Counting

Andreas Klappenecker
Counting

The art of counting is known as enumerative combinatorics. One tries to count the number of elements in a set (or, typically, simultaneously count the number of elements in a series of sets).

For example, let $S_1, S_2, S_3, ...$ be sets with 1, 2, 3, ... elements, respectively. Then the number of subsets of $S_i$ is given by $f(i) = |P(S_i)| = 2^i$.

The basic principles are extremely simple, but counting is a nontrivial task.
The Product Rule

Suppose that a task can be broken down into a sequence of two subtasks. If there are \( n_1 \) ways to solve subtask 1 and \( n_2 \) ways to solve subtask 2, then there must be \( n_1n_2 \) ways to solve the task.

Let \( S_1 \) and \( S_2 \) be sets describing the ways of the first and second subtasks, so \( n_1 = |S_1| \) and \( n_2 = |S_2| \). Then \( |S_1 \times S_2| = n_1n_2 \).
Product Rule: Example 1

How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits (and no sequence of letters are prohibited).

There are 26 choices for each of the three uppercase letters, and 10 choices for each of the three digits. Thus,

\[26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000\]

possible license plates. Since Texas has already a population of 25,674,681, this is perhaps not a good choice here.
How many functions are there from a set with $m$ elements to a set with $n$ elements?

For each of the $m$ elements in the domain, we can choose any element from the codomain as a function value. Hence, by the product rule, we get

$$n \times n \times \ldots \times n = n^m$$

different functions.
How many injective functions are there from a set with $m$ elements to a set with $n$ elements?

If $m > n$, then there are $0$ injective functions.

If $m \leq n$, then there are $n$ ways to choose the value for the first element in the domain, $n-1$ ways to choose the value for the second element (as one has to avoid the previously chosen value), $n-2$ for the third element of the domain and so forth. Thus, we have $n(n-1)...(n-m+1)$ injective functions in this case.
Sum Rule

If a task can be done either in one of \( n_1 \) ways or in one of \( n_2 \) ways, where none of the set of \( n_1 \) ways is the same as any of the set of \( n_2 \) ways, then there are \( n_1 + n_2 \) ways to do the task.

Let \( S_1 \) and \( S_2 \) be disjoint sets with \( n_1 = |S_1| \) and \( n_2 = |S_2| \). Then \( |S_1 \cup S_2| = n_1 + n_2 \). 
A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

There are 23 + 15 + 19 = 57 projects to choose from.
Sum Rule: Example 2 (1)

How many sequences of 1s and 2s sum to $n$?

Let us call the answer to this question $a_n$.

$a_0 = 1$  \{ one sequence, namely the empty sequence () \}

$a_1 = 1$  \{ one sequence, namely (1) \}

$a_2 = 2$  \{ the sequences (1,1) and (2) \}

$a_3 = 3$  \{ the sequences (1,1,1), (1,2), and (2,1) \}

$a_4 = 5$  \{ the sequences (1,1,1,1), (1,1,2), (1,2,1), (2,1,1), and (2,2) \}
How many sequences of 1s and 2s sum to \( n \)?

Let us call the answer to this question \( a_n \).

\( a_0 = 1, \ a_1 = 1 \)

\( a_n = a_{n-1} + a_{n-2} \) for \( n \geq 2 \)

Indeed, there are

- \( a_{n-1} \) sequences starting with 1 (remaining seq. summing to \( n-1 \))
- \( a_{n-2} \) sequences starting with 2 (remaining seq. summing to \( n-2 \))

Thus, by the sum rule \( a_n = a_{n-1} + a_{n-2} \)

Defining \( a_{-1} = 0 \), we get \( a_n = f_{n+1} \) where \( f_n \) is the Fibonacci sequence.
Computer addresses belong to one of the following 3 types:

- **Class A**: address contains a 7-bit “netid” ≠ 1^7, and a 24-bit “hostid”

- **Class B**: address has a 14-bit netid and a 16-bit hostid.

