An Approach to Location Tracking of Mobile Sensors based on Distributed Randomized Multisets

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***Abstract*— The tracking of location information in mobile ad hoc sensor networks is a well-studied problem for which many solutions have been proposed. The approach discussed in this paper is based on the systematic use of a distributed shared data structure. Specifically, randomized versions of replicated sets and multisets are introduced. It is shown that this probabilistic approach can improve the performance of infomation disemination applications in mobile ad hoc sensor networks. The extra layer of abstraction greatly simplifies the programming, since the library implementing the randomized sets and multisets can be re-used for other applications. A literate program is provided that allows the reader to experiment with our randomized sets and multisets.**

I. Introduction

The location information of mobile sensors is essential for many applications and usually a part of the name resolution in routing protocols. One of the obstacles is that the location information of a moving sensor might not be available at all times. To remedy this fact, it is advantageous to keep at least slightly outdated location values that allow the routing protocol to make an informed guess about the current location of the sensor. In this paper, we discuss an approach to implement such a location tracking scheme that is based on a distributed data structure that implements a randomized multiset.

The location information is replicated, and we use a quorum based scheme to access this information. In order to reduce the number of messages, we employ a *randomized* quorum scheme. Based on this randomized quorum scheme, we describe the implementation of randomized sets and multisets. These randomized versions approximate the behavior of their deterministic counter parts, but some care is required in choosing the parameters for an application. Therefore, we discuss and analyze the behavior of multiset operations in detail.

The implementation of such distributed algorithms is a nontrivial task. The abstract data structures that we introduce here provide an extra layer of abstraction that greatly simplifies the implementation of this location tracking application and other information dissemination applications, since the library implementing the data type operations hides much of the details, and one can focus on the implementation of the application, rather than on low-level issues.

Recall that a *multiset* is an unordered collection of elements, in which an element can occur multiple times; whereas a *set* is an unordered collection of distinct elements. Many algorithms use these basic data structures. In fact, they are so common that many programming languages support sets and multisets either directly or through standard libraries (for example, the STL of C++). We provide here distributed replicated versions of set and multisets. For the purpose of information dissemination in mobile sensor networks, it is often sufficient to provide the correct information (such as the most recent location) with a high probability. In this particular application one can tolerate to receive occasionally a slightly outdated location of a sensor. The benefit is that the overall number of messages is dramatically reduced by employing a randomized approach.

It might be helpful to explain the concept behind a replicated multiset by way of a (somewhat playful) example. Assume that we have a set of n servers (the term 'server' is used in a liberal sense; it might be a beacon or an arbitrary mobile sensor node). We number the servers, for simplicity, from 1 to n. We represent a set (or multiset) M by replicas M_r on server r for $r \in \{1, \ldots, n\}$. To give an idea how this is traditionally done, let's have a look at a small example with five servers:

Server 1 : $M_1 =$ {⚀, ⚅},
Server 2 : $M_2 =$ {⚁, ⚅, ⚇},
Server 3 : $M_3 =$ {⚀, ⚇},
Server 4 : $M_4 =$ {⚁, ⚅, ⚇},
Server 5 : $M_5 =$ {⚀, ⚁}.

The five replicas represent the set

$$M = \bigcup_{r=1}^{5} M_r = \{⚀, ⚁, ⚅, ⚇\}.$$

You will notice that each element of M is contained in exactly three replicas. This means that whenever a client process requests the replicas from 3 servers, then the client can reconstruct M by taking the union of the three sets. This scheme is based on a traditional quorum system where each client process is supposed to access $k > n/2$ replica servers to read a set or to write an element.

The drawback of this scheme is that it is fairly rigid and it might even be undesirable for a client process to contact

such a large number of servers. Imagine for instance a mobile sensor network, where some servers are not available at all times. A large quorum Q of $k > n/2$ replica servers is likely to contain some servers that are not reachable, and this can be a considerable bottleneck.

Let us digress a little bit from our discussion of replicated multisets and discuss the concept of a probabilistic quorum, which was introduced by Malkhi, Reiter, Wool, and Wright [8] as a means to replace traditional quorum systems. In this case, a client selects a quorum of, say, size $k = \Omega(n^{1/2+\varepsilon})$. Unfortunately, one may find two quorums that are disjoint. However, if the clients choose the two quorums uniformly at random, then the two quorums are likely to share at least one server thanks to the birthday paradox. A concrete numerical example might help the reader to appreciate this property. Suppose that we have $n = 50$ servers. A traditional quorum system chooses quorums of size $k \geq 26$. However, if we assume that the clients select merely $k = 16$ servers per quorum, then any two quorums that are chosen uniformly at random will be disjoint with probability less than $\varepsilon < 0.000448$. So it is unlikely that two quorums do not overlap, but at the same time the overhead incurred by querying the servers is significantly reduced.

