1)}$ and $S_t^{(a_2,b_2)}$ have no intersection. Note that $S_t^{(a_1,b_1)}$ is the union of $S_{t-1}^{(a_1,b_1)}$ and $S_{t-1}^{(a_1,(b_1+1) \bmod l_2)}$, and $S_t^{(a_2,b_2)}$ is the union of $S_{t-1}^{(a_2,b_2)}$ and $S_{t-1}^{(a_2,(b_2+1) \bmod l_2)}$. So we get the conclusion. ∎

THEOREM 2.5. *For an* $l_1 \times l_2$ *torus, where* $l_1 \geq t$ *and* $l_2 \geq t$*, if an interleaving on it is a perfect* t*-interleaving, then for every color the spheres* S_t *centered or left-centered at the vertices of that color form a perfect sphere-packing in the torus. The converse is also true when* $t \neq 2$*.*

Proof. Let us say that the torus is interleaved. We used I to denote the set of distinct colors used by the interleaving. For any color $i \in I$, we use N_i to denote the number of vertices of color i.

Let us first prove one direction. Assume that the interleaving is a perfect t-interleaving. Then $|I| = |S_t|$. By Lemma 2.4, for any $i \in I$, the spheres S_t centered or left-centered at vertices of color i do not overlap. By counting the number of vertices in the torus and in each sphere S_t, we get $N_i \leq \frac{l_1 l_2}{|S_t|}$ for any $i \in I$. Since $\sum_{i \in I} N_i = l_1 l_2$, we get $N_i = \frac{l_1 l_2}{|S_t|}$ for any $i \in I$. So for any color $i \in I$, the spheres S_t centered or left-centered at the vertices of color i form a perfect sphere-packing in the torus.

Now let us prove the converse direction. Assume $t \neq 2$. Also assume for every color that the spheres S_t centered or left-centered at the vertices of that color form a perfect sphere packing in the torus. Then $N_i = \frac{l_1 l_2}{|S_t|}$ for any $i \in I$. Since $\sum_{i \in I} N_i = l_1 l_2$, we get $|I| = |S_t|$. What is left to prove is that the interleaving is a t-interleaving. By Lemma 2.4, the interleaving can fail to be a t-interleaving only if the following situation becomes true: "t is even, and there exist two vertices (a_1, b_1) and (a_2, b_2) of the same color such that $b_1 = b_2$ and $a_1 - a_2 \equiv t - 1 \bmod l_1$." We will now show that such a situation cannot happen.

Assume that situation happens. Then it is straightforward to verify that the four vertices $(a_1 - (\frac{t}{2} - 1) \bmod l_1, b_1)$, $(a_2 + (\frac{t}{2} - 1) \bmod l_1, b_1)$, $(a_1 - (\frac{t}{2} - 2) \bmod l_1, b_1 - 1 \bmod l_2)$, and $(a_2 + (\frac{t}{2} - 2) \bmod l_1, b_1 - 1 \bmod l_2)$ are contained in either $S_t^{(a_1,b_1)}$ or $S_t^{(a_2,b_2)}$, while the two vertices $(a_1 - (\frac{t}{2} - 1) \bmod l_1, b_1 - 1 \bmod l_2)$ and $(a_2 + (\frac{t}{2} - 1) \bmod l_1, b_1 - 1 \bmod l_2)$ are neither contained in $S_t^{(a_1,b_1)}$ nor in $S_t^{(a_2,b_2)}$. The two vertices $(a_1 - (\frac{t}{2} - 1) \bmod l_1, b_1 - 1 \bmod l_2)$ and $(a_2 + (\frac{t}{2} - 1) \bmod l_1, b_1 - 1 \bmod l_2)$ cannot

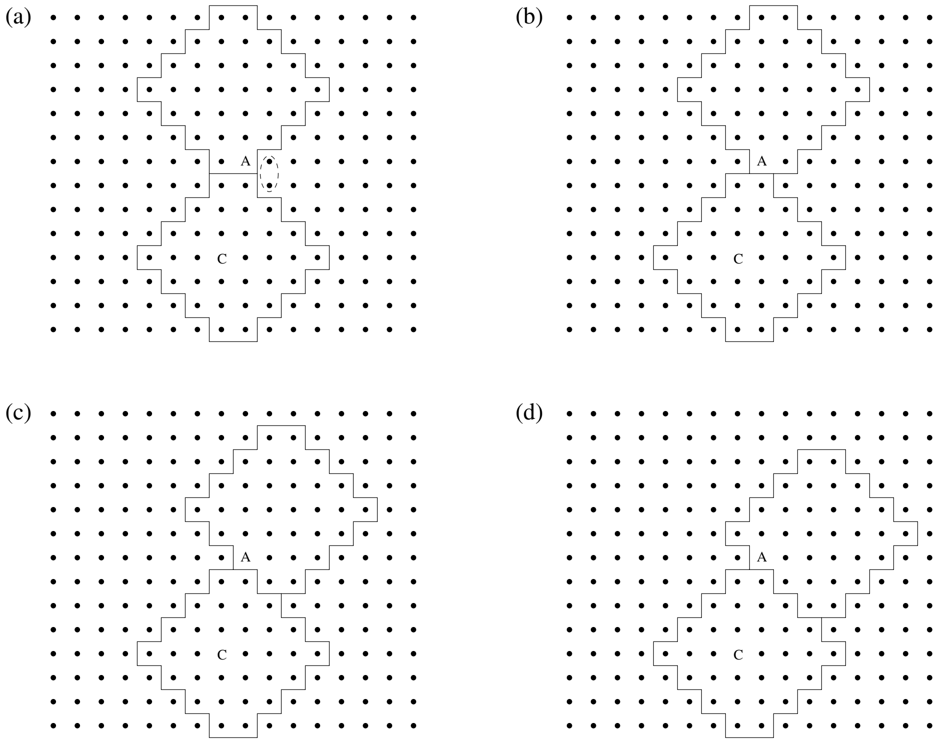

FIG. 2.2. *Relative positions of spheres and vertices.*

both be contained in spheres S_t that are left-centered at vertices having the color of (a_1, b_1) and (a_2, b_2), because they are vertically adjacent, and the vertices directly above them, below them, and to the right of them are all contained in two spheres that do not contain them, due to the shape of the sphere, as seen in Figure 2.2(a). This contradicts that fact that all the spheres S_t, left-centered at the vertices having the same color as (a_1, b_1), form a perfect sphere-packing in the torus. So the assumed situation cannot happen. Summarizing the above results, we see that the interleaving must be a perfect t-interleaving. ∎

THEOREM 2.6. *For an $l_1 \times l_2$ torus, where $l_1 \geq t$ and $l_2 \geq t$, if it can be perfectly t-interleaved, then the spheres S_t can be perfectly packed in it. The converse is also true when $t \neq 2$.*

Proof. Let G be an $l_1 \times l_2$ torus. For any t, Theorem 2.5 has shown that if G can be perfectly t-interleaved, then the spheres S_t can be perfectly packed in it. Now we prove the other direction. Assume $t \neq 2$, and that the spheres S_t can be perfectly packed in G. Let (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$ be a set of vertices such that the spheres S_t centered or left-centered at them form a perfect packing in G. The proof of Theorem 2.5 has essentially shown that for any i and j $(i \neq j)$, the Lee distance between (x_i, y_i) and (x_j, y_j) is at least t. Now we can interleave G is this way: Color each sphere S_t with $|S_t|$ distinct colors in the same way, so that every color is used in exactly the same position in every sphere. Clearly, for any two colors a and b, the two sets of vertices colored by a and b are translates of each other in the torus, and therefore the Lee distance between any two vertices of the same color is at least t. Thus G has a perfect t-interleaving. ∎

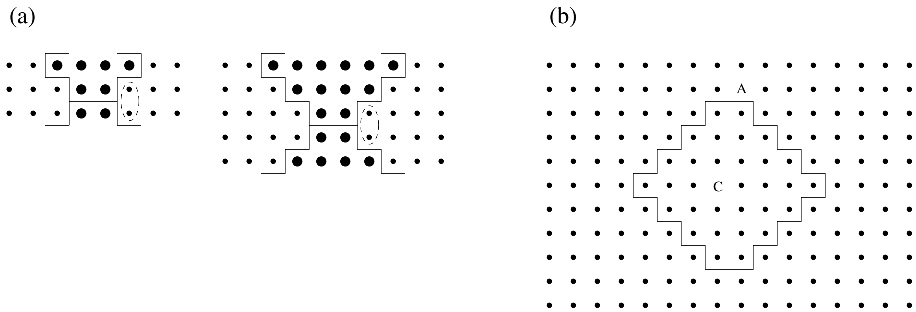

FIG. 2.3. *A sphere in a torus.*

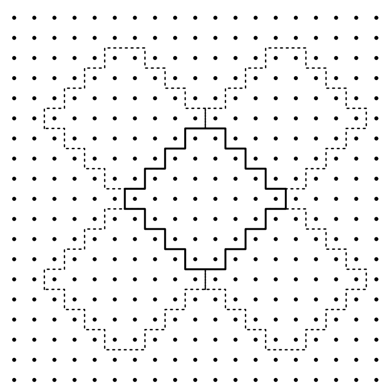

FIG. 2.4. *Four positions that a neighbor sphere might be in.*

2.2. Perfect t-interleaving and its construction. The following lemma is an important property of perfect sphere-packing. It will help us derive the necessary and sufficient condition for perfect t-interleaving.

LEMMA 2.7. *Let t be even and $t \geq 4$. When spheres S_t are perfectly packed in an $l_1 \times l_2$ torus, there exists an integer $a \in \{+1, -1\}$ such that if there is a sphere left-centered at the vertex (x, y), then there are two spheres respectively left-centered at $((x - \frac{t}{2}) \bmod l_1, (y - a \cdot \frac{t}{2}) \bmod l_2)$ and $((x + \frac{t}{2}) \bmod l_1, (y + a \cdot \frac{t}{2}) \bmod l_2)$.*

Proof. Assume that spheres S_t are perfectly packed in an $l_1 \times l_2$ torus, where $t \geq 4$ and t is even. First we observe that $l_1 \geq t$: Since a sphere S_t spans $t - 1$ rows when t is even, l_1 must be at least $t - 1$, but l_1 cannot be exactly $t - 1$ either, because then, as shown in Figure 2.3(a), the sphere will just touch itself, and it is clearly impossible to cover the two adjacent positions marked by dashed circles in Figure 2.3(a) using nonoverlapping spheres. Thus $l_1 \geq t$.

Clearly, one of the following two cases must be true, concerning the presence or absence of any of the four possible neighbor spheres shown in Figure 2.4:

- Case 1. Whenever there is a sphere left-centered at a vertex (x, y), there are four spheres respectively left-centered at the four vertices $((x - \frac{t}{2}) \bmod l_1, (y - \frac{t}{2}) \bmod l_2)$, $((x - \frac{t}{2}) \bmod l_1, (y + \frac{t}{2}) \bmod l_2)$, $((x + \frac{t}{2}) \bmod l_1, (y - \frac{t}{2}) \bmod l_2)$, and $((x + \frac{t}{2}) \bmod l_1, (y + \frac{t}{2}) \bmod l_2)$.
- Case 2. There exists a sphere left-centered at a vertex (x_0, y_0) such that there is no sphere left-centered at at least one of the following four vertices:

$((x_0 - \frac{t}{2}) \bmod l_1, (y_0 - \frac{t}{2}) \bmod l_2)$, $((x_0 - \frac{t}{2}) \bmod l_1, (y_0 + \frac{t}{2}) \bmod l_2)$, $((x_0 + \frac{t}{2}) \bmod l_1, (y_0 - \frac{t}{2}) \bmod l_2)$, and $((x_0 + \frac{t}{2}) \bmod l_1, (y_0 + \frac{t}{2}) \bmod l_2)$.

If Case 1 is true, then the conclusion of this lemma obviously holds. From now on, let us assume that Case 2 is true. Without loss of generality, we assume that there is one sphere left-centered at (x_0, y_0), but there is no sphere left-centered at $((x_0 - \frac{t}{2}) \bmod l_1, (y_0 + \frac{t}{2}) \bmod l_2)$.

Since $l_1 \geq t$, the vertex $((x_0 - \frac{t}{2}) \bmod l_1, (y_0 + 1) \bmod l_2)$, which we shall call vertex A, is not contained in the sphere left-centered at (x_0, y_0). An example is shown in Figure 2.3(b), where the sphere in consideration is an S_8, whose left-center (x_0, y_0) is labeled by C. The vertex A is contained in one of the perfectly packed spheres, which we shall call sphere B. The relative position of vertex A in sphere B can only be one of the following two possibilities:

- Possibility 1. The vertex A is the right-most vertex in the bottom row of the sphere B, as in Figure 2.2(a).
- Possibility 2. The vertex A is in the lower-left diagonal of the border of the sphere B, as in Figure 2.2(b), (c), and (d). Note that it cannot be the left-most vertex of the sphere B, because that is the location where we are assuming there is not a sphere.

Possibility 1, however, as we saw in Figure 2.2(a), is impossible. So we are left with Possibility 2. In the following proof we use the example of $t = 8$ for illustration, and assume that the relative position of the sphere B is as shown in Figure 2.2(b). We comment that when t takes other values or when the sphere B takes one of the three other positions, it is easy to see that the argument still holds.

Let the sphere left-centered at (x_0, y_0) be the sphere denoted by L_1 in Figure 2.5, and let sphere B be the sphere denoted by R_1 in Figure 2.5. We immediately see that the vertex denoted by E must be the right-most vertex of a sphere, so the sphere containing the vertex E must be the sphere denoted by L_2. Then we immediately see that the vertex denoted by F must be the right-most vertex in the bottom row of a sphere, so the sphere containing the vertex F must be the sphere denoted by R_2. With the same method we can fix the positions of a series of spheres L_1, L_2, L_3, L_4, ... and a series of spheres R_1, R_2, R_3, R_4, ... Since the torus is finite, we will get a series of spheres L_1, L_2, L_3, L_4, ..., L_n such that the relative position of L_n to L_1 is the same as the relative position of L_1 to L_2 (see Figure 2.5 for an illustration). Such a series of spheres forms a cycle in the torus. Since the spheres are perfectly packed in the torus, no two spheres in this cycle overlap. Similarly, the spheres R_1, R_2, ..., R_n also form a cycle in the torus. Note that we do not make any assumption about whether these two cycles overlap or not.

If those two cycles do not already contain all the spheres in the torus, then there must be some spheres outside the two cycles that are directly attached to the lower-left side of the cycle formed by $L_1, L_2, \ldots, L_n$. This is due to the very regular way the cycle is formed and the resulting shape of the cycle, which is invariant to horizontal and vertical shifts. Let D_1 be a sphere directly attached to the cycle formed by $L_1, L_2, \ldots, L_n$, as shown in Figure 2.5. Note that we do not care about the exact position of D_1, as long as it is directly attached to the lower-left side of the cycle. Then the vertex I immediately determines that the sphere containing it must be D_2, and similarly the vertex J determines the position of the sphere D_3, and so on. So we will get a series of spheres $D_1, D_2, D_3, \ldots, D_n$, which will again form a cycle. It is easy to see that this cycle does not overlap the previous two cycles. Continuing in this way, we can keep finding cycles until they cover the torus.

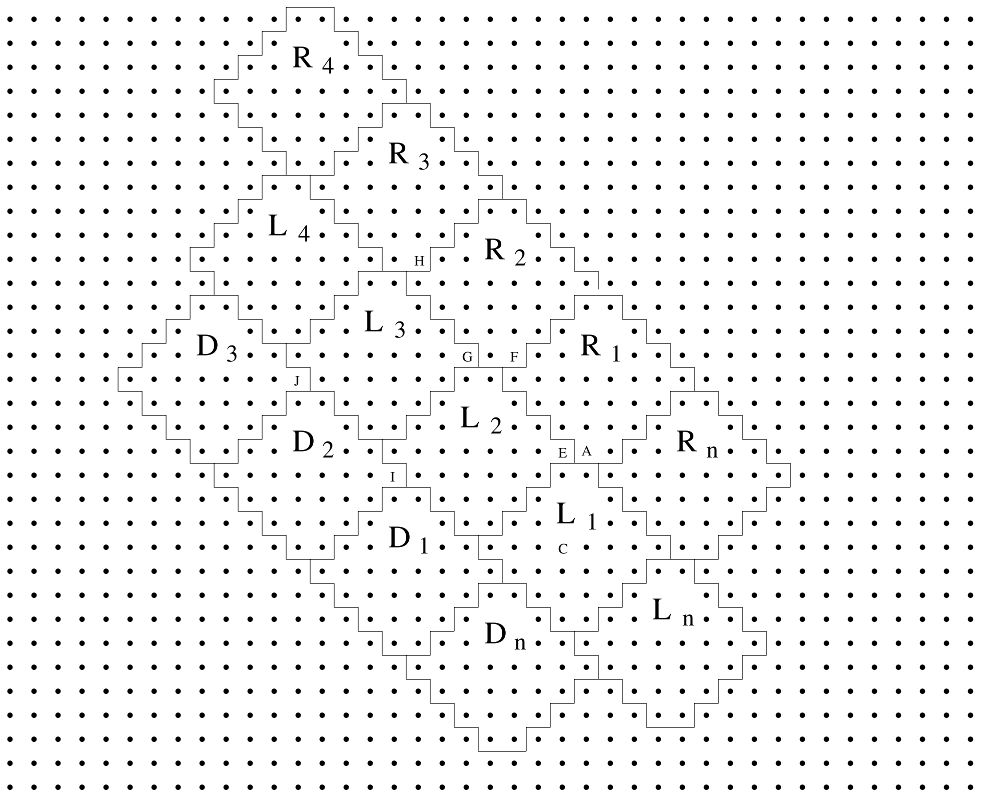

FIG. 2.5. *The packing of spheres in a torus.*

We can easily see that in each of the cycles here, if there is a sphere left-centered at a vertex (x, y), then there are two spheres respectively left-centered at $((x - \frac{t}{2}) \bmod l_1, (y - \frac{t}{2}) \bmod l_2)$ and $((x + \frac{t}{2}) \bmod l_1, (y + \frac{t}{2}) \bmod l_2)$. In the other instances of Case 2, we either find the same pattern of cycles or else we find the flipped pattern, in which whenever there is a sphere left-centered at a vertex (x, y), there are two spheres respectively left-centered at $((x - \frac{t}{2}) \bmod l_1, (y + \frac{t}{2}) \bmod l_2)$ and $((x + \frac{t}{2}) \bmod l_1, (y - \frac{t}{2}) \bmod l_2)$. The parameter a in the statement of the lemma represents which of the two patterns is being used. ∎

DEFINITION 2.8. *Let t be an even positive integer, let a be either $+1$ or -1, and let G be an $l_1 \times l_2$ torus. Let (x, y) be an arbitrary vertex in G. We define the* cycle containing (x, y) (*corresponding to the parameter a*) *to be the set of spheres S_t that are respectively left-centered at the vertices (x, y), $((x + \frac{t}{2}) \bmod l_1, (y + a \cdot \frac{t}{2}) \bmod l_2)$, $((x + 2 \cdot \frac{t}{2}) \bmod l_1, (y + 2a \cdot \frac{t}{2}) \bmod l_2)$, $((x + 3 \cdot \frac{t}{2}) \bmod l_1, (y + 3a \cdot \frac{t}{2}) \bmod l_2)$, ...*

The proof of the following lemma is omitted due to its simplicity.

