Effects of Varying the Delay Distribution in Random, Scale-free, and Small-World Networks

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Abstract

Graph-theory-based approaches have been used with great success when analyzing abstract properties of natural and artificial networks. However, these approaches have not factored in delay, which plays an important role in real-world networks. In this paper, we (1) developed a simple yet powerful method to include delay in graph-based analysis of networks, and (2) evaluated how different classes of networks (random, scale-free, and small-world) behave under different forms of delay (peaked, unimodal, or uniform delay distribution). We compared results from synthetically generated networks using two different sets of algorithms for network construction. In the first approach (naive), we generated directed graphs following the literal definition of the three types of networks. In the second approach (modified conventional), we adapted methods by Erdös–Rényi (random), Barabasi (scale-free), and Watts–Strogatz (small-world). With these networks, we investigated the effect of adding and varying the delay distribution. As a measure of robustness to added delay, we calculated the ratio between the sum of shortest path length between every node. Our main findings show that different types of network show different levels of robustness, but the shape of the delay distribution has more influence on the overall result, where uniformly randomly distributed delay showed the most robust result. Other network parameters such as neighborhood size in small-world networks were also found to play a key role in delay tolerance. These results are expected to extend our understanding of the relationship between network structure and delay.

1. Introduction

Friendships, ecosystems, food web, WWW, and even the brain are all networks we can find in our everyday life, with different scales, components, and connecting links. Thus, it is hard to imagine our life without networks. In spite of the importance and prevalence of networks around our environment, physical or functional structures and properties of networks are only partially understood.

Three classes of networks, random, scale-free, and small-world, have been studied extensively (see [7] for a review). However, most of the studies have not taken into account delay. If every connection has the same latency, the effects are trivial. Networks must deal with delay which may vary for each connection. So far, only few studies have included time delay in dynamic network analysis [20, 1, 10, 15, 19, 17]. These studies typically use delay differential equations. However, analysis based on delay differential equations rely on information other than connectivity (function of nodes, and weight of edges) and furthermore these studies do not assess the effect of added delay in different network classes. Thus, alternative approaches are required to compare multiple network classes with delay.

Graphs provide a significantly higher level of abstraction compared to delay differential equations and have been used to elegantly represent and analyze networks. Also, delay can be simulated in graphs by a temporal augmentation method as we have shown [6]. In this paper, we will investigate the effect of delay on the three different classes of networks under different delay distribution conditions. As a measure of robustness to delay, we calculated the increase in the sum of all pairs of shortest paths after delay has been added to the network.

Our main results show that different types of network respond differently to delay, and the shape of the delay distribution has great influence on the overall result. Other network parameters such as neighborhood size in small-world networks were also found to play a key role in delay tolerance.

The rest of this paper is organized as follows. In Sec.

2, we will review background material on network analysis. In the following Sec. 3, we will present our methods of network construction and delay robustness measure. In Sec. 4, we will provide our experimental setup and results, followed by discussion (Sec. 5) and conclusion (Sec. 6).

2. Background

2.1. Classes of networks

There are three broad categories of networks: random, regular, and complex networks. In random networks connections are placed randomly between nodes. Regular networks have a lattice organization. These two categorizations are extremes. Complex networks contain scale-free and small-world networks and are somewhere between random and regular networks in terms of their organization. Random networks are completely unorganized while regular networks are completely organized.

In network research three types of networks have commonly been experimented with: random, scale-free, and small-world networks [9, 2]. Random networks' edges randomly connect nodes in the network [3]. Scale-free networks are characterized by a few nodes with a large number of connections [4]. Small-world networks have clusters of tightly interconnected nodes which are sparsely connected to other clusters [22]. Among these, the properties of small-world networks are particularly interesting because biological networks, especially brain networks, have a small-world structure [5, 16, 18, 21, 13, 11].

Random networks were extensively studied before information about the connectivity of large-scale networks became available. For this reason they serve as a useful comparison against the other types of networks.