Coming back to our discussion of distributed multisets, it now seems natural to explore the possibility to build *randomized* set and multiset operations using probabilistic quorum constructions. An advantage of a smaller quorum size is a significantly reduced message complexity – a client has to communicate with considerably fewer servers to execute the operations. On the downside, all operations on randomized sets or multisets involve a certain (controllable) amount of error ε. The situation is comparable with floating point or fixed-point arithmetic where the operations also introduce inaccuracies.

In the section on *Location Tracking*, we describe an application of randomized multisets to the location tracking problem of sensor nodes, since this provides the motivation for the later sections. We introduce in the section on *Read and Write Operations* the read and write operations for randomized sets and multisets and analyze the behavior of these operations. We derive in the section *Further Operations* other set and multiset operations that are mostly based on these primitives. We find that the price that we have to pay for the significantly reduced quorum sizes is a small loss of values in most primitive operations.

Notations. If n is a positive integer, then we denote by $[n]$ the set $\{1, 2, \ldots, n\}$. The family of all k-subsets of the set $[n]$ is denoted by $\binom{[n]}{k}$.

II. Location Tracking

In this section, we illustrate how to use our randomized multisets for location tracking in mobile ad hoc sensor networks. Recall that the location tracking problem in a mobile ad hoc sensor network is to provide location information of mobile sensors as part of a name resolution protocol. Such a location service is important because a simple name resolution protocol does not work in a mobile environment without static infrastructure.

Lee, Welch and Vaidya [5] proposed a scheme based on replicated location tracking servers. Each server has a replica of the location database containing the most recent location information of the mobile nodes. Each mobile node is responsible to periodically update its location. Tseng et al. proposed a location tracking scheme using mobile agents [11]. A quadtree structured location service is used in the Grid system [7].

Here we sketch a location tracking scheme using our randomized multisets in implementing the location information database M. *The basic idea of the new scheme is that we keep a "history" of movement of mobile sensors rather than the single most recent location information.*

The reason why this scheme works well is because movement is continuous and it is always better to have at least some slightly outdated location information than to have no location information at all. Consequently, the errors due to our randomized multiset have less impact in a system keeping the location history.

Each element of M is a tuple

$$\langle \textit{location data}, \textit{sensor id}, \textit{timestamp} \rangle.$$

Each mobile sensor is uniquely identified by the *sensor id*, and associated with a client process. Furthermore, each client process has a local timestamp which is incremented by one for each `add` operation it performs.

a) • Add Location.: Whenever a sensor i wants to record its new location information loc, it increments its timestamp t by one and invokes `add(x,`M`)`, where the element `x` is of the form $\langle loc, i, t \rangle$. The implementation of the `add` operation is given in the section on *Read and Write Operations.*

Each replica server r keeps a replica M_r of M. When a replica server r receives $\langle$`add x`$\rangle$ message, it adds `x` to M_r. There is a system parameter *expire* that serves as a threshold of how long the old information should be kept. For example, if *expire* = 5, then each server will keep at most the five most recent location entries for each sensor. Thus, the server will check after each `add` operation whether it should remove old location data of the sensor.

b) • Lookup.: When a mobile sensor i wants to know the location of mobile sensor j, it performs `lookup(`j`,`M`)`. As a result of the `lookup` operation, sensor i will receive a list of location data of sensor j.

To implement the `lookup` operation, the client process does the following. It chooses a random quorum Q of replicas and sends the message $\langle$`lookup` $j\rangle$ to each replica server r in Q. The servers reply with lists of the location data of j, which the client merges and sorts according to increasing timestamps. Then it returns the sorted list of location data of j to the application.

When a server r receives $\langle$`lookup` $j\rangle$ message from client i, it creates a list (that is ordered by timestamp values) of j's location data in M_r and sends the list back to i. If no such data exists in M_r, it sends i the list with a single value $\perp$, indicating that it does not have location information on j.