LEMMA 2.9. *Let t be an even positive integer, let a be either $+1$ or -1, and let G be an $l_1 \times l_2$ torus. For any vertex (x, y) in G, the cycle containing it* (*corresponding to the parameter a*) *consists of $\frac{lcm(l_1, l_2, \frac{t}{2})}{\frac{t}{2}}$ distinct spheres S_t.*

The following theorem shows the necessary and sufficient condition for tori that can be perfectly t-interleaved.

THEOREM 2.10. *Let G be an $l_1 \times l_2$ torus, where $l_1 \geq t$ and $l_2 \geq t$. If t is odd, then G can be perfectly t-interleaved if and only if both l_1 and l_2 are multiples of $\frac{t^2+1}{2}$.*

If t is even, then G can be perfectly t-interleaved if and only if both l_1 and l_2 are multiples of t.

Proof. We consider the following three cases separately.

Case 1: $t = 2$. In this case, 2-interleaving is equivalent to vertex coloring, so the 2-interleaving number of G equals G's chromatic number $\chi(G)$. Let R_1 and R_2 each be a graph consisting of a single cycle, having l_1 and l_2 vertices, respectively. Then G is the Cartesian product of those two cycles, namely, $G = R_1 \otimes R_2$. It is well known [23] that for any two graphs H_1 and H_2, $\chi(H_1 \otimes H_2) = \max\{\chi(H_1), \chi(H_2)\}$. Since $l_1 \geq t = 2$ (respectively, $l_2 \geq t = 2$), we get that $\chi(R_1) \geq 2$ (respectively, $\chi(R_2) \geq 2$), and $\chi(R_1) = 2$ (respectively, $\chi(R_2) = 2$) if and only if l_1 (respectively, l_2) is a multiple of 2. So $\chi(G) = 2$ if and only if both l_1 and l_2 are multiples of 2. Since $|S_2| = 2$, we get the conclusion in this lemma.

Case 2: t is even but $t \neq 2$. First, we prove one direction. Assume that G can be perfectly t-interleaved. We will show that both l_1 and l_2 are multiples of t. Let i be a color used by a perfect t-interleaving on G. Then by Theorem 2.5, the spheres S_t left-centered at the vertices of color i form a perfect sphere-packing in G. By Lemma 2.7, there exists an integer $a \in \{+1, -1\}$ such that for any cycle containing a vertex of color i (corresponding to the parameter a), the spheres S_t in the cycle are all left-centered at vertices of color i, and therefore they do not overlap. By Lemma 2.9, the cycle containing a vertex of color i consists of $\frac{lcm(l_1,l_2,\frac{t}{2})}{\frac{t}{2}}$ distinct spheres S_t. So such a cycle consists of

$$\frac{lcm(l_1, l_2, \frac{t}{2})}{\frac{t}{2}} \cdot |S_t| = \frac{lcm(l_1, l_2, \frac{t}{2})}{\frac{t}{2}} \cdot \frac{t^2}{2} = lcm\left(l_1, l_2, \frac{t}{2}\right) \cdot t$$

vertices. Let (x_1, y_1) and (x_2, y_2) be any two vertices of color i. We can see that for the cycle containing (x_1, y_1) and the cycle containing (x_2, y_2), either they do not overlap or they are the same cycle. Therefore, the vertices in G can be partitioned into several such cycles, so $l_1 \cdot l_2$ is a multiple of $lcm(l_1, l_2, \frac{t}{2}) \cdot t$. Since $lcm(l_1, l_2, \frac{t}{2})$ is a multiple of l_1, l_2 must be a multiple of t. Similarly, l_1 must be a multiple of t, too. So if G can be perfectly t-interleaved, then both l_1 and l_2 are multiples of t.

Now we prove the other direction. Assume both l_1 and l_2 are multiples of t. Let W be such a set of vertices in G: $W = \{(x, y) | x \equiv 0 \bmod \frac{t}{2}, y \equiv 0 \bmod \frac{t}{2}, x + y \equiv 0 \bmod t\}$. It is easy to verify that the Lee distance between any two vertices in W is at least t. Now for $i = 0, 1, \ldots, \frac{t}{2} - 1$ and for $j = 0, 1, \ldots, t - 1$, define $W^{i,j}$ to be $W^{i,j} = \{((x + i) \bmod l_1, (y + j) \bmod l_2) | (x, y) \in W\}$. Clearly those $\frac{t}{2} \cdot t = |S_t|$ sets, $W^{0,0}$, $W^{0,1}$, ..., $W^{\frac{t}{2}-1,t-1}$, are a partition of the vertices in G. For each $W^{i,j}$, we color the vertices in it with the same distinct color. Clearly such an interleaving is a perfect t-interleaving. So if both l_1 and l_2 are multiples of t, then G can be perfectly t-interleaved.

Case 3: t is odd. First, we prove one direction. Assume that both l_1 and l_2 are multiples of $\frac{t^2+1}{2}$. Golomb and Welch have shown in [11] that a $\frac{t^2+1}{2} \times \frac{t^2+1}{2}$ torus can be perfectly packed by the spheres S_t for odd t. Therefore, G can also be perfectly packed by S_t because a torus has a toroidal topology and G can be folded onto itself into an $\frac{t^2+1}{2} \times \frac{t^2+1}{2}$ torus. Let C be a set of vertices in G such that the spheres S_t centered at the vertices in C form a perfect sphere-packing. Then the Lee distance between any two vertices in C is at least t. We call a set of vertices D a *translate* of C when the following condition is satisfied: "There exist integers a and b such that a vertex $(x, y) \in C$ if and only if $((x + a) \bmod l_1, (y + b) \bmod l_2) \in D$." C has $|S_t|$

different translates in total (including C itself), and those translates partition the vertices of G. For each translate, we color its vertices with one distinct color, and we get a perfect t-interleaving. So if both l_1 and l_2 are multiples of $\frac{t^2+1}{2}$, then G can be perfectly t-interleaved.

Now we prove the other direction. Assume that G can be perfectly t-interleaved. Let i be a color used by a perfect t-interleaving on G. Then by Theorem 2.5, the spheres S_t centered at the vertices of color i form a perfect sphere-packing in G. Golomb and Welch presented in [11] a way to perfectly pack spheres S_t in a torus when t is odd, which can be described as "either of the following two conditions is true: (1) Whenever there is a sphere S_t centered at a vertex (x, y), there are two spheres respectively centered at $((x+\frac{t+1}{2}) \bmod l_1, (y+\frac{t-1}{2}) \bmod l_2)$ and $((x-\frac{t-1}{2}) \bmod l_1, (y+\frac{t+1}{2}) \bmod l_2)$; (2) whenever there is a sphere S_t centered at a vertex (x, y), there are two spheres respectively centered at $((x + \frac{t-1}{2}) \bmod l_1, (y + \frac{t+1}{2}) \bmod l_2)$ and $((x - \frac{t+1}{2}) \bmod l_1, (y + \frac{t-1}{2}) \bmod l_2)$". It is easy to see that that way of packing is in fact the only way to perfectly pack S_t for odd t, whose feasibility requires both l_1 and l_2 to be multiples of $\frac{t^2+1}{2}$. Thus if G can be perfectly t-interleaved, then both l_1 and l_2 are multiples of $\frac{t^2+1}{2}$. ∎

Below we present the complete set of perfect sphere-packing constructions. But first let us explain a few concepts. Let G be an $l_1 \times l_2$ torus that is perfectly packed by spheres S_t, so there are $\frac{l_1 l_2}{|S_t|}$ such spheres. Define $e = \frac{l_1 l_2}{|S_t|}$, and let us say that those spheres are centered (or left-centered) at the vertices (x_1, y_1), (x_2, y_2), $\ldots$, (x_e, y_e). By *vertically* (respectively, *horizontally*) *shifting the spheres in* G, we mean to select some integer s, and get a new set of perfectly packed spheres that are centered (or left-centered) at $(x_1 + s \bmod l_1, y_1)$, $(x_2 + s \bmod l_1, y_2)$, $\ldots$, $(x_e + s \bmod l_1, y_e)$ (respectively, at $(x_1, y_1 + s \bmod l_2)$, $(x_2, y_2 + s \bmod l_2)$, $\ldots$, $(x_e, y_e + s \bmod l_2)$). By *vertically reversing the spheres in* G, we mean to get a new set of perfectly packed spheres that are centered (or left-centered) at $(-x_1 \bmod l_1, y_1)$, $(-x_2 \bmod l_1, y_2)$, $\ldots$, $(-x_e \bmod l_1, y_e)$. After such a shift or reverse operation, technically speaking, the way the spheres are perfectly packed in G is changed. However, the pattern of the sphere-packing essentially remains the same.

Construction 2.1. The complete set of perfect sphere-packing constructions.

Input: A positive integer t. An $l_1 \times l_2$ torus G, where (1) both l_1 and l_2 are multiples of t if t is even and $t \neq 2$, (2) l_2 is even if $t = 2$, and (3) both l_1 and l_2 are multiples of $\frac{t^2+1}{2}$ if t is odd.

Output: A perfect packing of the spheres S_t in G.

Construction:

1. If t is even and $t \neq 2$, then do the following:
   - Let A_1, A_2, $\ldots$, $A_{gcd(\frac{l_1}{t}, \frac{l_2}{t})-1}$ be $gcd(\frac{l_1}{t}, \frac{l_2}{t}) - 1$ integers, where A_i can be any integer in the set $\{0, 1, \ldots, \frac{t}{2} - 1\}$ for $i = 1, 2, \ldots, gcd(\frac{l_1}{t}, \frac{l_2}{t}) - 1$.
   - Find the $gcd(\frac{l_1}{t}, \frac{l_2}{t})$ cycles in G respectively containing the vertices $(0, 0)$, $(A_1, t+A_1)$, $(A_1+A_2, 2t+A_1+A_2)$, $\ldots$, and $(\sum_{i=1}^{gcd(\frac{l_1}{t}, \frac{l_2}{t})-1} A_i, \sum_{i=1}^{gcd(\frac{l_1}{t}, \frac{l_2}{t})-1} (t + A_i))$. The spheres S_t in those $gcd(\frac{l_1}{t}, \frac{l_2}{t})$ cycles form a perfect sphere-packing in the torus.
2. If $t = 2$, the do the following:
   - The $l_1 \times l_2$ torus G has l_1 rows, each of which can be seen as a ring of l_2 vertices. When $t = 2$, the sphere S_t simply consists of two horizontally adjacent vertices. Split each row of G into $\frac{l_2}{2}$ spheres in any way. The resulting $\frac{l_1 l_2}{2}$ spheres form a perfect sphere-packing in the torus.

3. If t is odd, then do the following:
   - Find a set of $\frac{l_1 l_2}{|S_t|}$ spheres S_t such that each of the spheres is centered at a vertex $(i \cdot \frac{t+1}{2} + j \cdot \frac{1-t}{2} \bmod l_1, i \cdot \frac{t-1}{2} + j \cdot \frac{t+1}{2} \bmod l_2)$ for some integers i and j. Those spheres form a perfect sphere-packing in the torus.
4. Horizontally shift, vertically shift, and/or vertically reverse the spheres in G in any way.

THEOREM 2.11. *Construction* 2.1 *is the* complete *set of perfect sphere-packing constructions.*

Proof. We consider the following three cases. For each case, we need to prove two things: First, the *input* part of Construction 2.1 sets the necessary and sufficient condition for a torus to have a perfect sphere-packing; second, the *construction* part of Construction 2.1 generates perfect sphere-packing correctly, and every perfect sphere-packing that exists is a possible output of it.

Case 1: t is even and $t \neq 2$. Lemma 2.7 and its proof have shown that when spheres are perfectly packed in a torus, those spheres can be partitioned into cycles. By observing the shape of the border of a cycle, we see that two adjacent cycles can freely slide along each other's border, and there are $\frac{t}{2}$ possible relative positions for two adjacent cycles. In Construction 2.1, the $\frac{t}{2}$ possible relative positions are determined by A_i, a variable that can take $\frac{t}{2}$ possible values. Now it is easy to see that step 1 of Construction 2.1 provides a perfect sphere-packing (which takes one of many possible forms, depending on the value of A_i), and step 4 changes the positions of the spheres to furthermore cover all the possible cases of perfect sphere-packing.

Case 2: $t = 2$. We skip the proof for this case due to its simplicity.

Case 3: t is odd. In this case, Construction 2.1 reproduces the sphere-packing method presented in [11], which is commonly known as the unique way to pack spheres for odd t (see the final paragraph of the proof of Theorem 2.10 for a more detailed introduction). ∎

Now we present perfect t-interleaving constructions that are based on perfect sphere-packing.

Construction 2.2. Perfect t-interleaving constructions

Input: A positive integer t. An $l_1 \times l_2$ torus G, where both l_1 and l_2 are multiples of t if t is even, and both l_1 and l_2 are multiples of $\frac{t^2+1}{2}$ if t is odd.

Output: A perfect t-interleaving on G.

Construction:

1 If $t \neq 2$, then do the following:
   - Use Construction 2.1 to get a perfect sphere-packing in G. Color each sphere in the same way, using $|S_t|$ distinct colors, so that each color is used exactly once in each sphere.

2 If $t = 2$, then do the following:
   - For every vertex (i, j) of G, if $i + j$ is even, color it with color 0; otherwise color it with color 1.

The following example illustrates how to use Construction 2.1 to obtain perfect sphere-packing, and how to use Construction 2.2 to obtain perfect t-interleaving.

Example 2.2. Let $t = 4$, and let G be a 12×24 torus. First, we use Construction 2.1 to find a perfect sphere-packing in G. Since t is even, step 1 of Construction 2.1 is executed. We choose $A_1, A_2, \ldots, A_{gcd(\frac{l_1}{t}, \frac{l_2}{t})-1}$ to be $A_1 = 0$, $A_2 = 1$. Note that here $gcd(\frac{l_1}{t}, \frac{l_2}{t}) - 1 = 2$. Then the $gcd(\frac{l_1}{t}, \frac{l_2}{t}) = 3$ cycles in G are as shown in Figure 2.6(a), which are three sets of spheres S_t respectively of three different background shades. The spheres in those three cycles form a perfect packing in G.

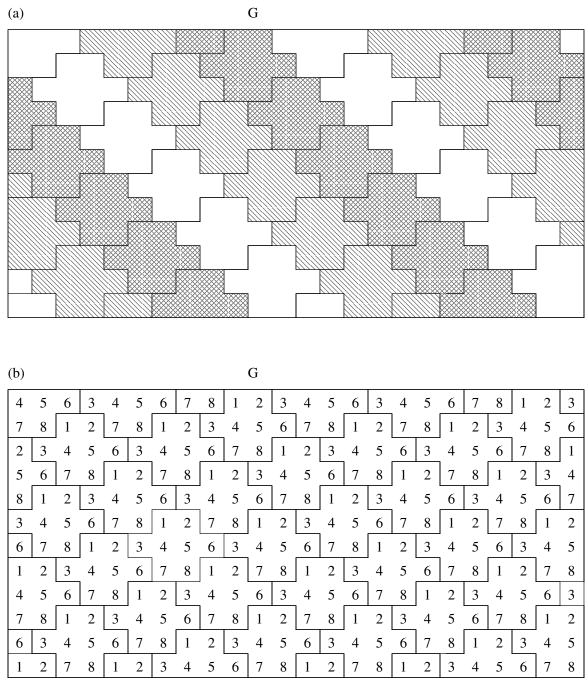

FIG. 2.6. *Example of perfect sphere-packing using Construction* 2.1*, and perfect t-interleaving using Construction* 2.2*.*

Next, we use Construction 2.2 to perfectly t-interleave G. Let the perfect sphere-packing remain as it is, and color all the spheres with the same pattern, using $|S_t| = 8$ distinct colors. The resulting perfect t-interleaving on G is shown in Figure 2.6(b).