2.2. Research on delay

All real networks have time delay due to various factors such as the number of steps between components, physical characteristics of the link, computation time at each component, and so on. However, researchers relying on graph theory commonly disregarded delay or considered it only as being homogeneous (see [8] for a rare exception). Even for those using dynamical equations, only few have considered non-uniform delay. Eurich et al. showed that dynamical systems converge to a stabilizing state and a simpler dynamical pattern with more uniformly distributed delay [10]. Several other studies showed similar results that distributed delay yields stabilizing state and simpler dynamical pattern [20, 19, 14]. Furthermore, networks with random delay were shown to have steady-state synchronization in coupled chaotic maps whereas those with fixed delay were shown to have chaotic synchronization [15]. Therefore, time delay having a broad distribution is not just noise but it seems to play an important functional role in networks.

2.3. Graph Theory

We use techniques adapted from graph theory in order to compare networks with distributed delay. Graphs are a set of vertices and edges which connect those vertices. Directed graphs are similar to graphs but lifts the constraint that edges are undirected (or symmetric). Networks are naturally represented as directed graphs where each network component is represented by a vertex and the relationships between the components are the edges connecting vertices. A graph (or network) with n vertices (components) can be represented by an $n \times n$ adjacency matrix G where the value 1 in the i^{th} row and j^{th} column indicates an edge connecting vertex j to the vertex i. An entry of 0 indicates that no edge connects j to i.

A path is a sequences of edges in a network which connect a vertex i to a vertex j. The shortest path is a path containing the least number of sequences connecting vertices i and j. Notice that a shortest path can never contain cycles.

In agreement with previous works representing biological networks as directed graphs self-connecting vertices were disallowed [18, 16].

2.4. Temporal Augmentation

Representing delay in a graph can be accomplished using temporal augmentation [6]. Delay is represented by the length of the shortest path from a vertex i to a vertex j. By replacing edges from the graph with a sequence of edges connecting dummy vertices the path between some pair of vertices can be lengthened and by analogy delay introduced into the network.

3. Methods

3.1. Network Construction

Two construction methods are used for creating random, scale-free, and small world networks: the naive method and what we refer to as modified conventional method.

NAIVE NETWORK CONSTRUCTION. The naive construction method constructs networks using as few parameters as possible, based on their literal definition. Random, scale-free, and small-world networks require the specification of some number of vertices and edges. However scale-free and small-world networks both require additional parameters. Scale-free networks require the number of hubs, which corresponds to the number of vertices which have above average connectivity to other vertices in the network.

Small-world networks are given a neighborhood size which indicates how many vertices are in a cluster. Starting with an empty adjacency matrix, a random network (RN) is constructed by randomly assigning edges to vertices until the network contains the desired number of edges. Scale-free networks (SFN) are constructed by first densely connecting the specified number of hub vertices to other vertices in the network and then randomly distributing the remaining edges. Small-world networks (SWN) are populated with the specified number of neighborhoods and the remaining edges are distributed randomly.

MODIFIED CONVENTIONAL NETWORK CONSTRUCTION. The conventional construction method for the three types of networks are based on Erdös–Rényi model of random networks (ER) [9], Barabasi model of scale-free network (BA) [3, 4], and Watts–Strogatz model of small-world networks (WS) [22], and have been adopted widely. However, modifications needed to be made to these conventional construction methods because they are meant for constructing undirected graphs rather than directed graphs. The definition of ER model is the same as the naive model. There is no difference between the random networks.The BA model does not need a parameter (the number of hubs) unlike the naive method. The number of hubs are determined naturally because the BA model has two mechanisms: ‘growth’ and ‘preferential attachment’. However the BA model needs to specify the ratio of in-degree/out-degree because directed graphs are determined by this ratio. As a result, the hubs do not have bi-directional connections to other vertices. WS model uses two steps: building a regular network (lattice) and rewiring the edges. The construction methods explained above all assume that edges are undirected. We extended these methods to allow directed edges, resulting in modified conventional network construction algorithms (pseudo code shown in Fig. 1).

```
RN(n,e)
make an empty n × n matrix A
while (e ≠ 0)
  i=random(1,n)
  j=random(1,n)
  if i ≠ j and A(i,j) ≠1
    A(i,j)=1
    e=e-1
  end
end
```

```
BA(n,e,r)
make an empty n×n matrix A
randomly pick two initial nodes
 A(i,j)=1,A(k,l)=1
e=e-2
while (e≠0)
ratio=rand()
  if ratio ≤ r
    choose a node X among existing nodes based on its indegree
    randomly choose a node Y≠X
    if A(X,Y)≠1
      A(X,Y)=1
      e=e-1
    end
  else
    choose a node Y among existing nodes based on its outdegree
    randomly choose a node X≠Y
    if A(X,Y)≠1
      A(X,Y)=1
      e=e-1
    end
  end
end
```

```
WS(n,e,h,r)
make a regular n×n matrix A with neighborhood h
R=e×r
while (R≠0)
  i=random(1,n), k=random(1,n)
  j=random(1,n), l=random(1,n)
  if i≠j and A(i,j)=1 and A(k,l)≠1
    A(i,j)=0
    A(k,l)=1
    R=R-1
  end
end
```