III. Read and Write Operations

We describe in this section the most fundamental operations to create, write, and read a randomized set or multiset. We analyze several key properties of these operations; in particular, we determine the expected size of the set returned by the read operation. The first operation concerns the genesis of sets:

c) • *Create.:* We can create a set or multiset M by

$$\texttt{create}(M).$$

This operation creates the replica $M_i = \emptyset$ on each server i in the range $1 \le i \le n$.

The read and write operations have the following behavior:

d) • *Add.:* The write operation

$$\texttt{add}(\texttt{x}, M)$$

chooses uniformly at random a k-subset W of $[n] := \{1, \ldots, n\}$ and adds the element x to the replicas M_w for all w in W. In other words, the operation has the effect

$$M_w := M_w \cup \{x\} \quad \text{for all } w \text{ in } W.$$

So the multiplicity of x is increased by one in the case of multisets; and the replica contains x in the case of sets.

e) • *Read.:* Similarly, the read operation

$$\texttt{read}(M)$$

chooses uniformly at random a k-subset R of $[n]$ and returns the union of the replicas M_r with r in R; in other words, the read operation returns the set $\bigcup_{r\in R} M_r$.

The replicas M_r with $1 \le r \le n$ represent a set M if and only if $M = \bigcup_{r\in[n]} M_r$.

Lemma 1: Suppose that we choose two quorums R and W in $\binom{[n]}{k}$ uniformly at random. Then

$$\Pr[R \cap W = \emptyset] = \binom{n-k}{k} \Big/ \binom{n}{k}.$$

In the terminology of Malkhi, Reiter, Wool and Wright [8], our operations are based on a probabilistic ε-intersecting quorum system $\mathcal{Q}_k = \binom{[n]}{k}$ of all k-subsets of the set $[n] = \{1, \ldots, n\}$ with a uniform access strategy. The previous lemma simply records the fact that two quorums in $\mathcal{Q}_k$ intersect with probability $1 - \varepsilon$, where $\varepsilon = \binom{n-k}{k}\binom{n}{k}^{-1}$. If we are conservative and choose $k > n/2$, then we recover a traditional, deterministic quorum system in which any two quorums intersect. An advantage of probabilistic quorum systems is that one can choose considerably smaller quorum sizes [8]. In our application, the benefit is a considerably reduced message complexity to access the distributed data structures. This leads to considerable savings of energy of the mobile sensors.

If we choose quorums of size $k \le \lfloor n/2 \rfloor$, then a read quorum can fail to intersect with a previous write quorum, and thus a set returned by the read operation might have smaller cardinality than it should have. Fortunately, it is possible to choose the quorum size k such that the probability of such an undesired event becomes negligible.

Let us first investigate the consequences of a particular quorum size selection before we make recommendations on the choice of k. Clearly, a crucial figure is the expected size of the set returned by a read operation.

Proposition 2: Suppose that the read and write quorums use the quorum system $\mathcal{Q}_k = \binom{[n]}{k}$ with uniform access probability. If the replica sets represent a set M of cardinality $m = |M|$, then the expected value of the size X of the set returned by read(M) is at least

$$\mathrm{E}[X] \ge m(1-\varepsilon) \quad \text{with} \quad \varepsilon = \binom{n-k}{k} \Big/ \binom{n}{k}.$$

Equality holds if and only if each element of M is represented by exactly k replicas.

Proof: Denote by M_r the replica at server r, where $1 \le r \le n$. It is clear from the definition of a write operation that an element x of M is contained in at least k replicas. It follows that the set $W_x = \{\, r \,|\, 1 \le r \le n, x \in M_r\}$ has cardinality $s_x = |W_x| \ge k$. Let R denote a read quorum of size k and denote by Y_x the indicator random variable for the event $W_x \cap R \ne \emptyset$. Then the probability $\Pr[Y_x = 1] = \Pr[W_x \cap R \ne \emptyset]$ is given by

$$\Pr[Y_x = 1] = 1 - \binom{n-s_x}{k} \Big/ \binom{n}{k} \ge 1 - \varepsilon. \tag{1}$$

We have $X = \sum_{x\in M} Y_x$. By linearity of expectation, we obtain

$$\mathrm{E}[X] = \mathrm{E}\Big[\sum_{x\in M} Y_x\Big] = \sum_{x\in M} \Pr[Y_x = 1] \ge m(1-\varepsilon),$$

which proves our claim. ∎

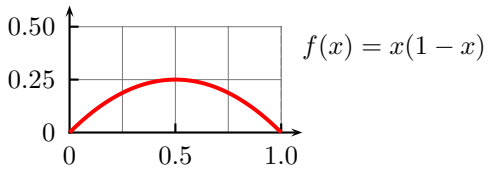
We would like to choose the quorum size k such that $\mathrm{E}[X]$ is close to the cardinality $|M|$ and such that it is unlikely that a particular read operation returns a set of size X that deviates much from the expected value $\mathrm{E}[X]$. Let us first derive a bound on $\mathrm{var}[X]$.