We comment that Construction 2.2 provides the *complete* set of perfect t-interleaving constructions that have the following property: For any two colors, the two sets of vertices respectively colored by those two colors are translates of each other in the torus. Observing the constructions, we note that every such interleaving pattern has at least one translational periodicity other than the identity. In the previous work of [8], three t-interleaving constructions for two-dimensional arrays were presented, all based on lattice interleavers. Those three constructions can also be applied to tori because of their periodic patterns. Our Construction 2.2 generalizes the results in [8] in two ways: First, it covers more constructions based on lattice interleavers, with the results of [8] included as special cases; secondly, when t is even, it also covers constructions that do not use lattice interleavers, which we can make happen by simply letting any A_i and A_j take different values.

3. Achieving an interleaving degree within one of the optimal. Recall that an optimal interleaving need not be a perfect interleaving. A perfect interleaving uses $|S_t|$ colors, which is possible only when the dimensions satisfy the divisibility conditions of Construction 2.2. Most dimensions do not satisfy these divisibility conditions, and thus most tori do not admit a perfect interleaving—any interleaving must use more than $|S_t|$ colors. Recall that an optimal interleaving uses the minimal number of necessary colors.

In this section, we present a novel t-interleaving construction, with which we can t-interleave any large enough torus with at most one more than the optimal number

0	2	4	0	3	5	1	4
1	3	5	1	4	0	2	5
2	4	0	2	5	1	3	0
3	5	1	3	0	2	4	1
4	0	2	4	1	3	5	2
5	1	3	5	2	4	0	3

FIG. 3.1. *An example of t-interleaving with the three features.*

of colors. The construction presented here will also be used as a building block in section 4 for optimal t-interleaving.

3.1. Interleaving construction. The following definition defines several types of integer strings that are crucial to the interleaving constructions to be presented.

DEFINITION 3.1.

- *Given a positive integer t, if t is odd, then P is defined to be a string of integers, "$a_1, a_2, \ldots, a_{\frac{t-1}{2}}$," where $a_{\frac{t-1}{2}} = t+1$ and $a_i = t$ for $1 \le i < \frac{t-1}{2}$; if t is even, then P is defined to be a string of integers, "$a_1, a_2, \ldots, a_{\frac{t}{2}}$," where $a_{\frac{t}{2}} = t$ and $a_i = t-1$ for $1 \le i < \frac{t}{2}$. For example, if $t = 3$, then $P =$ "4"; if $t = 4$, then $P =$ "3,4"; if $t = 5$, then $P =$ "5,6."*
- *Given a positive integer t, if t is odd, then Q is defined to be a string of integers "$b_1, b_2, \ldots, b_{\frac{t+1}{2}}$," where $b_{\frac{t+1}{2}} = t+1$ and $b_i = t$ for $1 \le i < \frac{t+1}{2}$; if t is even, then Q is defined to be a string of integers "$b_1, b_2, \ldots, b_{\frac{t}{2}+1}$," where $b_{\frac{t}{2}+1} = t$ and $b_i = t-1$ for $1 \le i < \frac{t}{2}+1$.*
- *Given a positive integer t, an* offset sequence *is a string of P's and Q's. For example, an offset sequence consisting of one P and two Q's can be "PQQ," "QPQ" or "QQP." The offset sequence is also naturally seen as a string of integers which is the concatenation of the integer strings in its P's and Q's. For example, when $t = 3$, if an offset sequence consisting of one P and two Q's is "PQQ," then the offset sequence is also seen as "4,3,4,3,4"; when $t = 4$, if an offset sequence consisting of three P's and two Q's is "PQPPQ," then the offset sequence is also seen as "3,4,3,3,4,3,4,3,4,3,3,4." The number of integers in an offset sequence is called its* length.

In this section, we are particularly interested in one kind of t-interleaving on an $l_1 \times l_2$ torus, which has the following features:

- Feature 1: $l_1 = |S_t| + 1$. In other words, if t is odd, then $l_1 = \frac{t^2+1}{2} + 1$; if t is even, then $l_1 = \frac{t^2}{2} + 1$.
- Feature 2: The number of colors in the t-interleaving equals l_1. Also, in every column of the torus, each of the l_1 colors is assigned to exactly one vertex.
- Feature 3: If the vertex (a_1, b_1) and the vertex (a_2, b_2) have the same color, then for $i = 1, 2, \ldots, l_1 - 1$, the vertex $((a_1 + i) \bmod l_1, b_1)$ and the vertex $((a_2 + i) \bmod l_1, b_2)$ have the same color.

Example 3.1. Figure 3.1 shows a t-interleaving on an $l_1 \times l_2$ torus which has the above three features. There $t = 3$, $l_1 = |S_t| + 1 = 6$, and $l_2 = 8$.

Now let us choose a color i, where $0 \le i \le 5$, and say that the set of vertices of color i is $\{(x_0, 0), (x_1, 1), \dots, (x_{l_2-1}, l_2-1)\}$. Then the string of integers "$(x_1-x_0) \bmod l_1, (x_2-x_1) \bmod l_1, \dots, (x_7-x_6) \bmod l_1, (x_0-x_7) \bmod l_1$" equals "4,4,4,3,4,4,3,4." Since when $t = 3$, $P =$ "4" and $Q =$ "3,4," the above string of integers actually equals "$PPPQPQ$," which is an offset sequence of length l_2. We comment that this phenomenon is not a pure coincidence: Offset sequences do help us find t-interleavings that have the above three features. In fact, we can prove that in many cases (e.g., when $t = 5$ or 7), for *any* t-interleaving on a torus that has the above three features, after horizontally shifting and/or vertically reversing the interleaving pattern, the resulting interleaving will exhibit the same phenomenon as the example shown here.

The following construction outputs a t-interleaving that has the three features.

Construction 3.1.

Input: A positive integer t. An $l_1 \times l_2$ torus, where $l_1 = |S_t| + 1$. An integer m that equals $\lfloor \frac{t}{2} \rfloor$. Two integers p and q that satisfy the following equation set if t is odd,

$$\left\{ \begin{array}{c} pm + q(m+1) = l_2, \\ p(2m^2+m+1) + q(2m^2+3m+2) \equiv 0 \bmod (2m^2+2m+2), \\ p \text{ and } q \text{ are nonnegative integers, } p+q > 0, \end{array} \right. \tag{3.1}$$

and satisfy the following equation set if t is even:

$$\left\{ \begin{array}{c} pm + q(m+1) = l_2, \\ p(2m^2-m+1) + q(2m^2+m) \equiv 0 \bmod (2m^2+1), \\ p \text{ and } q \text{ are nonnegative integers, } p+q > 0. \end{array} \right. \tag{3.2}$$

Output: A t-interleaving on the $l_1 \times l_2$ torus that satisfies Features 1, 2, and 3.

Construction: Let $S =$"$s_0, s_1, \dots, s_{l_2-1}$" be an arbitrary offset sequence consisting of p P's and q Q's. For $j = 1, 2, \dots, l_2$ and for $i = 0, 1, \dots, l_1 - 1$, color the vertex $((\sum_{k=0}^{j-1} s_k + i) \bmod l_1, j \bmod l_2)$ with color i.

Example 3.2. Let $t=3$, $l_1=6$, $l_2=8$, $m=1$, $p=4$, and $q=2$. We use Construction 3.1 to t-interleave an $l_1 \times l_2$ torus. Say the offset sequence S is chosen to be "$PPPQPQ$." Then Construction 3.1 outputs the t-interleaving shown in Figure 3.1.

We explain Construction 3.1 a little further. The equation set (3.1) (for odd t) and the equation set (3.2) (for even t) ensure that the offset sequence S, which consists of p P's and q Q's, exists. Furthermore, for any integer j ($0 \le j \le l_2 - 1$), if (a, j) and $(b, (j+1) \bmod l_2)$ are two vertices of the same color, then $b - a \equiv s_j \bmod l_1$. That is, the offset sequence S indicates the *vertical offsets* of any two vertices of the same color in adjacent columns. It is simple to verify that the t-interleaving output by Construction 3.1 satisfies all the three features listed earlier in this subsection.

The following lemma will be used to prove the correctness of Construction 3.1 and also in future analysis.

LEMMA 3.2. *Let $i \in \{0, 1, \dots, |S_t|\}$ be any of the colors used by Construction 3.1 to interleave the $l_1 \times l_2$ torus. Let $\{(b_0, 0), (b_1, 1), \dots, (b_{l_2-1}, l_2-1)\}$ be the set of vertices of color i in the torus. Let m and S have the same meaning as in Construction 3.1 (namely, $m = \lfloor \frac{t}{2} \rfloor$, and $S =$"$s_0, s_1, \dots, s_{l_2-1}$" is the offset sequence consisting of p P's and q Q's utilized by Construction 3.1). For any two integers j_1 and j_2 ($0 \le j_1 \ne j_2 \le l_2 - 1$), we define $L_{j_1 \to j_2}$ as $L_{j_1 \to j_2} = [(j_2 - j_1) \bmod l_2] + \min\{(b_{j_2} - b_{j_1}) \bmod l_1, (b_{j_1} - b_{j_2}) \bmod l_1\}$. Then we have the following conclusions:*

- *Case* 1. *t is odd, $j_2 - j_1 \equiv m \bmod l_2$, and s_{j_1}, $s_{(j_1+1) \bmod l_2}$, $s_{(j_1+2) \bmod l_2}$, $\dots$, $s_{(j_2-1) \bmod l_2}$ do not all equal t. In this case, $b_{j_2} - b_{j_1} \equiv -(m+1) \bmod l_1$ and $L_{j_1 \to j_2} = t$.*

- *Case* 2. *t is odd, $j_2 - j_1 \equiv m+1 \bmod l_2$, and exactly one of s_{j_1}, $s_{(j_1+1) \bmod l_2}$, $s_{(j_1+2) \bmod l_2}$, $\ldots$, $s_{(j_2-1) \bmod l_2}$ equals $t+1$. In this case, $b_{j_2} - b_{j_1} \equiv m \bmod l_1$ and $L_{j_1 \to j_2} = t$.*
- *Case* 3. *t is even, $j_2 - j_1 \equiv 1 \bmod l_2$, and $s_{j_1} = t-1$. In this case, $b_{j_2} - b_{j_1} \equiv t-1 \bmod l_1$ and $L_{j_1 \to j_2} = t$.*
- *Case* 4. *t is even, $j_2 - j_1 \equiv m \bmod l_2$, and s_{j_1}, $s_{(j_1+1) \bmod l_2}$, $s_{(j_1+2) \bmod l_2}$, $\ldots$, $s_{(j_2-1) \bmod l_2}$ do not all equal $t-1$. In this case, $b_{j_2} - b_{j_1} \equiv -m \bmod l_1$ and $L_{j_1 \to j_2} = t$.*
- *Case* 5. *t is even, $j_2 - j_1 \equiv m+1 \bmod l_2$, and exactly one of s_{j_1}, $s_{(j_1+1) \bmod l_2}$, $s_{(j_1+2) \bmod l_2}$, $\ldots$, $s_{(j_2-1) \bmod l_2}$ equals t. In this case, $b_{j_2} - b_{j_1} \equiv m-1 \bmod l_1$ and $L_{j_1 \to j_2} = t$.*
- *If none of the above five cases is true and $j_2 - j_1 \neq t \bmod l_2$, then $L_{j_1 \to j_2} > t$. If none of the above five cases is true and $j_2 - j_1 \equiv t \bmod l_2$, then $L_{j_1 \to j_2} \geq t$.*

Proof. Let $\Delta = t+1$ if t is odd, and let $\Delta = t$ if t is even. The offset sequence S consists of P's and Q's, so it has the following property: For any $k \in \{0, 1, \ldots, l_2 - 1\}$ such that $s_k = \Delta$, the $m-1$ integers $s_{(k+1) \bmod l_2}$, $s_{(k+2) \bmod l_2}$, $\ldots$, $s_{(k+m-1) \bmod l_2}$ are all equal to $\Delta - 1$, and either $s_{(k+m) \bmod l_2}$ or $s_{(k+m+1) \bmod l_2}$ equals Δ. Also note that $b_{j_2} - b_{j_1} \equiv s_{j_1} + s_{(j_1+1) \bmod l_2} + s_{(j_1+2) \bmod l_2} + \cdots + s_{(j_2-1) \bmod l_2} \bmod l_1$. Based on those two observations, this lemma can be proved with straightforward computation. ◻

THEOREM 3.3. *Construction* 3.1 *is correct.*

Proof. Let (b_{j_1}, j_1) and (b_{j_2}, j_2) be any two vertices of the same color in the $l_1 \times l_2$ torus that was interleaved by Construction 3.1. The Lee distance between them is $d((b_{j_1}, j_1), (b_{j_2}, j_2)) = \min\{(j_2 - j_1) \bmod l_2, (j_1 - j_2) \bmod l_2\} + \min\{(b_{j_2} - b_{j_1}) \bmod l_1, (b_{j_1} - b_{j_2}) \bmod l_1\} = \min\{L_{j_1 \to j_2}, L_{j_2 \to j_1}\}$. From Lemma 3.2, it is clear that neither $L_{j_1 \to j_2}$ nor $L_{j_2 \to j_1}$ is less than t. Therefore $d((b_{j_1}, j_1), (b_{j_2}, j_2)) \geq t$. So Construction 3.1 t-interleaved the torus. And as mentioned before, this t-interleaving satisfies Features 1, 2, and 3. ◻

3.2. Existence of offset sequences. The feasibility of Construction 3.1 depends only on one thing: whether the two input parameters p and q exist or not. The following theorem shows that when the width of the torus, l_2, exceeds a threshold, p and q are guaranteed to exist.

THEOREM 3.4. *Let t be an odd* (*respectively, even*) *positive integer. When $l_2 \geq \lfloor \frac{t}{2} \rfloor (\lfloor \frac{t}{2} \rfloor + 1)(|S_t| + 1)$, there exists at least one solution (p, q) to the equation set* (3.1) (*respectively, equation set* (3.2)), *which is shown in the* input *part of Construction* 3.1.

Proof. Firstly, let us assume that t is odd. The equation set (3.1) is as follows:

$$\begin{cases} pm + q(m+1) = l_2, \\ p(2m^2 + m + 1) + q(2m^2 + 3m + 2) \equiv 0 \bmod (2m^2 + 2m + 2), \\ p \text{ and } q \text{ are nonnegative integers, } p + q > 0, \end{cases}$$

where $m = \lfloor \frac{t}{2} \rfloor$. We introduce a new variable z, and transform the above equation set equivalently to be

$$\begin{cases} \begin{pmatrix} m & m+1 \\ 2m^2 + m + 1 & 2m^2 + 3m + 2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} l_2 \\ z(2m^2 + 2m + 2) \end{pmatrix}, \\ p \text{ and } q \text{ are nonnegative integers; } z \text{ is a positive integer,} \end{cases}$$

which is the same as

$$\begin{cases} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} m & m+1 \\ 2m^2+m+1 & 2m^2+3m+2 \end{pmatrix}^{-1} \begin{pmatrix} l_2 \\ z(2m^2+2m+2) \end{pmatrix}, \\ p \text{ and } q \text{ are nonnegative integers; } z \text{ is a positive integer,} \end{cases}$$

which equals

$$\begin{cases} p = 2(m+1)(m^2+m+1)z - (2m^2+3m+2)l_2, \\ q = (2m^2+m+1)l_2 - 2m(m^2+m+1)z, \\ p \text{ and } q \text{ are nonnegative integers; } z \text{ is a positive integer.} \end{cases}$$

There exists a solution for the variables p, q, and z in the above equation set if and only if the following conditions can be satisfied:

$$\begin{cases} 2(m+1)(m^2+m+1)z - (2m^2+3m+2)l_2 \geq 0, \\ (2m^2+m+1)l_2 - 2m(m^2+m+1)z \geq 0, \\ z \text{ is a positive integer,} \end{cases}$$

which is equivalent to

$$\begin{cases} \frac{(2m^2+3m+2)l_2}{2(m+1)(m^2+m+1)} \leq z \leq \frac{(2m^2+m+1)l_2}{2m(m^2+m+1)}, \\ z \text{ is a positive integer.} \end{cases}$$

To enable a value for z to exist that satisfies the above conditions, it is sufficient to make $\frac{(2m^2+m+1)l_2}{2m(m^2+m+1)} - \frac{(2m^2+3m+2)l_2}{2(m+1)(m^2+m+1)} \geq 1$, that is, to make $l_2 \geq 2m(m+1)(m^2+m+1) = \lfloor \frac{t}{2} \rfloor (\lfloor \frac{t}{2} \rfloor + 1)(|S_t|+1)$. Therefore when $l_2 \geq \lfloor \frac{t}{2} \rfloor (\lfloor \frac{t}{2} \rfloor + 1)(|S_t|+1)$, there exists at least one solution (p,q) to the equation set (3.1).

When t is even, the conclusion can be proved in a very similar way. We skip its details. ∎

COROLLARY 3.5. *When* $l_2 \geq \lfloor \frac{t}{2} \rfloor (\lfloor \frac{t}{2} \rfloor + 1)(|S_t|+1)$*, Construction* 3.1 *can be used to output a* t*-interleaving on an* $(|S_t|+1) \times l_2$ *torus.*

Proof. When $l_2 \geq \lfloor \frac{t}{2} \rfloor (\lfloor \frac{t}{2} \rfloor + 1)(|S_t|+1)$, all the parameters in the *input* part of Construction 3.1 exist, including p and q. ∎

3.3. Interleaving with degree within one of the optimal. In this subsection, we will show how to interleave a large enough torus with at most one more than the optimal number of colors.