Figure 1. Pseudo code for constructing random (top), scale-free (middle), and small-world networks (bottom).

3.2. Measurements

The measurement used to compare the networks was the average shortest path length and clustering coefficient.

The average shortest path length is a measurement of how efficiently, on average, information can be propagated from one neuron to another in the network.

$$L(G) = \frac{1}{N(N-1)} \sum_{i \neq j \in G} d_{ij}, \qquad (1)$$

where N is the number of nodes of graph G, and d_{ij} is the distance from node j to i where $i \neq j$.

Clustering coefficient describes how connected a single vertex is to all other vertices in the network. For a single vertex i the clustering coefficient is defined by

$$C_i = \frac{\text{number of existing edges in } G_i}{\text{maximum possible number of edges in } G_i}, \qquad (2)$$

where G_i is a subgraph of neighbors connecting to/from node i. The average of clustering of coefficients is defined as:

$$C(G) = \frac{1}{N} \sum_{i \in G} C_i, \qquad (3)$$

where N is the number of nodes in graph G.

Smallworldness measures whether a network contains tightly clustered groups with sparse connections between networks by comparing L and C. For a small-world network the average shortest path length should be about the same as a random networks average shortest path $\lambda = L/L_{\text{rand}} \approx 1$ and there should be higher clustering than for random networks $\gamma = C/C_{\text{rand}} > 1$. Smallworldness is defined as $\sigma = \gamma/\lambda$. A value of $\sigma > 1$ suggests that the network is small-world-like.

4. Experiments and Results

A series of experiments were performed on networks constructed using both the naive and the modified conven-

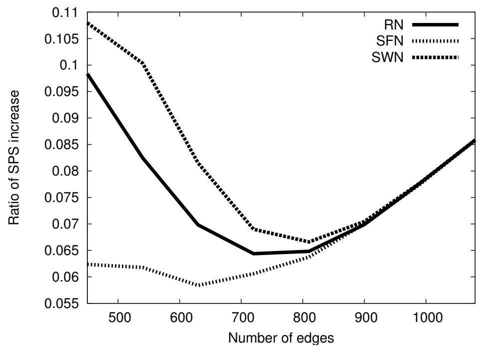


Figure 2. The ratio of increase in shortest path sum (SPS) of normal vs. delayed networks. The increase ratio in SPS under delay as the initial number of edge is increased is shown. Each network contained 45 nodes and $\frac{1}{4}$ of the edges were randomly delayed. SFNs have the smallest increase in SPS, SWNs have the largest increase, and RNs fall in between SFNs and SWNs.

