Counting

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Counting

k = 0;
for(int i=1; i<=m; i++) {
    for(int j=1; j<=n; j++) {
        k = k+1;
    }
}
// What is the value of k? Answer: k=mn
Product Rule

Suppose that a procedure can be broken down into a sequence of two tasks.

If there are $m$ ways to do the first task and for each of these ways there are $n$ ways to do the second task, then there are $mn$ ways to do the procedure.
How many bit strings of length seven are there?

Answer: Each of the seven bits can be chosen in 2 ways. Therefore, there is a total of $2^7 = 128$ different bit strings of length 7 by the product rule.
Number of Functions

Suppose that $f: A \to B$ is a function from a set $A$ with $m$ elements to a set $B$ with $n$ elements.

How many such functions exist?

Answer: For each argument, we can choose one of the $n$ elements of the codomain. Therefore, by the product rule, we have $n \times n \times \ldots \times n = n^m$ functions.
Recall that a function $f$ is injective if and only if its function values $f(a)$ and $f(b)$ are different whenever the arguments $a$ and $b$ are different. Thus, all function values of an injective function are different.

How many injective functions are there from a set $A$ with $m$ elements to a set $B$ with $n$ elements?

<Try to figure it out before we answer it on the next slide>
Case $|A|>|B|$. There are no injective functions from $A$ to $B$, as one cannot choose all function values to be different.

Case $|A|\leq|B|$. Suppose that $A = \{a_1, a_2, ..., a_m\}$ and $n=|B|$.

The value of $a_1$ can be chosen in $n$ different ways.

The value of $a_2$ can be chosen in $n-1$ different ways, as the value $f(a_1)$ cannot be used again.

In general, after values of $\{a_1, ..., a_{k-1}\}$ have been chosen, the value of $a_k$ can be chosen in just $n-(n-1) = n-k+1$ ways.

Thus, by the product rule there are $n(n-1)...(n-m+1)$ injective functions from $A$ to $B$. 
Suppose that $S$ is a disjoint union of two finite sets $A$ and $B$. Recall that $|S|$ denotes the size (cardinality) of the set.

The **sum rule** says that $|S| = |A| + |B|$. 
Counting

\[ k = 0; \]
\[ \text{for(int } i=1; i<=m; i++) \]
\[ \quad k = k+1; \]
\[ \text{// First loop completed} \]
\[ \text{for(int } j=1; j<=n; j++) \]
\[ \quad k = k+1; \]
\[ \text{// What is the value of } k? \quad k=m+n \]
Counting Principles

Counting is based on simple rules such as the sum and product rules. By themselves, they are trivial. These rules are typically used in a combination.

The hardest part is to figure out a good strategy to count elements.
Example

Each user on a computer system has a password which is 6-8 characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Let $P$ be the number of passwords, $P_k$ the number of passwords of length $k$. Hence, $P = P_6 + P_7 + P_8$.

Now, $P_k = 36^k - 26^k$

= number of strings of length $k$ with digits or uppercase letters - number of strings that just contain letters.

Hence, $P = P_6 + P_7 + P_8 = 36^6 - 26^6 + 36^7 - 26^7 + 36^8 - 26^8$
Inclusion-Exclusion Formula

Let $A$ and $B$ be finite sets.

$$|A \cup B| = |A| + |B| - |A \cap B|$$
How many bit strings of length 8 either start with 1 or end with 00?

Let $A$ be the set of bit strings of length 8 that start with 1.

Then $|A| = 2^7$.

Let $B$ be the set of bit strings of length 8 that end with 00.

Then $|B| = 2^6$.

The $|A \cap B| = |\text{bit strings of length 8 starting with 1 and ending with 00}| = 2^5$.

Therefore, $|A \cup B| = |A| + |B| - |A \cap B| = 2^7 + 2^6 - 2^5$
Pigeonhole Principle

If $k$ is a positive integer and $k+1$ or more objects are placed into $k$ boxes, then there is at least one box containing two or more objects.
Pigeonhole Principle: Example 1

Among any 367 people, there must be at least two with the same birthday.

[There are at most 366 possible birthdays]
Pigeonhole Principle: Example 2

For every positive integer $n$ there is a multiple of $n$ containing only 0s and 1s in its decimal expansion.

Proof: Consider the $n+1$ integers

$1, 11, ..., 11...1$ (with $n+1$ ones)

There are $n$ possible remainders when these integers are divided by $n$, so two must have the same remainder.

The larger minus the smaller of the two numbers is a multiple of $n$, which has a decimal expansion consisting entirely of 0s and 1s.
Among any N positive integers, there exists 2 whose difference is divisible by N-1.

Proof: Let $a_1, a_2, ..., a_N$ be the numbers. For each $a_i$, let $r_i$ be the remainder that results from dividing $a_i$ by $N - 1$. (So $r_i = a_i \mod (N-1)$ and $r_i$ can take on only the values 0, 1, ..., N-2.) There are N-1 possible values for each $r_i$, but there are N $r_i$'s. Thus, by the pigeon hole principle, there must be two of the $r_i$'s that are the same, $r_j = r_k$ for some pair $j$ and $k$. But then, the corresponding $a_i$'s have the same remainder when divided by N-1, and so their difference $a_j - a_k$ is evenly divisible by N-1.
Let $S = (a_1, \ldots, a_m)$ be a sequence of numbers. A subsequence of $S$ is obtained by deleting terms of the sequence $S$, but keeping the order among the elements.

Example: $(5, 4, 7, 1, 3, 2)$ contains subsequences $(5, 4, 3, 2)$ and $(5, 7)$

A sequence is called strictly decreasing if each term is smaller than the previous one.

A sequence is called strictly increasing if each term is larger than the previous one.
Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

[ A proof by contradiction seems like a good choice. But how can we arrive at a contradiction?]
Sequences (3)

Seeking a contradiction, we assume that there exists a sequence $S= (a_1,...,a_{n^2-1})$ does not contain any strictly increasing (or decreasing) subsequence of length $n+1$ or longer.

Associate with each term $a_k$ of the sequence two integers:

$i_k = \text{length of the longest increasing subsequence starting at } a_k$

$d_k = \text{length of the longest decreasing subsequence starting at } a_k$

Notice that $1 \leq i_k, d_k \leq n$, so there are $n^2$ distinct ordered pairs $(i_k, d_k)$ but the sequence contains $n^2+1$ elements, so there must be two terms of $a_s$ and $a_t$ of the sequence that contain the same pairs. We will show that this is impossible.
Since all elements of the sequence are distinct real numbers, we either have $a_s < a_t$ or $a_s > a_t$.

If $a_s < a_t$ then a strictly increasing subsequence of length $i_s+1$ can be formed by taking $a_s$ followed by the strictly increasing subsequence of $S$ starting at $a_t$, contradicting the fact that $i_s$ denote the length of the longest strictly increasing subsequence starting at $a_s$.

Similarly, if $a_s > a_t$ then a strictly decreasing subsequence of length $i_s+1$ starting at $a_s$ can be formed, contradicting the definition of $i_s$. Therefore, such a sequence $S$ cannot exist.