Undecidability

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[based on slides by Prof. Welch]
Understanding Limits of Computing

• So far, we have studied how efficiently various problems can be solved.
• There has been no question as to whether it is possible to solve the problem.
• If we want to explore the boundary between what can and what cannot be computed.
Church-Turing Thesis

• Conjecture: Anything we reasonably think of as an algorithm can be computed by a Turing Machine (specific formal model).

• So we might as well think in our favorite programming language, or in pseudocode.

• Frees us from the tedium of having to provide boring details
  • in principle, pseudocode descriptions can be converted into some appropriate formal model.
Short Review of some Basic Set Theory Concepts
Some Notation

If $A$ and $B$ are sets, then the set of all functions from $A$ to $B$ is denoted by $B^A$.

If $A$ is a set, then $P(A)$ denotes the power set, i.e., $P(A)$ is the set of all subsets of $A$. 
Cardinality

Two sets $A$ and $B$ are said to have the same cardinality if and only if there exists a bijective function from $A$ onto $B$.

[ A function is bijective if it is one-to-one and onto ]

We write $|A| = |B|$ if $A$ and $B$ have the same cardinality.

[Note that $|A| = |B|$ says that $A$ and $B$ have the same number of elements, even if we do not yet know about numbers!]
Set theorists count

• $0 = \{}$ // the empty set exists by axiom
  This set contains no elements
• $1 = \{0\} = \{{}\}$ // form the set containing {} 
  This set contains one element
• $2 = \{0,1\} = \{\{},\{{}\}\}$ 
  This set contains two elements
• Keep including all previously created sets as elements of the next set.
Example

Theorem: \(|P(X)| = |2^X|\)

Proof: The bijection from \(P(X)\) onto \(2^X\) is given by the characteristic function. \(q.e.d.\)

Example: \(X = \{a, b\}\)

\(\emptyset\) corresponds to \(f(a)=0, f(b)=0\)

\(\{a\}\) corresponds to \(f(a)=1, f(b)=0\)

\(\{b\}\) corresponds to \(f(a)=0, f(b)=1\)

\(\{a, b\}\) corresponds to \(f(a)=1, f(b)=1\)
More About Cardinality

Let A and B be sets.

We write $|A| \leq |B|$ if and only if there exists an injective function from A to B.

We write $|A| < |B|$ if and only if there exist an injective function from A to B, but no bijection exists from A to B.
Cardinality

Cantor’s Theorem: Let $S$ be any set. Then $|S| < |P(S)|$.

Proof: Since the function $i$ from $S$ to $P(S)$ given by $i(s) = \{s\}$ is injective, we have $|S| \leq |P(S)|$.

Claim: There does not exist any function $f$ from $S$ to $P(S)$ that is surjective.

Indeed, $T = \{ s \in S : s \notin f(s) \}$ is not contained in $f(S)$.

An element $s$ in $S$ is either contained in $T$ or not.

- If $s \in T$, then $s \notin f(s)$ by definition of $T$. Thus, $T \neq f(s)$.
- If $s \notin T$, then $s \in f(s)$ by definition of $T$. Thus, $T \neq f(s)$.

Therefore, $f$ is not surjective. This proves the claim.
Uncountable Sets and Uncomputable Functions
Countable Sets

Let $\mathbb{N}$ be the set of natural numbers.

A set $X$ is called \textit{countable} if and only if there exists a surjective function from $\mathbb{N}$ onto $X$.

Thus, finite sets are countable, $\mathbb{N}$ is countable, but the set of real numbers is not countable.
**An Uncountable Set**

**Theorem:** The set $\mathbb{N}^\mathbb{N} = \{ f \mid f: \mathbb{N} \rightarrow \mathbb{N} \}$ is not countable.

**Proof:** We have $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$ by Cantor’s theorem. Since $|\mathcal{P}(\mathbb{N})| = |2^\mathbb{N}|$ and $2^\mathbb{N}$ is a subset of $\mathbb{N}^\mathbb{N}$ we can conclude that

$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |2^\mathbb{N}| \leq |\mathbb{N}^\mathbb{N}|$. q.e.d.
Alternate Proof:
The Set $\mathbb{N}^\mathbb{N}$ is Uncountable

Seeking a contradiction, we assume that the set of functions from $\mathbb{N}$ to $\mathbb{N}$ is countable. Let the functions in the set be $f_0, f_1, f_2, \ldots$

We will obtain our contradiction by defining a function $f^d$ (using "diagonalization") that should be in the set but is not equal to any of the $f_i$'s.
Diagonalization

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<td>12</td>
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Diagonalization

- Define the function: \( f^d(n) = f_n(n) + 1 \)
- In the example:
  - \( f^d(0) = 4 + 1 = 5 \), so \( f^d \neq f_0 \)
  - \( f^d(1) = 32 + 1 = 33 \), so \( f^d \neq f_1 \)
  - \( f^d(2) = 5 + 1 = 6 \), so \( f^d \neq f_2 \)
  - \( f^d(3) = 7 + 1 = 8 \), so \( f^d \neq f_3 \)
  - \( f^d(4) = 3 + 1 = 4 \), so \( f^d \neq f_4 \)
  - etc.
Uncomputable Functions Exist!