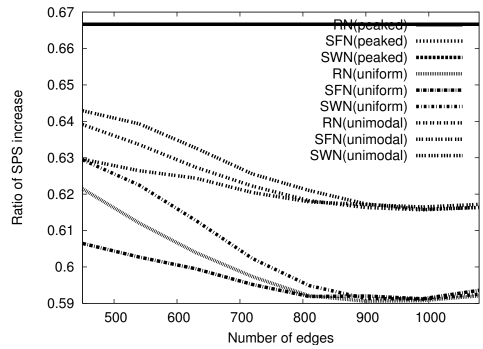


Figure 3. Increasing SPS in delayed networks with different delay distributions: Naive construction method. The increase ratio in SPS under delay is shown as the number of edges increases. Each network contained 45 nodes. Delay distribution types are indicated in the parentheses. Three networks with uniformly randomly distributed delay showed the smallest increase in the SPS and SFNs have the smallest increase in all delay distribution conditions.

tional methods. The purpose of these experiments was to uncover the relationships between network structure and delay. Comparing networks with and without delay was done using the increased shortest path sum (SPS). The SPS S_N of a network N was measured by calculating the lengths of all shortest paths and summing them together. Then N was temporally augmented to obtain a delayed network D. The SPS of D was calculated by adding up the shortest paths between all vertices of D which were originally in N to obtain S_D. The difference in the SPS is defined by $(S_D - S_N)/S_D$.

4.1. Fixed Delay Ratio

A fixed ratio ($\frac{1}{4}$) of the edges in naively constructed random, scale-free, and small-world networks were temporally augmented. The ratio of increase in shortest path length was used as a measure the effect of delay on each network type. All networks were constructed with 45 vertices and varying number of edges (Fig. 2).

With a small number of edges, small-world networks are the most affected by delay, scale-free networks are the least affected, and the effect of delay on random networks is between that of scale-free and small-world networks. As the number of edges becomes large the effects of delay converge for all three network types. This experiment demonstrates that network type has a definite impact on how delay affects network function.

The number of hubs in scale-free networks and neighborhood size in small-world networks were also tested to determine how much these parameters affect a networks robustness to delay. Increasing the number of hubs in scale-free networks resulted in a downward trend in ratio of increase in SPS, but the overall change was negligible (less than 1%). However, increasing the neighborhood size in small-world networks from 2 to 20 showed a significant increase in SPS of about 5%.

4.2. Differing Delay Distributions

Peaked, uniform, and unimodal distributions were used to introduce delay into random, scale-free, and small-world networks constructed using both the naive and the modified conventional construction methods. Figs. 3 and 4 show the ratio of increase in SPS after temporally augmenting the networks with peaked, uniform, and unimodal delay distributions for naive and the modified conventional constructions, respectively.

All networks were least affected by a uniform delay distribution and most affected by a peaked delay distribution. Among networks constructed with the naive method, scale-free networks showed the highest robustness (least affected by delay), followed by random and small-world networks. On the other hand, for those constructed with the modified construction method, random networks (ER) were the most robust, followed by scale-free (BA) and small-world (WS). This somehow contradictory result led to the analysis in the following section (Sec. 4.3).

4.3. Smallworldness

The results for the naively constructed networks and modified constructed networks are at odds because small-

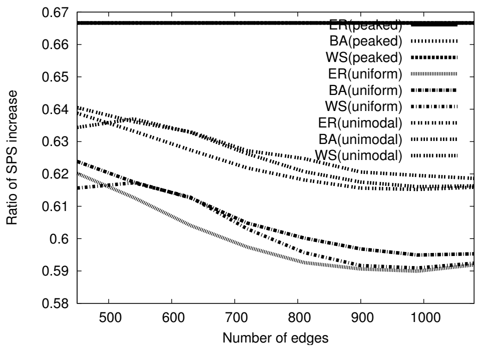
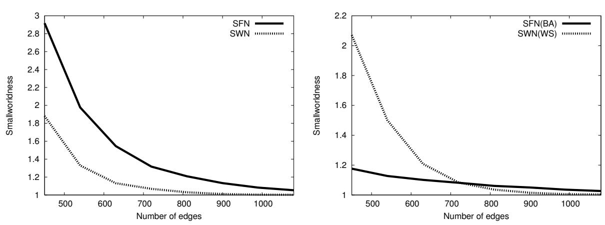


Figure 4. Increasing SPS ratio of delayed networks with different delay distributions: modified construction method. Increase ratio in SPS under delay is shown as the initial number of edges increased. Each network contained 45 nodes. Delay distribution types are indicated in the parentheses. Three networks with uniformly randomly distributed delay showed the smallest increase in SPS and WS had lower increase than BA in all delay distribution conditions.