We define the simple term of *tiling tori* here. By tiling several interleaved tori vertically or horizontally, we get a larger torus, whose interleaving is the straightforward combination of the interleaving on the smaller tori. It is best explained with an example.

Example 3.3. Three interleaved tori, A, B, and C, are shown in Figure 3.2. The torus D is a 5×4 torus, obtained by *tiling* A *and* B *vertically* in the form of $\begin{bmatrix} A \\ B \end{bmatrix}$. The torus E is a 2×8 torus, obtained by *tiling one copy of* A *and two copies of* C *horizontally* in the form of $[\ C \quad A \quad C\ ]$.

The following construction t-interleaves a large enough torus with at most $|S_t|+2$ distinct integers.

Construction 3.2. t-interleave an $l_1 \times l_2$ torus G, where $l_1 \geq |S_t|(|S_t|+1)$ and $l_2 \geq \lfloor \frac{t}{2} \rfloor (\lfloor \frac{t}{2} \rfloor + 1)(|S_t|+1)$, using at most $|S_t|+2$ distinct integers.

1. Let G_1 be an $(|S_t|+1) \times l_2$ torus that is t-interleaved by Construction 3.1, using colors $0, 1, \ldots, |S_t|$. Let $\{(c_0, 0), (c_1, 1), \ldots, (c_{l_2-1}, l_2-1)\}$ be the set of vertices in G_1 having color 0.

(a)

A

0	1	2	3
3	2	1	0

B

0	0	0	0
1	1	1	1
2	2	2	2

C

1	2
3	4

(b)

D

0	1	2	3
3	2	1	0
0	0	0	0
1	1	1	1
2	2	2	2

E

1	2	0	1	2	3	1	2
3	4	3	2	1	0	3	4

FIG. 3.2. *Examples of tiling tori.*

G_1

0	2	0	2	0	2
1	0	1	0	1	0
2	1	2	1	2	1

G_2

3	2	3	2	3	2
0	3	0	3	0	3
1	0	1	0	1	0
2	1	2	1	2	1

G

0	2	0	2	0	2
1	0	1	0	1	0
2	1	2	1	2	1
3	2	3	2	3	2
0	3	0	3	0	3
1	0	1	0	1	0
2	1	2	1	2	1

FIG. 3.3. *Examples of Construction* 3.2.

2. Let G_2 be an $(|S_t|+2) \times l_2$ torus. Color the vertices $\{(c_0, 0), (c_1, 1), \ldots, (c_{l_2-1}, l_2 - 1)\}$ in G_2 with color $|S_t| + 1$.

3. For $j = 0, 1, \ldots, l_2 - 1$ and for $i = 1, 2, \ldots, |S_t| + 1$, color vertex $((c_j + i) \bmod (|S_t| + 2), j)$ in G_2 with color $i - 1$.

4. Let x and y be two nonnegative integers such that $l_1 = x(|S_t|+1)+y(|S_t|+2)$. Tile x copies of G_1 and y copies of G_2 vertically to get an $l_1 \times l_2$ torus G. Note that then G has been t-interleaved using at most $|S_t| + 2$ distinct integers.

Example 3.4. We use Construction 3.2 to t-interleave a 7×6 torus G, where $t = 2$. The first step is to use Construction 3.1 to t-interleave a 3×6 torus G_1. Say the offset sequence selected in Construction 3.1 is $S =$ *"QQQ"* = "1,2,1,2,1,2"; then G_1 is as shown in Figure 3.3. Then the 4×6 torus G_2 is as shown in the figure. By tiling one copy of G_1 and one copy of G_2 vertically, we get the t-interleaved torus G. $|S_t| + 2 = 4$ distinct integers are used to interleave G.

THEOREM 3.6. *Construction* 3.2 *is correct.*

Proof. It is a known fact that for any two relatively prime positive integers A and B, any integer C no less than $(A-1)(B-1)$ can be expressed as $C = xA+yB$, where x and y are nonnegative integers. Therefore in Construction 3.2, since $l_1 \geq |S_t|(|S_t|+1)$, l_1 indeed can be expressed as $l_1 = x(|S_t| + 1) + y(|S_t| + 2)$, as shown in the last step of Construction 3.2. Thus the construction can be executed from beginning to end successfully. Now we prove that the construction does t-interleave G; that is, for any two vertices (a_1, b_1) and (a_2, b_2) both of color i in G, the Lee distance between them is at least t. We consider three cases.

Case 1: $b_1 = b_2$, which means that (a_1, b_1) and (a_2, b_2) are in the same column of G. We see every column of G as a ring of length l_1 (because it is toroidal). Then, observe the colors in a column of G, and we can see that on the column, the color following color $|S_t| + 1$ and before the next color $|S_t| + 1$ must be the following, where the pattern $0, 1, \ldots, |S_t|$ appears at least once:

$$0, 1, \ldots, |S_t|, 0, 1, \ldots, |S_t|, \ldots\ldots, 0, 1, \ldots, |S_t|.$$

Therefore since (a_1, b_1) and (a_2, b_2) have the same color, the Lee distance between them must be at least $|S_t| + 1 > t$.

Case 2: $b_1 \neq b_2$, and $i \neq |S_t|+1$. In this case, let us first observe two conclusions:

- The interleaving on G_2 (defined in Construction 3.2) is a t-interleaving. This can be proved as follows: Any two vertices of the same color in G_2 can be expressed as $((c_{j_1} + i_0) \bmod (|S_t| + 2), j_1)$ and $((c_{j_2} + i_0) \bmod (|S_t| + 2), j_2)$ (see steps 2 and 3 of Construction 3.2); then, $d_{G_2}(((c_{j_1} + i_0) \bmod (|S_t| + 2), j_1), ((c_{j_2} + i_0) \bmod (|S_t| + 2), j_2)) = d_{G_2}((c_{j_1}, j_1), (c_{j_2}, j_2)) \geq d_{G_1}((c_{j_1}, j_1), (c_{j_2}, j_2)) \geq t$.
- Let (α, j) and (β, j) be two vertices respectively in G_1 and G_2, which both have the same color. Then it is simple to see that $\beta = \alpha$ or $\beta = \alpha + 1$. Since G_1 has $|S_t| + 1$ rows and G_2 has $|S_t| + 2$ rows, we have $d_{G_2}((\beta, j), (0, j)) \geq d_{G_1}((\alpha, j), (0, j))$ and $d_{G_2}((\beta, j), (|S_t| + 1, j)) \geq d_{G_1}((\alpha, j), (|S_t|, j))$. That is, if u and v are two vertices respectively in G_1 and G_2, both of which are in the jth column and have the same color, then the vertical distance from v to either the top or bottom of G_2 is no less than the vertical distance from u to the top or bottom of G_1.

According to Construction 3.2, G is obtained by vertically tiling x copies of G_1 and y copies of G_2. Let us call each of those $x + y$ tori a *component torus* of G. Now, if (a_1, b_1) and (a_2, b_2) are in the same component torus of G, we know that the Lee distance between them *in* G is no less than the Lee distance between them *in that component torus*, which is at least t because that component torus is t-interleaved. If (a_1, b_1) and (a_2, b_2) are not in the same component torus of G, we do the following. We first construct a torus G', which is obtained by vertically tiling $x+y$ copies of G_1. It is simple to see that G' is t-interleaved. We call each of the $x+y$ copies of G_1 in G' a *component torus* of G'. Let us say that (a_1, b_1) and (a_2, b_2) are respectively in the k_1th and k_2th component torus *of* G. Let (c_1, b_1) and (c_2, b_2) be the two vertices of color i that are respectively in the k_1th and k_2th component torus *of* G'. Observe the shortest path between (a_1, b_1) and (a_2, b_2) *in* G, and we see that it can be split into such three intervals: from (a_1, b_1) to a border of the k_1th component torus, from the border of the k_1th component torus to the border of the k_2th component torus, and from the border of the k_2th component torus to (a_2, b_2). There is a corresponding (not necessarily shortest) path connecting (c_1, b_1) and (c_2, b_2) in G', which can be split into such three intervals similarly. Furthermore, each of the three intervals of the first path is at least as long as the corresponding interval of the second path. G' is t-interleaved, and so the second path's length is at least t. Thus the Lee distance between (a_1, b_1) and (a_2, b_2) *in* G is at least t.

Case 3: $b_1 \neq b_2$ and $i = |S_t| + 1$. In this case, it is simple to see that the two vertices in G, $(a_1 + 1 \bmod l_1, b_1)$ and $(a_2 + 1 \bmod l_1, b_2)$, both have color 0. Based on the conclusion of Case 2, $d_G((a_1 + 1 \bmod l_1, b_1), (a_2 + 1 \bmod l_1, b_2)) \geq t$. Thus $d_G((a_1, b_1), (a_2, b_2)) = d_G((a_1 + 1 \bmod l_1, b_1), (a_2 + 1 \bmod l_1, b_2)) \geq t$.

Thus Construction 3.2 correctly t-interleaved G. □

As a result of Construction 3.2, we get the following theorem.

THEOREM 3.7. *When* $l_1 \geq |S_t|(|S_t| + 1)$ *and* $l_2 \geq \lfloor \frac{t}{2} \rfloor (\lfloor \frac{t}{2} \rfloor + 1)(|S_t| + 1)$*, the* t*-interleaving number of an* $l_1 \times l_2$ *(or* $l_2 \times l_1$*) torus is at most* $|S_t| + 2$.

By combining Construction 2.2 (the construction for perfect t-interleaving) and

Construction 3.2, we can t-interleave any sufficiently large torus with at most one more than the optimal number of colors.

4. Optimal interleaving on large tori. In the previous section, it is shown that when l_2 is large enough, an $(|S_t|+1)\times l_2$ torus can be t-interleaved using $|S_t|+1$ integers. In this section, we will construct a $[k(|S_t|+1)-1]\times l_2$ torus (for some integer k) which is also t-interleaved using $|S_t|+1$ integers, by using an operation we call *removing a zigzag row*. Those two tori have a special property: When they (or multiple copies of them) are tiled vertically to get a larger torus, the larger torus is also t-interleaved with $|S_t|+1$ colors. Since $|S_t|+1$ and $k(|S_t|+1)-1$ are relatively prime, a large enough l_1 must be a linear combination of those two numbers with nonnegative integral coefficients, and therefore an $l_1 \times l_2$ torus can be t-interleaved using $|S_t|+1$ integers in this way. We present constructions to optimally t-interleave such tori, and as a parallel result, the existence of Region I (see the Introduction) is proved.

All the results of this section can be split into two parts: one for the case when t is odd, and the other for the case when t is even. Those two cases can be analyzed with very similar methods; however, their analysis and results differ in details. For succinctness, in this section, we only analyze in detail the case when t is odd, which should suffice for illustrating all the ideas. So in the first three subsections here (subsections 4.1, 4.2, and 4.3), we always assume that *t is odd*. In subsection 4.4, we present just the final result for the case when t is even. We list the major intermediate results for the case when t is even in Appendix II (section 8).

4.1. Removing a zigzag row in a torus. Below we define zigzag rows and the concept of removing a zigzag row in a torus.

DEFINITION 4.1. *A* zigzag row *in an $l_1\times l_2$ torus is a set of l_2 vertices of the torus: $\{(a_0,0),(a_1,1),\dots,(a_{l_2-1},l_2-1)\}$, where $0\le a_i\le l_1-1$ for $i=0,1,\dots,l_2-1$.*

For example, $\{(2,0),(3,1),(0,2),(0,3),(3,4)\}$ is a zigzag row in a 4×5 torus.

DEFINITION 4.2. *Let T be an $l_1\times l_2$ torus. Let $\{(a_0,0),(a_1,1),\dots,(a_{l_2-1},l_2-1)\}$ be a zigzag row in T. Let there be an interleaving on T, which colors T's vertex (b,c) with color $I(b,c)$, for $b=0,1,\dots,l_1-1$ and $c=0,1,\dots,l_2-1$. Then a torus G is said to be obtained by* removing *the zigzag row $\{(a_0,0),(a_1,1),\dots,(a_{l_2-1},l_2-1)\}$ in T if and only if these two conditions are satisfied:*

- *G is an $(l_1-1)\times l_2$ torus.*
- *For $i=0,1,\dots,l_1-2$ and $j=0,1,\dots,l_2-1$, the vertex (i,j) in G has color $I(i,j)$ if $i<a_j$, and color $I(i+1,j)$ if $i\ge a_j$.*

Example 4.1. In Figure 4.1, a 6×5 torus T is shown. A zigzag row $\{(3,0),\ (2,1),\ (1,2),\ (3,3),\ (1,4)\}$ in T is circled in the figure. Figure 4.1 shows a torus G obtained by removing the zigzag row $\{(3,0),\ (2,1),\ (1,2),\ (3,3),\ (1,4)\}$ in T.

It can be readily observed that G can be seen as being derived from T in the following way: First, delete the zigzag row in T that is circled in Figure 4.1; then in each column of T, move the vertices below the circled vertex upward.

In order to get our final results, we present three rules to follow for devising a zigzag row. Let B be an $l_0\times l_2$ torus which is t-interleaved by Construction 3.1. Note that this means $l_0=|S_t|+1$. Let $S=$"$s_0,s_1,\dots,s_{l_2-1}$" be the offset sequence utilized by Construction 3.1 when it was t-interleaving B. Let H be an $l_1\times l_2$ torus obtained by tiling several copies of B vertically. Let $m=\lfloor\frac{t}{2}\rfloor$. Then the three rules for devising a zigzag row *in H*, $\{(a_0,0),(a_1,1),\dots,(a_{l_2-1},l_2-1)\}$, are the following:

- Rule 1. For any j such that $0\le j\le l_2-1$, if the integers $s_j,\ s_{(j+1) \bmod l_2}$, $\dots,\ s_{(j+m-1) \bmod l_2}$ do not all equal t, then $a_j\ge a_{(j+m) \bmod l_2}+m$.

T

1	3	5	2	4
2	4	⑥	3	⑤
3	⑤	1	4	6
④	6	2	⑤	1
5	1	3	6	2
6	2	4	1	3

G

1	3	5	2	4
2	4	1	3	6
3	6	2	4	1
5	1	3	6	2
6	2	4	1	3

FIG. 4.1. *Removing a zigzag row* $\{(3,0),(2,1),(1,2),(3,3),(1,4)\}$ *in* T.

- Rule 2. For any j such that $0 \leq j \leq l_2 - 1$, if exactly one of the integers $s_j, s_{(j+1) \bmod l_2}, \ldots, s_{(j+m) \bmod l_2}$ equals $t+1$, then $a_j \leq a_{(j+m+1) \bmod l_2} - (m-1)$.
- Rule 3. For any j such that $0 \leq j \leq l_2 - 1$, $m \leq a_j \leq l_1 - m - 1$.

LEMMA 4.3. *Let B be a torus t-interleaved by Construction* 3.1. *Let H be a torus obtained by tiling copies of B vertically, and let T be a torus obtained by removing a zigzag row in H, where the zigzag row in H follows the three rules listed above. Let G be a torus obtained by tiling copies of B and T vertically. Then, both T and G are t-interleaved.*

Proof. When $t = 1$, the proof is trivial. So we assume $t \geq 3$ in the rest of the proof. It is simple to see that H is t-interleaved, because H is obtained by tiling B, a t-interleaved torus. We assume that B is an $l_0 \times l_2$ torus (where $l_0 = |S_t|+1$), H is an $l_1 \times l_2$ torus (where l_1 is a multiple of l_0), T is an $l_T \times l_2$ torus (where $l_T = l_1 - 1$), and G is an $l_G \times l_2$ torus. Let $m = \lfloor \frac{t}{2} \rfloor$. Let $S =$"$s_0, s_1, \ldots, s_{l_2-1}$" be the offset sequence utilized by Construction 3.1 when it was t-interleaving B.

(1) In this part, we will prove that T is t-interleaved. Let (x_1, y_1) and (x_2, y_2) be two vertices in T both of color r. We need to prove that $d_T((x_1,y_1),(x_2,y_2)) \geq t$.

Let $\{(a_0, 0), (a_1, 1), \ldots, (a_{l_2-1}, l_2-1)\}$ denote the zigzag row removed in H to get T. If $a_{y_1} \leq x_1$, then let $z_1 = x_1 + 1$; otherwise let $z_1 = x_1$. Similarly, if $a_{y_2} \leq x_2$, then let $z_2 = x_2 + 1$; otherwise let $z_2 = x_2$. Clearly, the two vertices in H, (z_1, y_1) and (z_2, y_2), also have color r.

We need to consider only the following three cases.

Case 1: $y_1 = y_2$. In this case, $d_H((z_1,y_1),(z_2,y_2))$ is a multiple of $|S_t|+1$ (the number of rows in B), and $d_T((x_1,y_1),(x_2,y_2)) \geq d_H((z_1,y_1),(z_2,y_2)) - 1 \geq |S_t| = \frac{t^2+1}{2} > t$.

Case 2: $y_1 \neq y_2$ and $d_T((x_1,y_1),(x_2,y_2)) \leq d_H((z_1,y_1),(z_2,y_2)) - 2$. Without loss of generality, we assume $x_1 \geq x_2$. Then, based on the definition of removing a zigzag row, it is simple to verify that the following must be true: $d_T((x_1,y_1),(x_2,y_2)) = d_H((z_1,y_1),(z_2,y_2)) - 2$, $a_{y_2} < z_2 < z_1 < a_{y_1}$, $(z_2 - z_1 \bmod l_1) \leq (z_1 - z_2 \bmod l_1)$. By Rule 3, any vertex in the removed zigzag row is neither in the first m rows nor in the last m rows of H, so $(z_2 - z_1 \bmod l_1) \geq 2m+3$. Thus $d_T((x_1,y_1),(x_2,y_2)) = d_H((z_1,y_1),(z_2,y_2)) - 2 > (z_2 - z_1 \bmod l_1) - 2 \geq 2m+1 = t$.

