Problem Set 1
CPSC 411 Analysis of Algorithms
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The assignment is due next Thursday, Sep 10, 2009, before class.

The limit and limit superior criteria give in Lecture 2 for determining whether 
\( f(n) = O(g(n)) \) can be convenient in some of the subsequent exercises.

Exercise 1. Suppose that 
\[ p(n) = a_0 + a_1 n + \cdots + a_m n^m \]
is a polynomial of degree \( m \) with complex coefficients. Show that 
\[ p(n) = O(n^k) \]
for all \( k \geq m \), and 
\[ p(n) \neq O(n^\ell) \]
for \( 0 \leq \ell < m \).

Exercise 2. Show that for fixed \( k \), we have
\[ \binom{n}{k} = \frac{n^k}{k!} + O(n^{k-1}) \]
and
\[ \binom{n+k}{k} = \frac{n^k}{k!} + O(n^{k-1}) , \]
where \( \binom{n}{k} = n(n-1)(n-2)\cdots(n-k+1)/k! \) is the binomial coefficient.

Exercise 3. Show that 
\[ n/(n+1) = 1 + O(1/n) . \]

Exercise 4. Prove or disprove: 
\[ e^n = O(2^n) . \]

Remark. Let \( n_0 \) and \( n_1 \) be positive integers with \( n_1 > n_0 \). If 
\[ L(n_0) \geq R(n_0) \]
holds, and the numbers \( L(n) \), \( R(n) \), \( L(n+1) \), and \( R(n+1) \) are positive and
\[ \frac{L(n+1)}{L(n)} \geq \frac{R(n+1)}{R(n)} \]
holds for all \( n \) in the range \( n_0 \leq n < n_1 \), then it follows that 
\( L(n) \geq R(n) \)
holds for all \( n \) in the range \( n_0 \leq n \leq n_1 \).

Exercise 5. (a) Prove by induction on \( k \) that
\[ \left( 1 + \frac{1}{n} \right)^k < 1 + \frac{k}{n} + \frac{k^2}{n^2} , \]
holds for all \( k \) in \{1, 2, \ldots, n\}.
(b) Prove by induction that \( n! \geq (n/3)^n \) holds for all \( n \geq 1 \). 
[Hint: Use part (a) of this exercise and the above remark].
(c) Deduce that \( \log(n!) = \Omega(n \log n) \) holds. [This is an alternative way to prove the result that we have shown in the lecture.]

Exercise 6 (CLRS). Exercise 2.3-3, page 36