(a) Naive　　(b) Modified Conventional

Figure 5. Smallworldness. Smallworldness for naive (a) and modified conventional (b) scale-free and small world networks show vertex–edge ratio dependence.

world networks are the most affected by delay in naively constructed networks but are less affected than scale-free networks in the modified constructed networks. However, the apparent contradiction is resolved by examining the smallworldness of the naively constructed networks. Naively constructed scale-free networks have a higher smallworldness value than the naively constructed small-world networks (Fig. 5(a)). Even in the modified conventional construction method the smallworldness of the scale-free network is higher for networks with more than 720 edges (Fig. 5(b)).

Because the modified conventional method for constructing small-world networks requires a lattice to be constructed and then rewired to make it a small-world network, a network with n vertices and q edges cannot be assigned an arbitrary neighborhood size. In order to account for this neighborhood size was increased proportional to the number of edges. This had the effect of increasing the smallworldness of the small-world networks generated using this method.

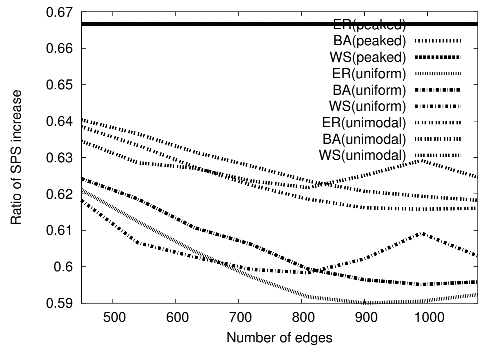


Figure 6. Increase in SPS in delayed networks with different delay distributions constructed by the MC method. The increase ratio in SPS as the initial number of edges increase is shown. Each network contained 45 nodes. Increase of SPS in WS is smaller than that of BA at low node-to-edge ratio (number of edges below 650) and larger than that of BA at high node-to-edge ratio. Note that here, for WS, neighborhood size was increased proportional to the number of edges.

After making this modification to the modified conventional small-world constructor the networks were tested again with peaked, uniform, and unimodal delay distributions. Fig. 6 demonstrates that sparse networks with high smallworldness are less influenced by delay than scale-free networks while dense small-world networks with high smallworldness are more influenced by delay than scale-free networks.

5. Discussion

Previous studies have represented networks as graphs, but prior to this study we are unaware of any systematic study of delay in networks from a graph perspective. In this study we proposed a strategy for comparing diverse network topologies under conditions of delay using temporal augmentation and demonstrated how this comparison strategy could be used to compare multiple network types with different constructions.

Through experiments, the role of the number of hubs in scale-free networks and that of the neighborhood size in small-world networks became apparent. Also, it was found that smallworldness and neighborhood size of non-augmented network affected the effects of delay. Therefore, we can say that network structure also influences the effects of delay.

The comparison strategy revealed an inconsistency in the naively constructed and modified conventional constructed small-world network. This was an unexpected result. Further investigation on the smallworldness of the networks revealed that the naively constructed scale-free network actually has a higher smallworldness property than the naively constructed small-world network. We can also speculate that there may exist a proper neighborhood size for a certain node-to-edge ratio that reduces the shortest path sum the most.

Finally, all networks were the least affected by uniformly distributed delay, but not all network classes were equally affected. This demonstrates that network structure does influence how delay will affect the network's function.

6. Conclusion

Delay is an important issue in the analysis of network function. By examining the effects of delay on different types of networks we can better understand why the natural networks like the brain have the topology that it has. A small-world topology is more robust to the effects of delay than a scale-free network and could have evolved because of this desirable property. That is, sparse networks with small-world structure are robust to delay and neighborhood size of small-world network was found to be a critical parameter. Based on this study, we were able to extend our understanding of network structures and delay, but we are still far from a complete understanding.