Case 3: $y_1 \neq y_2$ and $d_T((x_1,y_1),(x_2,y_2)) \geq d_H((z_1,y_1),(z_2,y_2)) - 1$. We know that $d_H((z_1,y_1),(z_2,y_2)) \geq t$. So to show that $d_T((x_1,y_1),(x_2,y_2)) \geq t$, we just need to prove that if $d_H((z_1,y_1),(z_2,y_2)) = t$, then $d_T((x_1,y_1),(x_2,y_2)) \geq d_H((z_1,y_1),(z_2,$

$y_2)$). By Lemma 3.2, there are only two nontrivial subcases to consider, without loss of generality, as follows.

Subcase 3.1: $y_2 - y_1 \equiv m \bmod l_2$, $z_2 - z_1 \equiv -(m+1) \bmod l_1$, $d_H((z_1,y_1),(z_2,y_2)) = (y_2 - y_1 \bmod l_2) + (z_1 - z_2 \bmod l_1) = t$, and $s_{y_1}, s_{(y_1+1) \bmod l_2}, s_{(y_1+2) \bmod l_2}, \dots, s_{(y_1+m-1) \bmod l_2}$ do not all equal t. If $z_1 > z_2$ (which means $z_1 = z_2 + (m+1)$), then from Rule 1, it is simple to see that $x_1 - x_2 = z_1 - z_2$, and so $d_T((x_1,y_1),(x_2,y_2)) = d_H((z_1,y_1),(z_2,y_2)) = t$. If $z_1 < z_2$ (which means that (z_1,y_1) and (z_2,y_2) are respectively in the first and last $m+1$ rows of H), since the first and last m rows of H and T must be the same, we get that $(x_1 - x_2 \bmod l_T) = (z_1 - z_2 \bmod l_1) = m+1$, and so $d_T((x_1,y_1),(x_2,y_2)) = d_H((z_1,y_1),(z_2,y_2)) = t$.

Subcase 3.2: $y_2 - y_1 \equiv m+1 \bmod l_2$, $z_2 - z_1 \equiv m \bmod l_1$, $d_H((z_1,y_1),(z_2,y_2)) = (y_2 - y_1 \bmod l_2) + (z_2 - z_1 \bmod l_1) = t$, and exactly one of s_{y_1}, $s_{(y_1+1) \bmod l_2}$, $s_{(y_1+2) \bmod l_2}, \dots, s_{(y_1+m) \bmod l_2}$ equals $t+1$. If $z_1 < z_2$ (which means $z_1 = z_2 - m$), then from Rule 2, it is simple to see that $x_2 - x_1 = z_2 - z_1$, and so $d_T((x_1,y_1),(x_2,y_2)) = d_H((z_1,y_1),(z_2,y_2)) = t$. If $z_1 > z_2$ (which means that (z_1,y_1) and (z_2,y_2) are respectively in the last and first m rows of H), since the first and last m rows of H and T must be the same, we get that $(x_2 - x_1 \bmod l_T) = (z_2 - z_1 \bmod l_1) = m$, and so $d_T((x_1,y_1),(x_2,y_2)) = d_H((z_1,y_1),(z_2,y_2)) = t$.

Thus T is t-interleaved.

(2) In this part, we will prove that G is t-interleaved. First let us make an observation: When a t-interleaved torus K is tiled with other tori vertically to get a larger torus $\hat{G}$, for any two vertices μ and ν in K (which are now also in $\hat{G}$) of the same color, the Lee distance between them *in* $\hat{G}$, $d_{\hat{G}}(\mu,\nu)$, is clearly no less than t. Let us also notice that the torus obtained by tiling one copy of B and one copy of T vertically is t-interleaved, which can be proved with exactly the same proof as in part (1).

G is obtained by tiling multiple copies of B and T. Let us call each copy of B or T in G a *component torus*. Let (x_1,y_1) and (x_2,y_2) be two vertices in G of the same color. Assume $d_G((x_1,y_1),(x_2,y_2)) \le t$. Then since both B and T have more than t rows, (x_1,y_1) and (x_2,y_2) must be either in the same component torus or in two adjacent component tori. Now if (x_1,y_1) and (x_2,y_2) are in the same component torus, let K denote that component torus; if (x_1,y_1) and (x_2,y_2) are in two adjacent component tori, let K be the torus obtained by vertically tiling those two component tori; let $\hat{G}$ be the same as G. By using the observation in the previous paragraph, we can readily prove that $d_G((x_1,y_1),(x_2,y_2)) \ge t$. Thus G is t-interleaved. ∎

4.2. Constructing the zigzag row. We presented three rules on devising a zigzag row in the previous subsection. But specifically, how can one construct a zigzag row that follows all those rules? In this subsection, we present such constructions.

Before the formal presentation, let us go over a few concepts. An offset sequence is a string of P's and Q's, where P and Q are strings of integers depending on t. For example, when $t = 5$, $P =$"5, 6" and $Q =$"5, 5, 6." Then an offset sequence "PPQ" can also be written as "5, 6, 5, 6, 5, 5, 6." Let us also express the offset sequence "PPQ" as "$s_0, s_1, s_2, s_3, s_4, s_5, s_6$," where $s_0 = 5$, $s_1 = 6$, ..., $s_6 = 6$. Then for $i = 0, 1, \dots, 6$ we will call s_i the $(i+1)$th element of the offset sequence. Also, we will say that s_2 is *the first element of a* P, because it is the first element of the second P in the offset sequence. For the same reason, s_0 is the first element of a P (this time, the first P in the offset sequence), s_1 is the second (or last) element of a P (the first P in the offset sequence), s_4 is the first element of a Q, and so on.

Now we begin the formal presentation of the constructions. Let B be an $l_0 \times l_2$ torus that is t-interleaved by Construction 3.1, so $l_0 = |S_t| + 1$. Let H be an $l_1 \times l_2$

torus obtained by tiling z copies of B vertically, so $l_1 = zl_0 = z(|S_t| + 1)$. Let $S =$"$s_0, s_1, \ldots, s_{l_2-1}$" be the offset sequence utilized by Construction 3.1 when it was t-interleaving B. We say that the offset sequence S consists of p P's and q Q's, where we require $p > 0$ and $q > 0$. We require that in the offset sequence the P's and Q's be interleaved very evenly. To be specific, in the offset sequence, between any two nearby P's (including between the last P and the first P, because we see the offset sequence as being toroidal), there must be either $\lceil \frac{q}{p} \rceil$ or $\lfloor \frac{q}{p} \rfloor$ consecutive Q's; and between any two nearby Q's (including between the last Q and the first Q), there must be either $\lceil \frac{p}{q} \rceil$ or $\lfloor \frac{p}{q} \rfloor$ consecutive P's. Also, we require the offset sequence to start with a P and to end with a Q. For example, an offset sequence consisting of three P's and five Q's that satisfies the above requirements is "$PQQPQQPQ$." Let $m = \frac{t-1}{2}$. Let $L = m + m\lceil \frac{p}{q} \rceil$ if $p \geq q$, and let $L = m + (m-1)\lceil \frac{q}{p} \rceil$ if $p < q$. Below we present two constructions—Constructions 4.1 and 4.2—for constructing a zigzag row *in* H, applicable respectively when $p \geq q$ and when $p < q$. If l_1 is too small, there may not exist a zigzag row *in* H that follows the three rules. To make our constructions work, we require that

$$l_1 \geq \left(\left\lceil \frac{p}{q} \right\rceil + 1\right)m^2 + 2m + 1$$

if $p \geq q$, and also that

$$l_1 \geq \left(\left\lceil \frac{q}{p} \right\rceil + 1\right)m^2 + m + \left(2 - \left\lceil \frac{q}{p} \right\rceil\right)$$

if $p < q$. Note that the constructed zigzag row is denoted by $\{(a_0, 0), (a_1, 1), \ldots, (a_{l_2-1}, l_2 - 1)\}$. Also note that both constructions require $t > 3$. The analysis for the case $t = 3$, a somewhat special case, is presented in Appendix I (section 7).

Construction 4.1. Constructing a zigzag row in H, when t is odd, $t > 3$, and $p \geq q > 0$.

1. Let $s_{x_1}, s_{x_2}, \ldots, s_{x_{p+q}}$ be the integers such that $0 = x_1 < x_2 < \cdots < x_{p+q} = l_2 - m - 1$ and each s_{x_i} $(1 \leq i \leq p + q)$ is the first element of a P or Q in the offset sequence S.

   Let $a_{x_1} = L$. For $i = 2$ to $p+q$, if $s_{x_{i-1}}$ is the first element of a Q, let $a_{x_i} = L$.
   For $i = 2$ to $p+q$, if $s_{x_{i-1}}$ is the first element of a P, then let $a_{x_i} = a_{x_{i-1}} - m$.
2. For $i = 2$ to m and for $j = 1$ to $p + q$, let $a_{x_j+i-1} = a_{x_j+i-2} + L$.
3. Let $s_{y_1}, s_{y_2}, \ldots, s_{y_q}$ be the integers such that $y_1 < y_2 < \cdots < y_q = l_2 - 1$ and each s_{y_i} $(1 \leq i \leq q)$ is the last element of a Q in the offset sequence S.

   For $i = 1$ to q, let $a_{y_i} = mL + m$.

   Now we have fully determined the zigzag row, $\{(a_0, 0), (a_1, 1), \ldots, (a_{l_2-1}, l_2 - 1)\}$, in the torus H.

The zigzag row constructed by Construction 4.1 has a quite regular structure. We show it with an example.

Example 4.2. We use this example to illustrate Construction 4.1. In this example, $t = 5$, and B is an 14×18 torus as shown in Figure 4.2(a). B is t-interleaved by Construction 3.1 by using the offset sequence $S =$"$PPPQPPPQ$"="5, 6, 5, 6, 5, 6, 5, 5, 6, 5, 6, 5, 6, 5, 6, 5, 5, 6." The torus H is shown in Figure 4.2(b). H is an 28×18 torus obtained by tiling two copies of B vertically. The rest of the parameters used by Construction 4.1 are $p = 6$, $q = 2$, $m = 2$, and $L = 8$. It is not difficult to verify that the zigzag row in H constructed by Construction 4.1 is $\{(8, 0), (16, 1), (6, 2), (14, 3), (4, 4), (12, 5), (2, 6), (10, 7), (18, 8), (8, 9), (16, 10), (6, 11), (14, 12), (4, 13), (12, 14),$

(a)

B

0	9	3	12	6	1	9	4	13	7	2	10	5	13	8	2	11	6
1	10	4	13	7	2	10	5	0	8	3	11	6	0	9	3	12	7
2	11	5	0	8	3	11	6	1	9	4	12	7	1	10	4	13	8
3	12	6	1	9	4	12	7	2	10	5	13	8	2	11	5	0	9
4	13	7	2	10	5	13	8	3	11	6	0	9	3	12	6	1	10
5	0	8	3	11	6	0	9	4	12	7	1	10	4	13	7	2	11
6	1	9	4	12	7	1	10	5	13	8	2	11	5	0	8	3	12
7	2	10	5	13	8	2	11	6	0	9	3	12	6	1	9	4	13
8	3	11	6	0	9	3	12	7	1	10	4	13	7	2	10	5	0
9	4	12	7	1	10	4	13	8	2	11	5	0	8	3	11	6	1
10	5	13	8	2	11	5	0	9	3	12	6	1	9	4	12	7	2
11	6	0	9	3	12	6	1	10	4	13	7	2	10	5	13	8	3
12	7	1	10	4	13	7	2	11	5	0	8	3	11	6	0	9	4
13	8	2	11	5	0	8	3	12	6	1	9	4	12	7	1	10	5

(b)

H

P P P Q P P P Q

0	9	3	12	6	1	9	4	13	7	2	10	5	13	8	2	11	6
1	10	4	13	7	2	10	5	0	8	3	11	6	0	9	3	12	7
2	11	5	0	8	3	(11)	6	1	9	4	12	7	1	10	(4)	13	8
3	12	6	1	9	4	12	7	2	10	5	13	8	2	11	5	0	9
4	13	7	2	(10)	5	13	8	3	11	6	0	9	(3)	12	6	1	10
5	0	8	3	11	6	0	9	4	12	7	1	10	4	13	7	2	11
6	1	(9)	4	12	7	1	10	5	13	8	(2)	11	5	0	8	3	12
7	2	10	5	13	8	2	11	6	0	9	3	12	6	1	9	4	13
(8)	3	11	6	0	9	3	12	7	(1)	10	4	13	7	2	10	5	0
9	4	12	7	1	10	4	13	8	2	11	5	0	8	3	11	6	1
10	5	13	8	2	11	5	⟨0⟩	9	3	12	6	1	9	4	12	⟨7⟩	2
11	6	0	9	3	12	6	1	10	4	13	7	2	10	5	13	8	3
12	7	1	10	4	⟨13⟩	7	2	11	5	0	8	3	11	⟨6⟩	0	9	4
13	8	2	11	5	0	8	3	12	6	1	9	4	12	7	1	10	5
0	9	3	⟨12⟩	6	1	9	4	13	7	2	10	⟨5⟩	13	8	2	11	6
1	10	4	13	7	2	10	5	0	8	3	11	6	0	9	3	12	7
2	⟨11⟩	5	0	8	3	11	6	1	9	⟨4⟩	12	7	1	10	4	13	8
3	12	6	1	9	4	12	7	2	10	5	13	8	2	11	5	0	9
4	13	7	2	10	5	13	8	{3}	11	6	0	9	3	12	6	1	{10}
5	0	8	3	11	6	0	9	4	12	7	1	10	4	13	7	2	11
6	1	9	4	12	7	1	10	5	13	8	2	11	5	0	8	3	12
7	2	10	5	13	8	2	11	6	0	9	3	12	6	1	9	4	13
8	3	11	6	0	9	3	12	7	1	10	4	13	7	2	10	5	0
9	4	12	7	1	10	4	13	8	2	11	5	0	8	3	11	6	1
10	5	13	8	2	11	5	0	9	3	12	6	1	9	4	12	7	2
11	6	0	9	3	12	6	1	10	4	13	7	2	10	5	13	8	3
12	7	1	10	4	13	7	2	11	5	0	8	3	11	6	0	9	4
13	8	2	11	5	0	8	3	12	6	1	9	4	12	7	1	10	5

FIG. 4.2. *An example of Construction* 4.1.

$(2,15)$, $(10,16)$, $(18,17)\}$. In Figure 4.2(b), the vertices in the zigzag row are shown in solid circles, solid hexagons, or dashed circles.

Now we briefly analyze the structure of the zigzag row in H. Let us write the offset sequence S as $S =$"$s_0, s_1, \dots, s_{17}$." Then for $i = 0, 1, \dots, 17$, we can see that s_i actually shows the *offset* between the ith column and the $(i+1)$th column of H. In other words, if we shift the integers in the ith column of H down (toroidally) by s_i units, we get the $(i+1)$th column of H, so we can think of s_i as spanning from the ith column to the $(i+1)$th column of H. And let us say that a P or Q in the offset sequence spans the columns that all its elements span. Then, since the offset sequence here is "$PPPQPPPQ$," the range spanned by each is as indicated in Figure 4.2(b).

Let us observe the vertices in the zigzag row that are in solid circles. If we indicate them by (a_{x_1}, x_1), $(a_{x_2}, x_2), \dots, (a_{x_{p+q}}, x_{p+q})$, where $x_1 < x_2 \cdots < x_{p+q}$, then we can see that $s_{x_1}, s_{x_2}, \dots, s_{x_{p+q}}$ are the first elements of the P's and Q's in the offset sequence (namely, each of them is the first element of a P or a Q in the offset sequence). And we can see that the vertices in solid circles have a regular structure: The vertical position climbs up by $m = 2$ units from one vertex to the next, and drops to a base-position if it is between the spanned ranges of a Q and a P. Now let us observe the vertices in solid hexagons. We can see that they correspond to the second elements of the P's and Q's in the offset sequence, and they also have a regular structure. To be specific, the positions of the vertices in solid hexagons can be obtained by shifting the positions of the vertices in solid circles horizontally by one unit and then down by $L = 8$ units. In general, those vertices in a zigzag row that correspond to the $(i+1)$th elements of P's and Q's can be obtained by shifting the positions of the vertices that correspond to the ith elements of P's and Q's horizontally by one unit and down by L unit (here $0 \le i < m$). As for the vertices in dashed circles, they correspond to the last elements of the Q's in the offset sequence, and they are all in the same row. The above observations can be extended in an obvious way to the general outputs of Construction 4.1.

Now we present the second construction.

Construction 4.2. Constructing a zigzag row in H, when t is odd, $t > 3$, and $0 < p < q$.

1. Let s_{x_1}, s_{x_2}, $\dots$, $s_{x_{p+q}}$ be the integers such that $0 = x_1 < x_2 < \cdots < x_{p+q} = l_2 - m - 1$, and each s_{x_i} $(1 \le i \le p+q)$ is the first element of a P or Q in the offset sequence S.

   Let $a_{x_1} = L$.

   For $i = 2$ to $p+q$, if s_{x_i} is the first element of a P, let $a_{x_i} = L$; if $s_{x_{i-1}}$ is the first element of a P, let $a_{x_i} = L - \lceil \frac{q}{p} \rceil (m-1)$; otherwise, let $a_{x_i} = a_{x_{i-1}} + (m-1)$.