Lemma 3: We keep the notation of the previous proposition. If the quorum size k is chosen[1] such that $\varepsilon \le 1/2$, then $\mathrm{var}[X] \le m\varepsilon(1-\varepsilon)$.

Proof: Recall that X is the sum of independent random variables Y_k; therefore, the variance sum theorem yields

$$\mathrm{var}[X] = \mathrm{var}[Y_1] + \mathrm{var}[Y_2] + \cdots + \mathrm{var}[Y_m]. \tag{2}$$

The random variable Y_k is a Bernoulli random variable with probability of success given in (1). Recall that the function $x(1-x)$ is monotonically decreasing on the interval $[1/2, 1]$, as is illustrated by the function plot:



[1] Actually, we will of course choose k such that $\varepsilon \ll 1/2$.

Let $p_k := \Pr[Y_k = 1]$. Then $p_k \geq 1-\varepsilon \geq 1/2$, we obtain that the variance $\mathrm{var}[Y_k] = p_k(1-p_k)$ is bounded from above by

$$\mathrm{var}[Y_k] = f(p_k) \leq f(1-\varepsilon) = \varepsilon(1-\varepsilon).$$

Therefore, we can conclude from equation (2) that $\mathrm{var}[X] \leq m\varepsilon(1-\varepsilon)$, as claimed. ∎

f) Tail Estimates.: We have shown that the size X of a set returned by the read operation `read(M)` has expectation value $\mathrm{E}[X] \geq m(1-\varepsilon)$. We can select the quorum size k such that ε becomes as small as we please, as we will show in the next paragraph. We will demonstrate now that it is unlikely that the value X deviates much from $\mathrm{E}[X]$.

Lemma 4: If X denotes the size of a set returned by `read(M)` and m is the cardinality of M, then

$$\Pr\left[\,|X - \mathrm{E}[X]| \geq \varepsilon^{1/2}\,\mathrm{E}[X]\,\right] \leq \frac{1}{(1-\varepsilon)m}.$$

Proof: Recall that Chebychev's inequality states that the probability $\Pr[\,|X-\mathrm{E}[X]| \geq \lambda]$ is at most $\mathrm{var}[X]/\lambda^2$. Thus, if we set $\lambda = \varepsilon^{1/2}\,\mathrm{E}[X]$, then we obtain the claim with the help of Proposition 2 and Lemma 3. ∎

Lemma 5: If X denotes the size of a set returned by `read(M)` and the cardinality of M is m, then

$$\Pr\left[\,|X - \mathrm{E}[X]| > d\right] < 2e^{-2d^2/m}.$$

In particular, if we set $d = \varepsilon^{1/2}\,\mathrm{E}[X]$, then

$$\Pr\left[\,|X - \mathrm{E}[X]| > \varepsilon^{1/2}\,\mathrm{E}[X]\right] < 2e^{-2\varepsilon(1-\varepsilon)^2 m}.$$

Proof: Recall that the random variable X is given by the sum of indicator random variables $X = \sum_{x\in M} Y_x$. Define new random variables $Z_x := Y_x - p_x$ with $p_x = \Pr[Y_x = 1]$ for all $x \in M$. We have $\Pr[Z_x = 1-p_x] = p_x$ and $\Pr[Z_x = -p_x] = 1-p_x$. The main point of this definition is that the sum of the random variables Z_x is given by $X - \mathrm{E}[X] = \sum_{x\in M} Z_x$, and we can estimate the tails of the right hand side by

