Challenge Problem 2
CPSC 489/689 Quantum Algorithms
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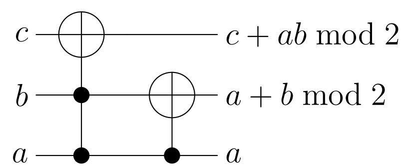
The following quantum circuit represents a half-adder; it calculates the sum $a + b \bmod 2$, and the carry ab of the inputs a and b:



The circuit implements a unitary matrix U_{add}, which is determined by

$$\begin{array}{rclcrcl} U_{add}|000\rangle &=& |000\rangle, & & U_{add}|100\rangle &=& |100\rangle, \\ U_{add}|001\rangle &=& |011\rangle, & & U_{add}|101\rangle &=& |111\rangle, \\ U_{add}|010\rangle &=& |010\rangle, & & U_{add}|110\rangle &=& |110\rangle, \\ U_{add}|011\rangle &=& |101\rangle, & & U_{add}|111\rangle &=& |001\rangle. \end{array}$$

Let $m(U)$ denote the minimal number of controlled-not and single qubit gates, which are needed to realize $U \in \mathcal{U}(2^n)$. The challenge is to determine $m(U_{add})$. In other words, how many controlled-not gates and single qubits gates are needed in an optimal implementation of U_{add}? You need to prove your result.

Remark. Let T denote the unitary matrix corresponding to the Toffoli gate. Notice that $|m(T) - m(U_{add})| \leq 1$.

I offer a **Challenges in Quantum Computing Award**, worth US$ 100, for the first correct solution to this problem.

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