Consider all programs (e.g. in the Turing machine model) that compute functions in $\mathbb{N}^\mathbb{N}$. The set $\mathbb{N}^\mathbb{N}$ is uncountable, hence cannot be enumerated.

However, the set of all programs can be enumerated (i.e., is countable).

Thus there must exist some functions in $\mathbb{N}^\mathbb{N}$ that cannot be computed by a program.
Set of All Programs is Countable

- Fix your computational model (e.g., Turing machines).
- Every program is finite in length.
- For every integer $n$, there is a finite number of programs of length $n$.
- Enumerate programs of length 1, then programs of length 2, then programs of length 3, etc.
Uncomputable Functions

- Previous proof just showed there must exist uncomputable functions
- Did not exhibit any particular uncomputable function
- Maybe the functions that are uncomputable are uninteresting...
- But actually there are some VERY interesting functions (problems) that are uncomputable
The Halting Problem
The Function Halt

• Consider this function, called Halt:
  • input: code for a program P and an input X for P
  • output: 1 if P terminates (halts) when executed on input X, and 0 if P doesn't terminate (goes into an infinite loop) when executed on input X

• By the way, a compiler is a program that takes as input the code for another program

• Note that the input X to P could be (the code for) P itself
  • in the compiler example, a compiler can be run on its own code
The Function Halt

- We can view Halt as a function from \( \mathbb{N} \) to \( \mathbb{N} \):
  - \( P \) and \( X \) can be represented in ASCII, which is a string of bits.
  - This string of bits can also be interpreted as a natural number.
- The function Halt would be a useful diagnostic tool in debugging programs.
Halt is Uncomputable

- Suppose in contradiction there is a program $P_{\text{halt}}$ that computes Halt.
- Use $P_{\text{halt}}$ as a subroutine in another program, $P_{\text{self}}$.
- Description of $P_{\text{self}}$:
  - input: code for any program $P$
  - constructs pair $(P, P)$ and calls $P_{\text{halt}}$ on $(P, P)$
  - returns same answer as $P_{\text{halt}}$
If $P$ halts on input $P$, then $P_{\text{self}}(P, P) = 0$.

If $P$ doesn't halt on input $P$, then $P_{\text{self}}(P, P) = 1$. 

$P_{\text{halts}}$ receives $(P, P)$ as input and outputs 0 if $P$ halts on input $P$, and 1 if $P$ doesn't halt on input $P$. 

$P_{\text{self}}$ takes in $P$ and outputs $(P, P)$. 

$P_{\text{self}}$ takes in $(P, P)$ and checks if $P$ halts on input $P$. If $P$ halts, it outputs 0; if $P$ doesn't halt, it outputs 1.
Halt is Uncomputable

- Now use $P_{self}$ as a subroutine inside another program $P_{diag}$.
- Description of $P_{diag}$:
  - input: code for any program $P$
  - call $P_{self}$ on input $P$
  - if $P_{self}$ returns 1 then go into an infinite loop
  - if $P_{self}$ returns 0 then output 0
- $P_{diag}$ on input $P$ does the opposite of what program $P$ does on input $P$. 
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P_{\text{diag}}

0 if P doesn't halt on input P

1 if P halts on input P

P \rightarrow P_{\text{diag}} \rightarrow P_{\text{self}} \rightarrow P_{\text{halt}} \rightarrow 0
Halt is Uncomputable

• Review behavior of $P_{\text{diag}}$ on input $P$:
  • If $P$ halts when executed on input $P$, then $P_{\text{diag}}$ goes into an infinite loop
  • If $P$ does not halt when executed on input $P$, then $P_{\text{diag}}$ halts (and outputs 0)

• What happens if $P_{\text{diag}}$ is given its own code as input?
  It either halts or doesn’t.
  • If $P_{\text{diag}}$ halts when executed on input $P_{\text{diag}}$, then $P_{\text{diag}}$ goes into an infinite loop
  • If $P_{\text{diag}}$ doesn’t halt when executed on input $P_{\text{diag}}$, then $P_{\text{diag}}$ halts

Contradiction
Halt is Uncomputable

• What went wrong?
• Our assumption that there is an algorithm to compute Halt was incorrect.
• So there is no algorithm that can correctly determine if an arbitrary program halts on an arbitrary input.
Undecidability
Undecidability

• The analog of an uncomputable function is an **undecidable set**.
• The theory of what can and cannot be computed focuses on identifying sets of strings:
  • an algorithm is required to "decide" if a given input string is in the set of interest
  • similar to deciding if the input to some NP-complete problem is a YES or NO instance
Undecidability