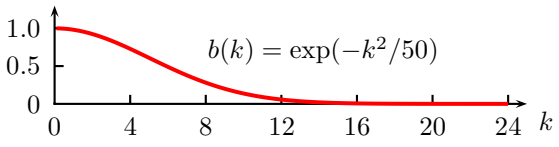
$$\Pr[|X-\mathrm{E}[X]| > d] = \Pr[|\sum_{x\in M} Z_x| > d] \leq 2e^{-2d^2/m},$$

where the last inequality is a bound of Chernoff, see [2] or [1, Lemma A.4]. ∎

g) Quorum Size.: Suppose that we have n servers and select a quorum of size $k < n/2$, then we can estimate

$$\varepsilon = \binom{n-k}{k}\binom{n}{k}^{-1} \leq e^{-k^2/n}, \tag{3}$$

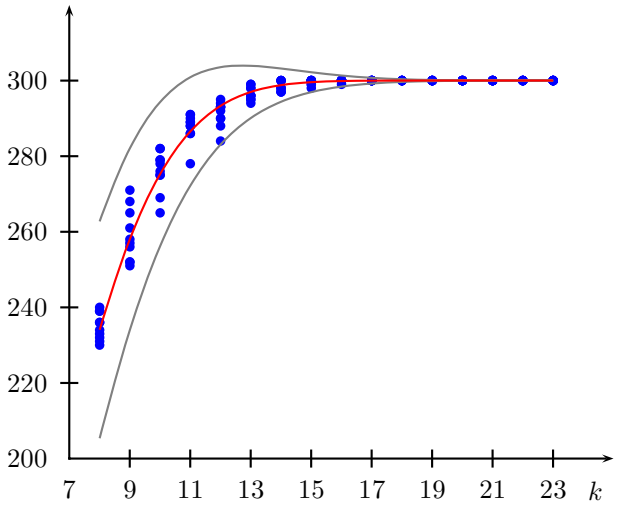
see, for instance, Jukna [3, p. 21]. Therefore, the probability $\varepsilon = \varepsilon(k)$ that two quorums of size k fail to intersect *decreases rapidly*, especially for quorum sizes larger than $n^{1/2}$. The next figure illustrates the bound for $n = 50$ servers and a range of quorum sizes:



Example 1: If we choose each quorum to be of size $k = 2\sqrt{n}$, where n denotes the number of servers, then the probability ε that two quorums do not intersect is bounded by $\varepsilon \leq 1/e^4 \approx 0.018$ by equation (3). If we increase the quorum size to $k = 3\sqrt{n}$, then $\varepsilon \leq 1/e^9 < 0.00013$.

Recall that the size X of a set returned by a read operation `read(M)` is the sum of m independent random variables $X = Y_1 + \cdots + Y_m$ provided that the set M has cardinality m. If m is large, then the central limit theorem tells us that we can approximate X by the normal distribution $N(\mathrm{E}[X], \mathrm{var}[X])$, see [6]. Thus, as a rule of thumb, the value X will lie within the range $m(1-\varepsilon) \pm 4\sqrt{m\varepsilon(1-\varepsilon)}$ about 99.99% of the time.

We bolster this claim by giving some experimental results of a randomized set with $m = 300$ elements which is realized with $n = 50$ replica servers. We plot the quorum size k in the range $8 \leq k \leq 23$ against the number of elements that have been returned by a read operation. The solid curve in the middle shows the expected number of elements $300(1-\varepsilon)$, with $\varepsilon = \binom{50-k}{k}/\binom{50}{k}$, that are returned by a read when the quorum size is k. The gray curves show $300(1-\varepsilon) \pm 4\sqrt{300\varepsilon(1-\varepsilon)}$. The dots represent the measured number of elements. For each quorum size k we have repeated the experiment 10 times.



The graph illustrates that for a quorum size of $k \geq 17$ all 300 elements were returned in each run. Moreover, it illustrates that all experimental results are within the region that we predicted with our theoretical considerations.

IV. Further Operations

We describe in this section further operations on randomized sets, most of which are based on the primitives that we have introduced and analyzed in the previous section.

h) • Cardinality.: We can obtain an estimate for the cardinality of the set M by

$$\mathrm{X} := \texttt{size(read(}M\texttt{))}.$$

Due to the probabilistic nature of the read operation, we cannot expect to get a precise answer and need to be contend with an estimate. It follows from Proposition 2 and Lemma 3 that

$X \approx m$ assuming that the quorum size k is chosen such that $\varepsilon \ll 1$.

i) • *Union.:* We can obtain a union of two sets (or multisets) B and C by

$$\texttt{read(B) union read}(C). \tag{4}$$

We expect that `read(`B`)` and `read(`C`)` will return at least $(1-\varepsilon)|B|$ and $(1-\varepsilon)|C|$ elements, respectively. Therefore, we can expect that the operation (4) yields at least $|B \cup C|(1-\varepsilon)$ elements. Furthermore, the result of (4) is contained in $B \cup C$.