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References

[1] F. M. Atay, J. Jost, and A. Wende. Delays, connection topology, and synchronization of coupled chaotic maps. *Physical Review Letters*, 92:144101.1–144101.4, 2004.

[2] A. Barabasi. *Linked: The new science of networks*. Perseus, Cambridge, Massachusetts, 2002.

[3] A. Barabasi and R. Albert. Emergency of scaling in random networks. *Science*, 286:509–512, 1999.

[4] A. Barabasi and E. Bonabeau. Scale-free networks. *Scientific American*, 288:50–59, 2003.

[5] D. S. Bassett and E. Bullmore. Small-world brain networks. *Neuroscientist*, 10(6):512–523, 2006.

[6] Y. Choe, W. Koh, and B. H. McCormick. Network connectivity analysis on the temporally augmented C. elegans web: a pilot study. *Society of Neuroscience Abstracts*, 30:921.9, 2004.

[7] L. Costa, R. A. Rodrigues, G. Travieso, and P. R. V. Boas. Characterization of complex network: A survey of measurements. *Arxiv preprint cond-mat/0505185*, June 2005.

[8] E. M. Daly and M. Haahr. Social network analysis for routing in disconnected delay-tolerant MANETs. In *Proceedings of the 8th ACM international symposium on Mobile ad hoc networking and computing*, pages 32–40, 2007.

[9] P. Erdös and A. Rényi. On the evolution of random graphs. *Publ. Math. Inst. Hung. Acad. Sci*, 5:17–61, 1960.

[10] C. W. Eurich, A. Thiel, and L. Fahse. Distributed delays stabilize ecological feedback systems. *Physical Review Letters*, 94:158104.1–158104.4, 2005.

[11] M. D. Humphries, K. Gurney, and T. J. Prescott. The brainstem reticular formation is a small-world, not scale-free, network. *Royal Society of London. Biological Sciences*, 273:503–511, 2006.

[12] B. S. Jang. Effect of varying the delay distribution in different classes of networks: Random, scale-free, and small-world. Master's thesis, Department of Computer Science, Texas A&M University, 2007.

[13] M. Kaiser and C. C. Hilgetag. Nonoptimal component placement, but short processing paths,due to long-distance projection in neural system. *Public Library of Science Computational Biology*, 2:e95, 2006. [Online]. Available: http://dx.doi.org/10.1371/journal.pcbi.0020095.

[14] C. M. Marcus and R. M. Westervelt. Stability of analog neural networks with delay. *Physical Review A*, 39:347–359, 1989.

[15] C. Masoller and A. C. Marti. Random delay and the synchronization of chaotic maps. *Physical Review Letters*, 94:134102.1–134102.4, 2004.

[16] O. Sporns and C. J. Honey. Small worlds inside big brains. *Proceedings of the National Academy of Science USA*, 103:19219–10220, 2006.

[17] O. Sporns, G. Tononi, and G. M. Edelman. Connectivity and complexity: the relationship between neuroanatomy and brain dynamics. *Neural Networks*, 13:909–922, 2000.

[18] O. Sporns and J. D. Zwi. The small world of the cerebral cortex. *Neuroinformatics*, 2:145–162, 2004.

[19] A. Thiel, C. W. Eurich, and H. Schwegler. Stabilized dynamics in physiological and neural systems despite strongly delayed feedback. *International Conference of Artificial Neural Network*, 2415:15–20, 2002.

[20] A. Thiel, H. Schwegler, and C. W. Eurich. Complex dynamics is abolished in delayed recurrent systems with distributed feedback times. *Complexity*, 8:102–108, 2003.

[21] P. A. Tsonis and A. A. Tsonis. A 'small-world' network hypothesis for memory and dreams. *Perspective in Biology and Medicine*, 47:176–180, 2004.

[22] D. J. Watts and S. H. Strogatz. Collective dynamics of 'small-world' networks. *Nature*, 393:409–410, 1998.