• Recall that a (formal) language is a set of strings, assuming some encoding.
• Analogous to the function Halt is the set $H$ of all strings that encode a program $P$ and an input $X$ such that $P$ halts when executed on $X$.
• There is no algorithm that can correctly identify for every string whether it belongs to $H$ or not.
Many-One Reduction

\[ \text{all strings over } L_1 \text{'s alphabet} \rightarrow \text{all strings over } L_2 \text{'s alphabet} \]

\[ L_1 \rightarrow f \rightarrow L_2 \]
Many-One Reduction

- YES instances map to YES instances
- NO instances map to NO instances
- computable (doesn't matter how slow)
- Notation: $L_1 \leq_m L_2$
- Think: $L_2$ is at least as hard to compute as $L_1$
Many-One Reduction Theorem

**Theorem:** If $L_1 \leq_m L_2$ and $L_2$ is computable, then $L_1$ is computable.

**Proof:** Let $f$ be the many-one reduction from $L_1$ to $L_2$. Let $A_2$ be an algorithm for $L_2$. Here is an algorithm $A_1$ for $L_1$.

- **input:** $x$
- **compute** $f(x)$
- **run** $A_2$ on input $f(x)$
Implication

- If there is no algorithm for $L_1$, then there is no algorithm for $L_2$.
- In other words, if $L_1$ is undecidable, then $L_2$ is also undecidable.
- Pay attention to the direction!
Example of a Reduction

- Consider the language $L_{NE}$ consisting of all strings that encode a program that halts (does not go into an infinite loop) on at least one input.
- Use a reduction to show that $L_{NE}$ is not decidable:
  - Show some known undecidable language $\leq_m L_{NE}$.
  - Our only choice for the known undecidable language is $H$ (the language corresponding to the halting problem)
  - So show $H \leq_m L_{NE}$. 
Example of a Reduction

• Given an arbitrary H input (encoding of a program P and an input X for P), compute an $L_{NE}$ input (encoding of a program $P'$)
  • such that P halts on input X if and only if $P'$ halts on at least one input.

• Construction consists of writing code to describe $P'$.

• What should $P'$ do? It's allowed to use P and X
Example of a Reduction

• The code for $P'$ does this:
  • input $X'$:
  • ignore $X'$
  • call program $P$ on input $X$
  • if $P$ halts on input $X$ then return whatever $P$ returns

• How does $P'$ behave?
  • If $P$ halts on $X$, then $P'$ halts on every input
  • If $P$ does not halt on $X$, then $P'$ does not halt on any input
Example of a Reduction

- Thus if \((P,X)\) is a YES input for \(H\) (meaning \(P\) halts on input \(X\)), then \(P'\) is a YES input for \(L_{\text{NE}}\) (meaning \(P'\) halts on at least one input).
- Similarly, if \((P,X)\) is NO input for \(H\) (meaning \(P\) does not halt on input \(X\)), then \(P'\) is a NO input for \(L_{\text{NE}}\) (meaning \(P'\) does not halt on even one input).
- Since \(H\) is undecidable, and we showed \(H \leq_m L_{\text{NE}}\), 
  \(L_{\text{NE}}\) is also undecidable.
Generalizing Such Reductions

- There is a way to generalize the reduction we just did, to show that lots of other languages that describe properties of programs are also undecidable.

- Focus just on programs that accept languages (sets of strings):
  - I.e., programs that say YES or NO about their inputs
  - Ex: a compiler tells you YES or NO whether its input is syntactically correct
Properties About Programs

- Define a property about programs to be a set of strings that encode some programs.
  - The "property" corresponds to whatever it is that all the programs have in common
- Example:
  - Program terminates in 10 steps on input y
  - Program never goes into an infinite loop
  - Program accepts a finite number of strings
  - Program contains 15 variables
  - Program accepts 0 or more inputs
Functional Properties

- A property about programs is called **functional** if it just refers to the language accepted by the program and not about the specific code of the program
  - Program terminates in 10 steps on input y (n.f.)
  - Program never goes into an infinite loop (f.)
  - Program accepts a finite number of strings (f.)
  - Program contains 15 variables (n.f.)
  - Program accepts 0 or more inputs (f.)
Nontrivial Properties

• A functional property about programs is nontrivial if some programs have the property and some do not

• Example of nontrivial programs:
  • Program never goes into an infinite loop
  • Program accepts a finite number of strings

• Example of a trivial program:
  • Program accepts 0 or more inputs
Rice's Theorem

- Every nontrivial (functional) property about programs is undecidable.
- The proof is a generalization of the reduction shown earlier.
- Very powerful and useful theorem:
  - To show that some property is undecidable, only need to show that is nontrivial and functional, then appeal to Rice's Theorem
Applying Rice's Theorem

- Consider the property "program accepts a finite number of strings".
- This property is functional:
  - it is about the language accepted by the program and not the details of the code of the program
- This property is nontrivial:
  - Some programs accept a finite number of strings (for instance, the program that accepts no input)
  - some accept an infinite number (for instance, the program that accepts every input)
- By Rice’s theorem, the property is undecidable.
Implications of Undecidable Program Property

• It is not possible to design an algorithm (write a program) that can analyze any input program and decide whether the input program satisfies the property!
• Essentially all you can do is simulate the input program and see how it behaves
  • but this leaves you vulnerable to an infinite loop
• Thought question: Then how can compilers be correct?