j) • *Intersection.:* The intersection

$$\texttt{read(B) intersection read}(C) \tag{5}$$

of two randomized sets (or multisets) has an expected value of at least $(1-\varepsilon)|B \cap C|$ elements, because the operation `read(`B`)` is expected to omit at most $\varepsilon|B|$ elements. And the result of (5) is contained in $B \cap C$, so one can boost the probability of success by several repetitions.

k) • *Difference.:* The most subtle behavior has the operation

$$\texttt{read(B) minus read}(C).$$

The expected number of elements returned by this operation is between $(1-\varepsilon)|B \backslash C|$ and $(1+\varepsilon)|B \backslash C|$; the potential increase of elements is a result of the fact that `read(`C`)` might return too few elements. We suggest to use a larger read quorum for `read(`C`)` to remedy this effect.

l) • *Containment.:* The operation

$$\texttt{x in } M \tag{6}$$

is realized by sending the request $\langle$is x in M?$\rangle$ to a quorum of servers that is selected uniformly at random from $\binom{[n]}{k}$. If at least one server replies with `true` then the result is `true`, otherwise it is `false`.

The answer has one-sided error. If `x` is not an element of M, then (6) will always correctly return `false`. If `x` is an element of M, then we get the incorrect answer `false` with probability $\leq \varepsilon$.

V. Conclusions

We proposed a location tracking scheme for mobile ad hoc sensor networks that utilizes distributed shared data structures such as randomized sets and multisets. We designed randomized set and multiset data structures and analyzed their basic operations.

In our location tracking application, we took into account that a slightly outdated location information can be still valuable information. Suppose that the last ℓ positions that have been disseminated by a sensor give a tolerable approximation to its location. If an ε-intersecting quorum system is used (meaning that the probability that a read and a write quorum fail to intersect is ε), then the probability that a location query will yield none of the last ℓ location values is given by ε^ℓ.

For example, if we choose a quorum size of $k = 2\sqrt{n}$, then $\varepsilon \leq 1/e^4 \approx 0.018$. If we assume that the last $\ell = 5$ locations of the sensor give a meaningful approximation to the current location, then the probability that we do not get any of the $\ell = 5$ most recent locations in a location query is at most 2.062×10^{-9}.

As a companion to this paper, we provide a generic C++ implementation that illustrates the fundamental principles of randomized sets. Our program is specified in Knuth's *literate programming* style that is easy to read [4]. The documentation contains a self-contained explanation of all implementation details. The reader can experiment with our program to explore the practical consequences of quorum size choices and other details.

Randomization can lower the message complexity and increase the energy efficiency of mobile sensor networks. Other applications in mobile sensor networks, such as data aggregation and information dissemination, can benefit from such properties as well. As another type of application, we note that it is possible to modify Rabin's Byzantine agreement algorithm [10], [9] to take advantage of randomized multisets.

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Appendix

As a companion to this paper, we have written routines in C++ that illustrate the randomized multiset operations. We have chosen a literate programming style, so that the documentation can be easily appreciated. We provide a small excerpt

from the documentation of our program, which illustrates the use of some elementary operations. An interested reader can download the literate program (from which the LaTeX documentation and the C++ library and sample files can be extracted) from the authors' home pages.

A small excerpt from the literate program
[*the preceding 15 pages are omitted*]

We have restricted ourselves so far to the creation of a randomized set and subsequent read operations. Using these primitives, we can define probabilistic versions of set operations such as union, intersection, and set difference in the following way:

```
A.read() + B.read(),
A.read() * B.read(),
A.read() - B.read().
```

The next program illustrates the use of these operations.

⟨⟨textE.cc⟩⟩ ≡

```
#include <iostream>
#include <set>
#include <vector>
#include "rset.h"
using namespace std;

int main () {
  int n = 6;
  int k = 3;
  rset<int> A(n,k);
  rset<int> B(n,k);
  A.insert(1);
  A.insert(2);
  A.insert(3);

  B.insert(3);
  B.insert(4);
  B.insert(5);

  cout << "A = {1,2,3} " << endl;
  cout << "B = {3,4,5} " << endl;

  cout << "A union B = {1,2,3,4,5}" << endl;
  rset<int> C = A + B;
  cout << C;

  cout << "A intersection B = {3}" << endl;
  rset<int> D = A * B;
  cout << D;

  cout << "A minus B = {1,2}" << endl;
  rset<int> E = A - B;
  cout << E;
}
```

The reader can easily develop a good intuition about the randomized multiset operations by experimenting with our generic C++ library.