2. For $i = 2$ to m and for $j = 1$ to $p+q$, let $a_{x_j+i-1} = a_{x_j+i-2} + L$.

3. Let s_{y_1}, s_{y_2}, $\dots$, s_{y_q} be the integers such that $y_1 < y_2 < \cdots < y_q = l_2 - 1$ and each s_{y_i} $(1 \le i \le q)$ is the last element of a Q in the offset sequence S.

   For $i = 1$ to q, let $a_{y_i} = a_{y_i - 1} + L$.

   Now we have fully determined the zigzag row, $\{(a_0, 0), (a_1, 1), \dots, (a_{l_2-1}, l_2 - 1)\}$, in the torus H.

Like Construction 4.1, the zigzag row constructed by Construction 4.2 also has a regular (and similar) structure.

THEOREM 4.4. *The zigzag rows constructed by Constructions* 4.1 *and* 4.2 *follow all the three rules listed above* (*Rules* 1, 2, *and* 3).

The above theorem can be proved with straightforward verification. So we skip its proof.

4.3. Optimal interleaving when t is odd. In this subsection, we prove that when t is odd, for a torus whose size is large enough in both dimensions, its t-interleaving number is at most one more than the sphere packing lower bound, $|S_t|$. We also present the corresponding optimal t-interleaving construction.

LEMMA 4.5. *In equation set* (3.1) (*the equation set in Construction* 3.1)*, let the values of t, m, and l_2 be fixed. Let $p = p_0, q = q_0$ be a solution that satisfies the equation set* (3.1)*. Then, another solution, $p = p_1, q = q_1$, also satisfies the equation set* (3.1) *if and only if there exists an integer c such that $p_1 = p_0 + c(m+1)(2m^2 + 2m + 2) \geq 0$ and $q_1 = q_0 - cm(2m^2 + 2m + 2) \geq 0$.*

Proof. We can easily prove that "$p = p_1, q = q_1$ is a solution that satisfies the equation set (3.1) if $p_1 = p_0 + c(m+1)(2m^2 + 2m + 2) \geq 0$ and $q_1 = q_0 - cm(2m^2 + 2m + 2) \geq 0$ for some integer c," by plugging $p = p_1, q = q_1$ into the equation set (3.1). Now let us prove the other direction.

Assume that $p = p_1, q = q_1$ is a solution that satisfies the equation set (3.1). Let $x = p_1 - p_0$ and $y = q_1 - q_0$. By the first equation in (3.1), $p_1 m + q_1(m+1) = l_2 = p_0 m + q_0(m+1)$, and therefore $(p_1 - p_0)m = -(q_1 - q_0)(m+1)$, which is $xm = -y(m+1)$. So x is a multiple of $m+1$, and y is a multiple of m. Thus there exists an integer a such that $x = a(m+1)$ and $y = -am$.

Now let us look at the second equation in (3.1), $p_1(2m^2+m+1)+q_1(2m^2+3m+2) \equiv 0 \bmod (2m^2+2m+2)$. Note that $2m^2+m+1 \equiv -(m+1) \bmod (2m^2+2m+2)$ and $2m^2+3m+2 \equiv m \bmod (2m^2+2m+2)$. So $-p_1(m+1)+q_1 m \equiv 0 \bmod (2m^2 + 2m + 2)$. Since $p_1 = p_0 + x = p_0 + a(m+1)$ and $q_1 = q_0 + y = q_0 - am$, we get $-[p_0 + a(m+1)](m+1) + (q_0 - am)m \equiv [-p_0(m+1) + q_0 m] - [a(m+1)^2 + am^2] \equiv -a(2m^2 + 2m + 1) \equiv 0 \bmod (2m^2 + 2m + 2)$. Since $2m^2 + 2m + 1$ and $2m^2 + 2m + 2$ must be relatively prime, we get $2m^2+2m+2|a$. So there exists an integer c such that $a = c(2m^2+2m+2)$. Then $p_1 = p_0+x = p_0+a(m+1) = p_0+c(m+1)(2m^2+2m+2) \geq 0$ and $q_1 = q_0+y = q_0-am = q_0-cm(2m^2+2m+2) \geq 0$, these two inequalities coming from the last condition in (3.1). That completes the proof of the other direction of this lemma. ∎

LEMMA 4.6. *In equation set* (3.1) (*the equation set in Construction* 3.1)*, let the values of t, m, and l_2 be fixed. Let $\Delta_P = (m+1)(2m^2 + 2m + 2)$ and $\Delta_Q = m(2m^2 + 2m + 2)$. If there exists a solution of p and q that satisfies the equation set* (3.1)*, then there exists a solution $p = p^*, q = q^*$ that satisfies not only* (3.1) *but also one of the following two inequalities:*

$$\frac{l_2}{2m+1} - \frac{\Delta_Q}{2} < q^* \leq p^* < \frac{l_2}{2m+1} + \frac{\Delta_P}{2}, \tag{4.1}$$

$$\frac{l_2}{2m+1} - \frac{\Delta_P}{2} \leq p^* < q^* \leq \frac{l_2}{2m+1} + \frac{\Delta_Q}{2}. \tag{4.2}$$

Proof. Assume that there is a solution $p=p_0, q=q_0$ that satisfies equation set (3.1). Trivially, either $p_0 \geq q_0$ or $p_0 < q_0$. First, let us assume that $p_0 \geq q_0$. If $p_0 \geq \frac{l_2}{2m+1} + \Delta_P$, then $q_0 = \frac{l_2 - p_0 m}{m+1} \leq \frac{l_2 - [l_2/(2m+1)+\Delta_P]m}{m+1} = \frac{l_2 - [l_2/(2m+1)+(m+1)(2m^2+2m+2)]m}{m+1} = \frac{l_2}{2m+1} - \Delta_Q$ (and vice versa), so then by Lemma 4.5, $p = p_0 - \Delta_P, q = q_0 + \Delta_Q$ is also a solution to (3.1), and, what is more, $p_0 - \Delta_P \geq \frac{l_1}{2m+1} \geq q_0 + \Delta_Q$. Based on the above observation, we can see that there must exist a solution $p = p_1, q = q_1$ such that $\frac{l_2}{2m+1} - \Delta_Q < q_1 \leq p_1 < \frac{l_2}{2m+1} + \Delta_P$. If $p_1 < \frac{l_2}{2m+1} + \frac{\Delta_P}{2}$, then $q_1 > \frac{l_2}{2m+1} - \frac{\Delta_Q}{2}$, so then we can simply let $p^* = p_1$ and let $q^* = q_1$. If $p_1 \geq \frac{l_2}{2m+1} + \frac{\Delta_P}{2}$, then $q_1 \leq \frac{l_2}{2m+1} - \frac{\Delta_Q}{2}$, so then we will let $p^* = p_1 - \Delta_P$ and let $q^* = q_1 + \Delta_Q$, in which case we will have $\frac{l_2}{2m+1} - \frac{\Delta_P}{2} \le p^* < \frac{l_2}{2m+1} < q^* \le \frac{l_2}{2m+1} + \frac{\Delta_Q}{2}$. So when $p_0 \ge q_0$, this lemma holds. The case that $p_0 < q_0$ can be analyzed similarly. ∎

THEOREM 4.7. *Let t be a positive odd integer. Let $m = \frac{t-1}{2}$. Define A as*

$$\max\Big\{ \big(\lceil \tfrac{l_2+(m+1)(2m+1)(m^2+m+1)}{l_2-m(2m+1)(m^2+m+1)} \rceil + 1\big)m^2 + 2m + 1, \big(\lceil \tfrac{l_2+m(2m+1)(m^2+m+1)}{l_2-(m+1)(2m+1)(m^2+m+1)} \rceil + 1\big)m^2 + m + 2 - \lceil \tfrac{l_2+m(2m+1)(m^2+m+1)}{l_2-(m+1)(2m+1)(m^2+m+1)} \rceil \Big\}.$$

Then when

$$l_2 \ge (m+1)(2m+1)(m^2+m+1) + 1$$

and

$$l_1 \ge (2m^2+2m+1)\left(\left\lceil \frac{A}{2m^2+2m+2} \right\rceil (2m^2+2m+2) - 2\right),$$

the t-interleaving number of an $l_1 \times l_2$ (or $l_2 \times l_1$) torus is either $|S_t|$ or $|S_t|+1$.

Proof. This theorem is trivially correct when $t = 1$. When $t = 3$, by the result of Appendix I (Theorem 7.1), we can also easily verify that this theorem is correct. Thus in the following analysis, we assume that $t > 3$.

Let us first define a few variables for the ease of expression. Let $\Delta_P = (m+1)(2m^2+2m+2)$, $\Delta_Q = m(2m^2+2m+2)$, $B = \frac{l_2+(m+1)(2m+1)(m^2+m+1)}{l_2-m(2m+1)(m^2+m+1)}$, $C = \frac{l_2+m(2m+1)(m^2+m+1)}{l_2-(m+1)(2m+1)(m^2+m+1)}$, $D = (\lceil B \rceil + 1)m^2 + 2m + 1$, and $E = (\lceil C \rceil + 1)m^2 + m + 2 - \lceil C \rceil$. Then clearly $A = \max\{D, E\}$.

When $l_2 \ge (m+1)(2m+1)(m^2+m+1)+1 = (m+\frac{1}{2})(m+1)(2m^2+2m+2)+1 > m(m+1)(2m^2+2m+2) = \lfloor \frac{t}{2} \rfloor (\lfloor \frac{t}{2} \rfloor + 1)(|S_t|+1)$, by Theorem 3.4, there exists at least one solution of p and q that satisfies equation set (3.1). Then by Lemma 4.6, there exists a solution $p = p^*, q = q^*$ to (3.1) that satisfies either the condition $\frac{l_2}{2m+1} - \frac{\Delta_Q}{2} < q^* \le p^* < \frac{l_2}{2m+1} + \frac{\Delta_P}{2}$ or the condition $\frac{l_2}{2m+1} - \frac{\Delta_P}{2} \le p^* < q^* \le \frac{l_2}{2m+1} + \frac{\Delta_Q}{2}$. We analyze the two cases below.

- Case 1. There is a solution $p = p^*, q = q^*$ to equation set (3.1) that satisfies the condition $\frac{l_2}{2m+1} - \frac{\Delta_Q}{2} < q^* \le p^* < \frac{l_2}{2m+1} + \frac{\Delta_P}{2}$. We use Construction 3.1 to t-interleave an $(|S_t|+1) \times l_2$ torus G_1. Note that when $l_2 \ge (m+1)(2m+1)(m^2+m+1)+1$, $\frac{l_2}{2m+1} - \frac{\Delta_Q}{2} > 0$, so $q^* > 0$. Also note that $\frac{p^*}{q^*} < \frac{l_2/(2m+1)+\Delta_P/2}{l_2/(2m+1)-\Delta_Q/2} = B$, so $D \ge (\lceil \frac{p^*}{q^*} \rceil + 1)m^2 + 2m + 1$. Let G_2 be a $[\lceil \frac{D}{|S_t|+1} \rceil (|S_t|+1)] \times l_2$ torus obtained by tiling $\lceil \frac{D}{|S_t|+1} \rceil$ copies of G_1 vertically. We use Construction 4.1 to find a zigzag row in G_2; then by removing the zigzag row in G_2, we get a torus G_3 whose size is $[\lceil \frac{D}{|S_t|+1} \rceil (|S_t|+1) - 1] \times l_2$. Clearly the number of rows in G_1, $|S_t|+1$, and the number of rows in G_3, $\lceil \frac{D}{|S_t|+1} \rceil (|S_t|+1) - 1$, are relatively prime. So for any $l_0 \times l_2$ torus G where $l_0 \ge (|S_t|+1-1)(\lceil \frac{D}{|S_t|+1} \rceil (|S_t|+1)-1-1) = |S_t|(\lceil \frac{D}{|S_t|+1} \rceil (|S_t|+1)-2)$, it can be obtained by tiling copies of G_1 and G_3 vertically, and so by Lemma 4.3, G is t-interleaved, using $|S_t|+1$ colors.
- Case 2. There is a solution $p = p^*, q = q^*$ to equation set (3.1) that satisfies the condition $\frac{l_2}{2m+1} - \frac{\Delta_P}{2} \le p^* < q^* \le \frac{l_2}{2m+1} + \frac{\Delta_Q}{2}$. We use Construction 3.1 to t-interleave an $(|S_t|+1) \times l_2$ torus G_1. Note that when $l_2 \ge (m+1) \cdot$

$(2m+1)(m^2+m+1)+1$, $\frac{l_2}{2m+1} - \frac{\Delta_P}{2} > 0$, so $p^* > 0$. Also note that $\frac{q^*}{p^*} \le \frac{l_2/(2m+1)+\Delta_Q/2}{l_2/(2m+1)-\Delta_P/2} = C$, so $E \ge (\lceil \frac{q^*}{p^*} \rceil + 1)m^2 + m + (2 - \lceil \frac{q^*}{p^*} \rceil)$. Let G_2 be an $[\lceil \frac{E}{|S_t|+1} \rceil (|S_t|+1)] \times l_2$ torus obtained by tiling $\lceil \frac{E}{|S_t|+1} \rceil$ copies of G_1 vertically. We use Construction 4.2 to find a zigzag row in G_2; then by removing the zigzag row in G_2, we get a torus G_3 whose size is $[\lceil \frac{E}{|S_t|+1} \rceil (|S_t|+1) - 1] \times l_2$. Clearly the number of rows in G_1, $|S_t|+1$, and the number of rows in G_3, $\lceil \frac{E}{|S_t|+1} \rceil (|S_t|+1) - 1$, are relatively prime. So for any $l_0 \times l_2$ torus G where $l_0 \ge (|S_t|+1-1)(\lceil \frac{E}{|S_t|+1} \rceil (|S_t|+1) - 1 - 1) = |S_t|(\lceil \frac{E}{|S_t|+1} \rceil (|S_t|+1) - 2)$, it can be obtained by tiling copies of G_1 and G_3 vertically, and so by Lemma 4.3, G is t-interleaved, using $|S_t|+1$ colors.

Now let G be an $l_1 \times l_2$ torus, where $l_2 \ge (m+1)(2m+1)(m^2+m+1)+1$ and $l_1 \ge (2m^2+2m+1)(\lceil \frac{A}{2m^2+2m+2} \rceil (2m^2+2m+2) - 2) = |S_t|(\lceil \frac{\max\{D,E\}}{|S_t|+1} \rceil (|S_t|+1) - 2)$. Based on the analysis for Cases 1 and 2, we know that G's t-interleaving number is at most $|S_t|+1$. By the sphere-packing lower bound, G's t-interleaving number is at least $|S_t|$. So G's t-interleaving number is either $|S_t|$ or $|S_t|+1$. ∎

For easy reference, we show the method for optimally t-interleaving a large torus as a construction below. Note that the construction below is applicable only when $t \ge 5$ (and, by default, t is odd). When $t = 1$, any torus can be t-interleaved with 1 integer in a trivial way. When $t = 3$, the torus can be t-interleaved with the construction to be presented in Appendix I.

Construction 4.3. Optimal t-interleaving on a large torus.

Input: An odd integer t such that $t \ge 5$. An integer m such that $m = \frac{t-1}{2}$. An $l_1 \times l_2$ torus, where

$$l_2 \ge (m+1)(2m+1)(m^2+m+1)+1$$

and

$$l_1 \ge (2m^2+2m+1)\left(\left\lceil \frac{A}{2m^2+2m+2} \right\rceil (2m^2+2m+2) - 2\right).$$

The parameter A is as defined in Theorem 4.7.

Output: An optimal t-interleaving on the $l_1 \times l_2$ torus.

Construction:

1. If both l_1 and l_2 are multiples of $|S_t|$, then the $l_1 \times l_2$ torus' t-interleaving number is $|S_t|$. In this case, we use Construction 2.2 to t-interleave the $l_1 \times l_2$ torus with $|S_t|$ distinct integers.

2. If either l_1 or l_2 is not a multiple of $|S_t|$, then the $l_1 \times l_2$ torus' t-interleaving number is $|S_t|+1$. In this case, we t-interleave the torus with $|S_t|+1$ integers in the following way: First, we t-interleave an $(|S_t|+1) \times l_2$ torus, B, by using Construction 3.1 (note that $|S_t|+1 = 2m^2+2m+2$); second, we let H be an $[\lceil \frac{A}{|S_t|+1} \rceil (|S_t|+1)] \times l_2$ torus, which is obtained by tiling $\lceil \frac{A}{|S_t|+1} \rceil$ copies of B vertically, and use Construction 4.1 or Construction 4.2 (depending on which is applicable) to find a zigzag row in H; third, we remove the zigzag row in H to get a $[\lceil \frac{A}{|S_t|+1} \rceil (|S_t|+1) - 1] \times l_2$ torus T; and finally, we find nonnegative integers x and y such that $l_1 = x(|S_t|+1) + y[\lceil \frac{A}{|S_t|+1} \rceil (|S_t|+1) - 1]$ and get an $l_1 \times l_2$ torus by tiling x copies of B and y copies of T vertically. The resulting interleaving on the $l_1 \times l_2$ torus is a t-interleaving.

4.4. Optimal interleaving when *t* is even. When t is even, the optimal t-interleaving on large tori can be analyzed in a very similar way as in the case of odd

t. The main result for even t is shown in the following theorem. For succinctness, we leave the major steps and intermediate results of the corresponding analysis to Appendix II.

THEOREM 4.8. *Let t be a positive even integer. Let $m = \frac{t}{2}$. Define A as*

$$\max\Big\{ \Big(\Big\lceil \tfrac{2l_2+(m+1)(2m+1)(2m^2+1)}{2l_2-m(2m+1)(2m^2+1)} \Big\rceil + 1\Big)m^2 + \Big(3 - \Big\lceil \tfrac{2l_2+(m+1)(2m+1)(2m^2+1)}{2l_2-m(2m+1)(2m^2+1)} \Big\rceil\Big)m - 3,$$
$$\Big(\Big\lceil \tfrac{2l_2+m(2m+1)(2m^2+1)}{2l_2-(m+1)(2m+1)(2m^2+1)} \Big\rceil + 1\Big)m^2 + \Big(3 - \Big\lceil \tfrac{2l_2+m(2m+1)(2m^2+1)}{2l_2-(m+1)(2m+1)(2m^2+1)} \Big\rceil\Big)m - 1$$
$$- 2\Big\lceil \tfrac{2l_2+m(2m+1)(2m^2+1)}{2l_2-(m+1)(2m+1)(2m^2+1)} \Big\rceil \Big\}.$$

Then when

$$l_2 > \frac{(m+1)(2m+1)(2m^2+1)}{2}$$

and

$$l_1 \geq 2m^2 \left(\left\lceil \frac{A}{2m^2+1} \right\rceil (2m^2+1) - 2 \right),$$

the t-interleaving number of an $l_1 \times l_2$ (or $l_2 \times l_1$) torus is either $|S_t|$ or $|S_t| + 1$.

5. General bounds on interleaving numbers. We have shown that for a torus whose size is large enough in both dimensions (Theorems 4.7 and 4.8), its t-interleaving number is at most $|S_t|+1$. If the requirement on the torus' size is loosened to some extent (Theorem 3.7), then its t-interleaving number is at most $|S_t|+2$. Does that mean that for a torus of any size its t-interleaving number is always at most $|S_t|$ plus a small constant? The answer is no. The following theorem shows bounds on t-interleaving numbers.

THEOREM 5.1. (1) *The t-interleaving numbers of two-dimensional tori are $|S_t| + O(t^2)$ in general. And that upper bound is tight, even if the number of rows or the number of columns of the torus approaches infinity.* (2) *When both l_1 and l_2 are of the order $\Omega(t^2)$, the t-interleaving number of an $l_1 \times l_2$ torus is $|S_t| + O(t)$.*

Proof. (1) First, let us show that the t-interleaving numbers of two-dimensional tori are $|S_t| + O(t^2)$ in general. Let G be an $l_1 \times l_2$ torus. First we assume that t is even and $l_1 \geq t$, $l_2 \geq t$. Let $K_1 = \lfloor \frac{l_1}{t} \rfloor$, $K_2 = \lfloor \frac{l_2}{t} \rfloor$. We see G as being tiled by small blocks in the way shown in Figure 5.1, where the blocks are labeled by A or B. Note

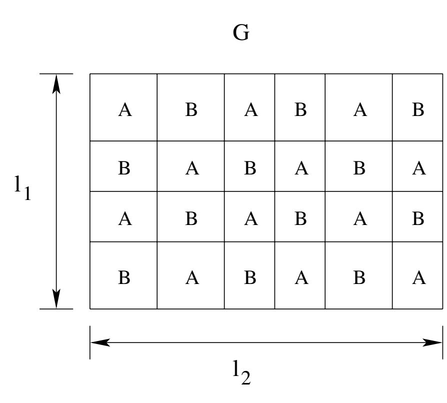

FIG. 5.1. *See G as being tiled by small blocks.*

that two blocks both labeled A are not necessarily of the same size, nor are two blocks both labeled B necessarily of the same size. For every block labeled as A (respectively, B), the four blocks around it (to its left, right, above, and below) are all labeled as B (respectively, A). Each block consists of either $\lceil \frac{l_1}{2K_1} \rceil$ or $\lfloor \frac{l_1}{2K_1} \rfloor$ rows and either $\lceil \frac{l_2}{2K_2} \rceil$ or $\lfloor \frac{l_2}{2K_2} \rfloor$ columns. Note that $\lceil \frac{l_1}{2K_1} \rceil = \lceil \frac{K_1 t + (l_1 \bmod t)}{2K_1} \rceil = \frac{t}{2} + \lceil \frac{l_1 \bmod t}{2K_1} \rceil$, $\lfloor \frac{l_1}{2K_1} \rfloor = \frac{t}{2} + \lfloor \frac{l_1 \bmod t}{2K_1} \rfloor$, $\lceil \frac{l_2}{2K_2} \rceil = \frac{t}{2} + \lceil \frac{l_2 \bmod t}{2K_2} \rceil$, and $\lfloor \frac{l_2}{2K_2} \rfloor = \frac{t}{2} + \lfloor \frac{l_2 \bmod t}{2K_2} \rfloor$. We see each block as a torus of its corresponding size. Thus for a block whose size is $\alpha \times \beta$, its vertices are denoted by (i,j) for $i = 0, 1, \ldots, \alpha - 1$ and $j = 0, 1, \ldots, \beta - 1$, just as a torus' vertices are normally denoted. Now we interleave all the blocks following these two rules: (i) only integers in the set $\{1, 2, \ldots, \lceil \frac{l_1}{2K_1} \rceil \cdot \lceil \frac{l_2}{2K_2} \rceil\}$ are used to interleave any block A, and only integers in the set $\{\lceil \frac{l_1}{2K_1} \rceil \cdot \lceil \frac{l_2}{2K_2} \rceil + 1, \lceil \frac{l_1}{2K_1} \rceil \cdot \lceil \frac{l_2}{2K_2} \rceil + 2, \ldots, 2 \cdot \lceil \frac{l_1}{2K_1} \rceil \cdot \lceil \frac{l_2}{2K_2} \rceil\}$ are used to interleave any block B; (ii) for all the blocks labeled by A (respectively, B) and for any i and j, the vertices denoted by (i,j) in them (provided they exist) all have the same color. It is very easy to see that G is t-interleaved in this way, using $2 \cdot \lceil \frac{l_1}{2K_1} \rceil \cdot \lceil \frac{l_2}{2K_2} \rceil = 2(\frac{t}{2} + \lceil \frac{l_1 \bmod t}{2K_1} \rceil)(\frac{t}{2} + \lceil \frac{l_2 \bmod t}{2K_2} \rceil) \leq 2(\frac{t}{2} + \lceil \frac{t-1}{2} \rceil)(\frac{t}{2} + \lceil \frac{t-1}{2} \rceil) = 2t^2 = |S_t| + \frac{3}{2}t^2$ distinct colors. So G's t-interleaving number is $|S_t| + O(t^2)$.

Now we assume that t is even and $l_1 < t$ or $l_2 < t$. Without loss of generality, let us say $l_1 < t$. Then we see G as being tiled horizontally by smaller tori $A_1, A_2, \ldots, A_n$, where each A_i (for $i = 1, 2, \ldots, n-1$) is an $l_1 \times t$ torus, and A_n is an $l_1 \times (l_2 \bmod t)$ torus. We interleave A_1, $A_2, \ldots, A_{n-1}$ in exactly the same way and assign $l_1 \times t$ distinct colors to each of them. We interleave A_n with a disjoint set of $l_1 \times (l_2 \bmod t)$ colors. Clearly G is t-interleaved in this way, using $l_1 \cdot t + l_1 \cdot (l_2 \bmod t) = |S_t| + O(t^2)$ distinct colors. So again, G's t-interleaving number is $|S_t| + O(t^2)$.

Finally we assume that t is odd. We can $(t+1)$-interleave G using $|S_{t+1}| + O((t+1)^2) = \frac{(t+1)^2}{2} + O((t+1)^2) = \frac{t^2+1}{2} + O(t^2) = |S_t| + O(t^2)$ distinct colors. $t+1$ is even, and a $(t+1)$-interleaving is also a t-interleaving. So G's t-interleaving number is still $|S_t| + O(t^2)$.

Now let us show that the above bound on t-interleaving numbers, $|S_t| + O(t^2)$, is tight, no matter whether t is even or odd. Consider an $l_1 \times l_2$ torus, where l_1 is the largest even integer that is no greater than $\lfloor \frac{3}{2}t \rfloor$ and l_2 is any integer greater than or equal to $\lfloor \frac{3}{4}t \rfloor$. We are first going to show that a t-interleaving can place a color at most twice in any $\lfloor \frac{3}{4}t \rfloor$ consecutive columns of the torus.

Assume that a t-interleaving places the same color on three vertices in $\lfloor \frac{3}{4}t \rfloor$ consecutive columns of the torus. Without loss of generality, let us say that those three vertices are $(0,0)$, (a,b), and (c,d), where $0 \leq b \leq \lfloor \frac{3}{4}t \rfloor - 1$ and $0 \leq d \leq \lfloor \frac{3}{4}t \rfloor - 1$; see Figure 5.2. Since the interleaving is a t-interleaving, the Lee distance between any two of those three vertices is at least t. Let $e = \frac{l_1}{2}$ and $f = \lfloor \frac{3}{4}t \rfloor - 1$. It is not difficult to see that the Lee distance between (a,b) and (e,f) is at most $\min\{(e - a) \bmod l_1, (a - e) \bmod l_1\} + (f - b) = \frac{l_1}{2} - \min\{(0 - a) \bmod l_1, (a - 0) \bmod l_1\} + (f - b)$ $= \frac{l_1}{2} + f - [\min\{(0 - a) \bmod l_1, (a - 0) \bmod l_1\} + b]$. Since the Lee distance between $(0,0)$ and (a,b) is at most $\min\{(0 - a) \bmod l_1, (a - 0) \bmod l_1\} + b$, we know that $\min\{(0 - a) \bmod l_1, (a - 0) \bmod l_1\} + b \geq t$. Therefore the Lee distance between (a,b) and (e,f) is at most $\frac{l_1}{2} + f - t \leq \lfloor \frac{3}{2}t \rfloor / 2 + \lfloor \frac{3}{4}t \rfloor - 1 - t < \frac{t}{2}$. Similarly, the Lee distance between (c,d) and (e,f) is also less than $\frac{t}{2}$. Therefore the Lee distance between (a,b) and (c,d) is less than t, which is a contradiction. So a t-interleaving can place each color on at most two vertices in $\lfloor \frac{3}{4}t \rfloor$ consecutive columns of the torus.

Any $\lfloor \frac{3}{4}t \rfloor$ consecutive columns of the $l_1 \times l_2$ torus contain $l_1 \times \lfloor \frac{3}{4}t \rfloor \geq (\frac{3}{2}t - 2) \times$

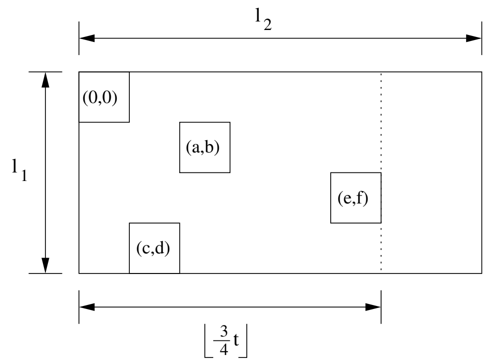

FIG. 5.2. *Four vertices in an $l_1 \times l_2$ torus.*

$(\frac{3}{4}t - 1) = \frac{9}{8}t^2 - 3t + 2$ vertices, where each color is placed at most twice by a t-interleaving. Therefore the t-interleaving number of the torus is at least $\frac{\frac{9}{8}t^2-3t+2}{2} = \frac{9}{16}t^2 - \frac{3}{2}t + 1 = \frac{t^2+1}{2} + \frac{1}{16}t^2 - \frac{3}{2}t + \frac{1}{2} \geq |S_t| + \frac{1}{16}t^2 - \frac{3}{2}t + \frac{1}{2} = |S_t| + \Theta(t^2)$, which matches the upper bound $|S_t| + O(t^2)$. Since here l_2 can be *any* integer that is no less than $\lfloor \frac{3}{4}t \rfloor$, the upper bound is tight even if the number of columns (or equivalently, the number of rows) of the torus approaches infinity. The first part of this theorem has been proved by now.

(2) Let us prove the second part of this theorem. In the previous part of this proof, a method for t-interleaving an $l_1 \times l_2$ torus has been proposed for the case when t is even and $l_1 \geq t$, $l_2 \geq t$. That method uses $2(\frac{t}{2} + \lceil \frac{l_1 \bmod t}{2K_1} \rceil)(\frac{t}{2} + \lceil \frac{l_2 \bmod t}{2K_2} \rceil)$ colors. Note that $K_1 = \lfloor \frac{l_1}{t} \rfloor$ and $K_2 = \lfloor \frac{l_2}{t} \rfloor$. When both l_1 and l_2 are of the order $\Omega(t^2)$, both K_1 and K_2 are of the order of $\Omega(t)$, and then $2(\frac{t}{2} + \lceil \frac{l_1 \bmod t}{2K_1} \rceil)(\frac{t}{2} + \lceil \frac{l_2 \bmod t}{2K_2} \rceil) = 2(\frac{t}{2} + O(1))(\frac{t}{2} + O(1)) = \frac{t^2}{2} + O(t) = |S_t| + O(t)$. When t is odd, we can t-interleave an $l_1 \times l_2$ torus, where $l_1 = \Omega(t^2) = \Omega((t+1)^2)$ and $l_2 = \Omega(t^2) = \Omega((t+1)^2)$, by $(t+1)$-interleaving it using $|S_{t+1}| + O(t+1) = \frac{(t+1)^2}{2} + O(t) = \frac{t^2+1}{2} + O(t) = |S_t| + O(t)$ colors. So no matter whether t is even or odd, when both l_1 and l_2 are of the order $\Omega(t^2)$, the t-interleaving number of an $l_1 \times l_2$ torus is $|S_t| + O(t)$. □

6. Discussion. In this paper, we have studied the t-interleaving problem for two-dimensional tori. It has applications in both distributed data storage and burst error correction. This is the first time that the t-interleaving problem has been studied for graphs with modular structures, and consequently, novel interleaving methods different from traditional techniques (e.g., the widely used lattice-interleaver schemes in early works [8], [10], [17]) have been developed for optimal t-interleaving. The necessary and sufficient condition for tori that can be perfectly t-interleaved was proved, and the corresponding perfect t-interleaving construction was presented, based on the method of sphere-packing. The most important contribution of this paper is to prove that for tori whose sizes are large in both dimensions, which constitute by far the majority of all existing cases, their t-interleaving numbers are at most one more than the sphere-packing lower bound. Optimal t-interleaving constructions for such tori were presented, based on the method of removing-a-zigzag-row and tori-tiling. Then, some additional bounds on the t-interleaving numbers were shown. Those results together give a general characterization of the t-interleaving problem for two-dimensional tori.

The importance of the t-interleaving method based on removing-a-zigzag-row and tori-tiling is not limited to the results in Theorems 4.7 and 4.8. Those two theorems should be seen as a lower bound for the performance of the t-interleaving method. By analyzing the performance of the corresponding t-interleaving constructions more carefully, and furthermore, by keeping the main idea of the t-interleaving method but tuning its specific parameters on a case-by-case basis, we can improve the bounds derived in Theorems 4.7 and 4.8. The content of Appendix I can serve as an example in this regard. What is more, the t-interleaving method can be used to optimally t-interleave some tori whose sizes do not fall within the derived bounds.

We are interested in studying the t-interleaving problem for higher-dimensional tori, as well as finding more t-interleaving constructions. Those remain as our future research.

7. Appendix I. The optimal t-interleaving construction for odd t, Construction 4.3, if applicable only when $t \geq 5$. In this appendix, we present the optimal t-interleaving construction when $t = 3$, thus completing the result for t-interleaving on large tori while t is odd. We also use this case, $t = 3$, as an example to show how previous results can be improved if the t-interleaving problem is analyzed case by case and more carefully.

We will show that when $l_1 \geq 20$ and $l_2 \geq 15$ (or equivalently, when $l_1 \geq 15$ and $l_2 \geq 20$), an $l_1 \times l_2$ torus' 3-interleaving number is either 5 or 6. Note that $|S_3| = 5$. Below we present an construction that can optimally 3-interleave any $l_1 \times l_2$ torus where $l_1 \geq 20$ and $l_2 \geq 15$, except when $l_2 = 19$.

Construction 7.1. Optimally 3-interleave an $l_1 \times l_2$ torus, where $l_1 \geq 20$, $l_2 \geq 15$, and $l_2 \neq 19$.

1. If both l_1 and l_2 are multiples of 5, then the $l_1 \times l_2$ torus' 3-interleaving number is $|S_t| = 5$. In this case, 3-interleave the $l_1 \times l_2$ torus with five colors by using Construction 2.2.

   If l_1 or l_2 is not a multiple of 5, then use steps 2–4 below to 3-interleave the $l_1 \times l_2$ torus with six colors.

2. Find nonnegative integers x_1 and x_2 such that $l_1 = 5x_1+6x_2$. Find nonnegative integers y_1, y_2, and y_3 such that $l_2 = 5y_1 + 8y_2 + 12y_3$.

3. There are six tori shown in Figure 7.1(a): a 5×5 torus A, a 5×8 torus B, a 5×12 torus C, a 6×5 torus A', a 6×8 torus B', and a 6×12 torus C'.

   Get a $5 \times l_2$ torus M_1 by tiling horizontally y_1 copies of A, y_2 copies of B, and y_3 copies of C (whose order can be arbitrary).

   Get a $6 \times l_2$ torus M_2 by tiling horizontally y_1 copies of A', y_2 copies of B', and y_3 copies of C', whose order needs to satisfy this rule: for $i = 1$ to $y_1 + y_2 + y_3$, if the ith module-torus in M_1 is an A (respectively, a B or a C), then the ith module in M_2 is an A' (respectively, a B' or a C').

4. Get an $l_1 \times l_2$ torus by tiling x_1 copies of M_1 and x_2 copies of M_2 (whose order can be arbitrary) vertically. The interleaving on the $l_1 \times l_2$ torus is a 3-interleaving.

Example 7.1. We use Construction 7.1 to 3-interleave an $l_1 \times l_2$ torus, where $l_1 = 11$ and $l_2 = 25$. l_1 is not a multiple of $|S_t|$, so the torus' 3-interleaving number is greater than 5. Since $l_1 = 5 + 6$ and $l_2 = 5 + 8 + 12$, the variables in Construction 7.1 can be set as follows: $x_1 = 1$, $x_2 = 1$, $y_1 = 1$, $y_2 = 1$, and $y_3 = 1$. Furthermore, we can let the torus M_1 have the form of $[ABC]$ and let the torus M_2 have the form of $[A'B'C']$. We then tile M_1 and M_2 to get the $l_1 \times l_2$ torus, which is of the form $\left[\begin{smallmatrix} A & B & C \\ A' & B' & C' \end{smallmatrix}\right]$. This 3-interleaved torus is shown in Figure 7.1(b). The interleaving used $6 = |S_3| + 1$ colors.

(a) Modules

A

0	2	4	1	3
1	3	0	2	5
2	5	1	3	0
4	0	2	5	1
5	1	3	0	2

B

0	2	4	0	3	5	1	3
1	3	5	2	4	0	2	5
2	4	1	3	5	1	4	0
3	0	2	4	0	3	5	1
5	1	3	5	2	4	0	2

C

0	2	5	1	4	0	3	5	2	4	1	3
1	3	0	2	5	1	4	0	3	5	2	4
2	5	1	4	0	3	5	2	4	1	3	0
4	0	3	5	2	4	1	3	0	2	5	1
5	1	4	0	3	5	2	4	1	3	0	2

A’

0	2	4	1	3
1	3	⑤	2	④
2	④	0	3	5
③	5	1	④	0
4	0	2	5	1
5	1	3	0	2

B’

0	2	4	0	3	5	1	3
1	3	5	①	4	0	2	④
2	4	⓪	2	5	1	③	5
3	⑤	1	3	0	②	4	0
④	0	2	4	①	3	5	1
5	1	3	5	2	4	0	2

C’

0	2	5	1	4	0	3	5	2	4	1	3
1	3	0	2	5	1	4	0	3	5	2	4
2	④	1	③	0	②	5	①	4	⓪	3	⑤
③	5	②	4	①	3	⓪	2	⑤	1	④	0
4	0	3	5	2	4	1	3	0	2	5	1
5	1	4	0	3	5	2	4	1	3	0	2

(b) Tiling of modules

0	2	4	1	3	0	2	4	0	3	5	1	3	0	2	5	1	4	0	3	5	2	4	1	3
1	3	0	2	5	1	3	5	2	4	0	2	5	1	3	0	2	5	1	4	0	3	5	2	4
2	5	1	3	0	2	4	1	3	5	1	4	0	2	5	1	4	0	3	5	2	4	1	3	0
4	0	2	5	1	3	0	2	4	0	3	5	1	4	0	3	5	2	4	1	3	0	2	5	1
5	1	3	0	2	5	1	3	5	2	4	0	2	5	1	4	0	3	5	2	4	1	3	0	2
0	2	4	1	3	0	2	4	0	3	5	1	3	0	2	5	1	4	0	3	5	2	4	1	3
1	3	5	2	4	1	3	5	1	4	0	2	4	1	3	0	2	5	1	4	0	3	5	2	4
2	4	0	3	5	2	4	0	2	5	1	3	5	2	4	1	3	0	2	5	1	4	0	3	5
3	5	1	4	0	3	5	1	3	0	2	4	0	3	5	2	4	1	3	0	2	5	1	4	0
4	0	2	5	1	4	0	2	4	1	3	5	1	4	0	3	5	2	4	1	3	0	2	5	1
5	1	3	0	2	5	1	3	5	2	4	0	2	5	1	4	0	3	5	2	4	1	3	0	2

FIG. 7.1. *Using modules for* 3*-interleaving.* (a) *The* 6 *modules,* (b) *tiling the modules.*

Clearly, since $25 = 5 \times 5 + 8 \times 0 + 12 \times 0$, another choice for tiling the 11×25 torus is $\left[\begin{smallmatrix} A & A & A & A & A \\ A' & A' & A' & A' & A' \end{smallmatrix}\right]$.

Construction 7.1 constructs a 3-interleaved $l_1 \times l_2$ torus by tiling copies of the six module-tori shown in Figure 7.1(a). It can be readily verified that when those six tori are tiled following the rule in Construction 7.1, the resulting interleaving on the $l_1 \times l_2$ torus is indeed a 3-interleaving. There are only a limited number of cases to analyze for the verification, so we skip the details. We comment that Construction 7.1 does not work for the case $l_2 = 19$, because 19 cannot be written as a linear combination of 5, 8, and 12 with nonnegative coefficients, and therefore an $l_1 \times 19$ torus cannot be obtained by tiling the module-tori. We present the construction for the case $l_2 = 19$ below.

Construction 7.2. Optimally 3-interleave an $l_1 \times 19$ torus, where $l_1 \geq 20$.

F

0	2	4	1	3	5	1	3	0	2	4	0	2	5	1	3	5	1	4
1	3	0	2	4	0	2	5	1	3	5	1	4	0	2	4	0	3	5
2	5	1	3	5	1	4	0	2	4	0	3	5	1	3	5	2	4	0
4	0	2	4	0	3	5	1	3	5	2	4	0	2	4	1	3	5	1
5	1	3	5	2	4	0	2	4	1	3	5	1	3	0	2	4	0	3

F’

0	2	4	⓪	3	5	1	3	⑤	2	4	0	2	④	1	3	5	1	4
1	3	⑤	1	4	0	2	④	0	3	5	1	③	5	2	4	0	②	5
2	④	0	2	5	1	③	5	1	4	0	②	4	0	3	5	①	3	0
③	5	1	3	0	②	4	0	2	5	①	3	5	1	4	⓪	2	4	1
4	0	2	4	①	3	5	1	3	⓪	2	4	0	2	⑤	1	3	5	②
5	1	3	5	2	4	0	2	4	1	3	5	1	3	0	2	4	0	3

FIG. 7.2. *Two modules used for* 3*-Interleaving an* $l_1 \times 19$ *torus, where* $l_1 \geq 20$.

Construction: Find nonnegative integers x_1 and x_2 such that $l_1 = 5x_1 + 6x_2$. There are two tori shown in Figure 7.2: a 5×19 torus F and a 6×19 torus F'. Construct an $l_1 \times 19$ torus by tiling x_1 copies of F and x_2 copies of F' vertically (whose order can be arbitrary). The resulting interleaving on the $l_1 \times 19$ torus is a 3-interleaving.

The correctness of Construction 4.5 can be easily verified, so we skip the details. Based on the previous two constructions, we readily get the following conclusion for 3-interleaving.

THEOREM 7.1. *When* $l_1 \geq 20$ *and* $l_2 \geq 15$*, or when* $l_1 \geq 15$ *and* $l_2 \geq 20$*, an* $l_1 \times l_2$ *torus'* 3*-interleaving number is either* $|S_3|$ *or* $|S_3| + 1$.

We comment that the result obtained here is comparatively better than the result derived in section 4. For example, if Theorem 4.7 is applied for the case $t = 3$, then the bound for l_2 would be 19, but here our bound for l_2 is 15. However, we should notice that the t-interleaving method used here is the same as the method used for $t > 3$ per se. We can see that the module-tori A, B, C in Figure 7.1(a) and F in Figure 7.2 are obtained by removing a zigzag row from A', B', C', and F'. The zigzag rows are shown in circles in those two figures. Both the interleaving method here and the method in section 4 are based on torus tiling. The improvement attained here is made by better tuning of construction parameters and more careful analysis of the bounds. The construction used for $t = 3$ does not follow all the requirements used in section 4. For example, the zigzag row in Figure 7.2 does not follow Rule 3. In section 4, while endeavoring to optimally tune all the parameters, we also need to ensure that the construction will work for all the cases of $t > 3$. If the interleaving problem is analyzed case by case (specifically, for each value of t, l_1, and l_2), the interleaving construction has room for further optimization.

8. Appendix II. In this appendix, we show how to optimally t-interleave large tori when t is even. The process is similar to the case where t is odd, differing only in details. For this reason, we just present a succinct description of the process and results. This appendix's content is parallel to that of the first three subsections of section 4, so comparative reading should help the understanding greatly.

We assume that t is even throughout the remainder of this appendix. The definitions of a *zigzag row* and *removing a zigzag row* are the same as in Definitions 4.1 and 4.2.

Let B be an $l_0 \times l_2$ torus which is t-interleaved by Construction 3.1 utilizing the offset sequence $S =$ "$s_0, s_1, \ldots, s_{l_2-1}$." Let H be an $l_1 \times l_2$ torus obtained by tiling several copies of B vertically. Let $m = \frac{t}{2}$. There are four rules to follow for devising a zigzag row (denoted by $\{(a_0, 0), (a_1, 1), \ldots, (a_{l_2-1}, l_2 - 1)\}$) in H:

- Rule 1. For any j such that $0 \le j \le l_2 - 1$, if the integers s_j, $s_{(j+1) \bmod l_2}$, $\ldots$, $s_{(j+m-1) \bmod l_2}$ do not all equal $t - 1$, then $a_j \ge a_{(j+m) \bmod l_2} + m - 1$.
- Rule 2. For any j such that $0 \le j \le l_2 - 1$, if exactly one of the integers $s_j, s_{(j+1) \bmod l_2}, \ldots, s_{(j+m) \bmod l_2}$ equals t, then $a_j \le a_{(j+m+1) \bmod l_2} - (m-2)$.
- Rule 3. For any j such that $0 \le j \le l_2 - 1$, if $s_j = t - 1$, then $a_j \le a_{(j+1) \bmod l_2} - (2m - 2)$.
- Rule 4. For any j such that $0 \le j \le l_2 - 1$, $2m - 2 \le a_j \le l_1 - 1 - (2m - 2)$.

LEMMA 8.1. *Let B be a torus t-interleaved by Construction* 3.1. *Let H be a torus obtained by tiling copies of B vertically, and let T be a torus obtained by removing a zigzag row in H, where the zigzag row in H follows the four rules listed above. Let G be a torus obtained by tiling copies of B and T vertically. Then, both T and G are t-interleaved.*

Now we present two constructions for finding a zigzag row, which are the counterparts of Construction 4.1 and 4.2. Let B be an $l_0 \times l_2$ torus which is t-interleaved by Construction 3.1 utilizing the offset sequence $S =$ "$s_0, s_1, \ldots, s_{l_2-1}$." Let H be an $l_1 \times l_2$ torus obtained by tiling z copies of B vertically. We say the offset sequence S consists of p P's and q Q's, where $p > 0$ and $q > 0$. We require that in S the P's and Q's are interleaved very evenly, and that S starts with a P and ends with a Q. Let $m = \frac{t}{2}$. Let $L = (2m-2) + (m-1)\lceil \frac{p}{q} \rceil$ if $p \ge q$, and let $L = (2m-2) + (m-2)\lceil \frac{q}{p} \rceil + 1$ if $p < q$. We require that $l_1 \ge (\lceil \frac{p}{q} \rceil + 1)m^2 + (3 - \lceil \frac{p}{q} \rceil)m - 3$ if $p \ge q$ and that $l_1 \ge (\lceil \frac{q}{p} \rceil + 1)m^2 + (3 - \lceil \frac{q}{p} \rceil)m - (2\lceil \frac{q}{p} \rceil + 1)$ if $p < q$. Below we present two constructions for constructing a zigzag row, which is denoted by $\{(a_0, 0), (a_1, 1), \ldots, (a_{l_2-1}, l_2 - 1)\}$, in H, applicable respectively when $p \ge q$ and $p < q$.

Construction 8.1. Constructing a zigzag row in H, when t is even, $t > 2$, and $p \ge q > 0$.

1. Let s_{x_1}, s_{x_2}, $\ldots$, $s_{x_{p+q}}$ be the integers such that $0 = x_1 < x_2 < \cdots < x_{p+q} = l_2 - m - 1$ and each s_{x_i} $(1 \le i \le p + q)$ is the first element of a P or Q in the offset sequence S.

Let $a_{x_1} = L$. For $i = 2$ to $p+q$, if $s_{x_{i-1}}$ is the first element of a Q, let $a_{x_i} = L$.

For $i = 2$ to $p + q$, if $s_{x_{i-1}}$ is the first element of a P, then let $a_{x_i} = a_{x_{i-1}} - (m - 1)$.

2. For $i = 2$ to m and for $j = 1$ to $p + q$, let $a_{x_j+i-1} = a_{x_j+i-2} + L - m + 1$.

3. Let s_{y_1}, s_{y_2}, $\ldots$, s_{y_q} be the integers such that $y_1 < y_2 < \cdots < y_q = l_2 - 1$ and each s_{y_i} $(1 \le i \le q)$ is the last element of a Q in the offset sequence S.

For $i = 1$ to q, $a_{y_i} = L + (m - 1)(L - m + 1) + (m - 1)$.

Now we have fully determined the zigzag row, $\{(a_0, 0), (a_1, 1), \ldots, (a_{l_2-1}, l_2 - 1)\}$, in the torus H.

Construction 8.2. Constructing a zigzag row in H, when t is even, $t > 2$, and $0 < p < q$.

1. Let s_{x_1}, s_{x_2}, $\ldots$, $s_{x_{p+q}}$ be the integers such that $0 = x_1 < x_2 < \cdots < x_{p+q} = l_2 - m - 1$ and each s_{x_i} $(1 \le i \le p + q)$ is the first element of a P or Q in the offset sequence S.

Let $a_{x_1} = L$. For $i = 2$ to $p + q$, if s_{x_i} is the first element of a P, then let $a_{x_i} = L$; if $s_{x_{i-1}}$ is the first element of a P, then let $a_{x_i} = L - \lceil \frac{q}{p} \rceil (m-2) - 1$; otherwise, let $a_{x_i} = a_{x_{i-1}} + (m-2)$.

2. For $i = 2$ to m and for $j = 1$ to $p + q$, let $a_{x_j+i-1} = a_{x_j+i-2} + L - m + 1$.

3. Let $s_{y_1}, s_{y_2}, \ldots, s_{y_q}$ be the integers such that $y_1 < y_2 < \cdots < y_q = l_2 - 1$ and each s_{y_i} is the last element of a Q in the offset sequence S.

For $i = 1$ to q, $a_{y_i} = a_{y_i - 1} + L - m + 1$.

Now we have fully determined the zigzag row, $\{(a_0, 0), (a_1, 1), \ldots, (a_{l_2-1}, l_2 - 1)\}$, in the torus H.

THEOREM 8.2. *The zigzag rows constructed by Constructions* 8.1 *and* 8.2 *follow all four rules: Rules* 1, 2, 3, *and* 4.

LEMMA 8.3. *In equation set* (3.2) (*which is in Construction* 3.1)*, let the values of* t*,* m*, and* l_2 *be fixed. Let* $p = p_0, q = q_0$ *be a solution that satisfies* (3.2)*. Then, another solution* $p = p_1, q = q_1$ *also satisfies* (3.2) *if and only if there exists an integer* c *such that* $p_1 = p_0 + c(m+1)(2m^2+1) \geq 0$ *and* $q_1 = q_0 - cm(2m^2+1) \geq 0$.

LEMMA 8.4. *In equation set* (3.2) (*which is in Construction* 3.1)*, let the values of* t*,* m*, and* l_2 *be fixed. Let* $\Delta_P = (m+1)(2m^2+1)$ *and* $\Delta_Q = m(2m^2+1)$*. If there exists a solution of* p *and* q *that satisfies* (3.2)*, then there exists a solution* $p = p^*, q = q^*$ *that satisfies not only* (3.2) *but also one of the following two inequalities:*

$$\frac{l_2}{2m+1} - \frac{\Delta_Q}{2} < q^* \leq p^* < \frac{l_2}{2m+1} + \frac{\Delta_P}{2}, \tag{8.1}$$

$$\frac{l_2}{2m+1} - \frac{\Delta_P}{2} \leq p^* < q^* \leq \frac{l_2}{2m+1} + \frac{\Delta_Q}{2}. \tag{8.2}$$

The above results lead to the main conclusion, Theorem 4.8.

We skip the specific construction of optimally t-interleaving large tori here, because of its similarity to Construction 4.3. But we present its sketch: If the torus can be perfectly t-interleaved, then it can be optimally t-interleaved using Construction 2.2. If the torus cannot be perfectly t-interleaved and $t \geq 4$, then it can be optimally t-interleaved using the tori-tiling method. The only remaining case is if the torus cannot be perfectly t-interleaved and $t = 2$. In that case, we can optimally t-interleave the torus (say it is an $l_1 \times l_2$ torus) using $|S_t| + 1 = 3$ distinct colors in the following way: First, interleave a ring of l_1 vertices and a ring of l_2 vertices using three colors (0, 1, and 2) such that no two adjacent vertices in those two rings are assigned the same color. Second, for $i = 1, 2, \ldots, l_1$ (respectively, for $i = 1, 2, \ldots, l_2$), use $I(i)$ (respectively, use $J(i)$) to denote the color assigned to the ith vertex in the ring of l_1 (respectively, l_2) vertices. Third, for $i = 0, 1, \ldots, l_1 - 1$ and $j = 0, 1, \ldots, l_2 - 1$, color the vertex (i, j) in the $l_1 \times l_2$ torus with color $(I(i+1) + J(j+1)) \bmod 3$. This yields an optimal 2-interleaving of the torus.

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