- **Class C**: address has 21-bit netid and an 8-bit hostid.

<table>
<thead>
<tr>
<th>Class</th>
<th>Bit 0</th>
<th>Bit 1</th>
<th>Bit 2</th>
<th>Bit 3</th>
<th>Bit 4</th>
<th>Bit 8</th>
<th>Bit 16</th>
<th>Bit 24</th>
<th>Bit 31</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class A</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>netid</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Class B</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td>netid</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Class C</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td>netid</td>
<td></td>
<td></td>
<td>hostid</td>
</tr>
</tbody>
</table>

Hostids that are all 0s or all 1s are not allowed.

How many valid computer addresses are there?
IPv4 Address Example (2)

- \( (# \text{ addr}) \)
  \[ = (# \text{ class A}) + (# \text{ class B}) + (# \text{ class C}) \]
  \( \text{(by sum rule)} \)

- \# \text{ class A} = (# \text{ valid netids}) \cdot (# \text{ valid hostids})
  \( \text{(by product rule)} \)

- (# valid class A netids) = \(2^7 - 1 = 127\).

- (# valid class A hostids) = \(2^{24} - 2 = 16,777,214\).

- Continuing in this fashion we find the answer is:
  
  \[3,737,091,842 \text{ (3.7 billion IP addresses)}\]
Subtraction Rule

If a task can be done in either \( n_1 \) ways or \( n_2 \) ways, then the number of ways to do the task is \( n_1+n_2 \) minus the number of ways to do the task that is common to the two different ways.

Principle of Inclusion-and-Exclusion:

Let \( S_1 \) and \( S_2 \) be sets. Then

\[
|S_1 \cup S_2| = |S_1|+|S_2| - |S_1 \cap S_2|
\]
Subtraction Rule: Example 1

How many bit strings of length 8 either start with a 1 bit or end with the last two bits equal to 00?

Let $S_1$ be the set of bit strings of length 8 that start with 1. Then $|S_1| = 2^7 = 128$.

Let $S_2$ be the set of bit strings of length 8 that end with 00. Then $|S_2| = 2^6 = 64$.

Furthermore, the set of bit strings of length 8 that start with bit 1 and end with bits 00 has cardinality $|S_1 \cap S_2| = 2^5 = 32$.

Thus, the answer is $|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2| = 128 + 64 - 32 = 160$. 
Let us consider some (slightly simplified) rules for passwords:

Passwords must be 2 characters long.

Each character must be

a) a letter [a-z],

b) a digit [0-9], or

c) one of the 10 punctuation characters [!@#$%^&*()].

Each password must contain at least 1 digit or punctuation character.
A legal password has a digit or a punctuation character in position 1 or position 2.

These cases overlap, so the subtraction rule applies.

(# of passwords with valid symbol in position #1) = (10+10)·(10+10+26) = 20·46

(# of passwords with valid symbol in position #2) = 20·46

(# of passwords with valid symbols in both places): 20·20

Answer: 920+920−400 = 1,440
Pigeonhole Principle

If $k+1$ objects are assigned to $k$ places, then at least one place must be assigned at least two objects.
Generalized Pigeonhole Principle

**Theorem:** If $N > k$ objects are assigned to $k$ places, then at least one place must be assigned at least $\lceil N/k \rceil$ objects.

**Proof:** Seeking a contradiction, suppose every place has less than $\lceil N/k \rceil$ objects; so at most $\leq \lceil N/k \rceil - 1$ objects per place.

Then the total number of objects is at most

$$k \left( \left\lfloor \frac{N}{k} \right\rfloor - 1 \right) < k \left( \left( \frac{N}{k} + 1 \right) - 1 \right) = k \left( \frac{N}{k} \right) = N$$

Thus, there are fewer than $N$ objects total, contradicting our assumption on the total number of objects.
Pigeonhole Principle: Example 1

Suppose that there are 120+54 students in two sections of a class. Then there must exist a week during which

\[ \left\lceil \frac{174}{52} \right\rceil = 4 \]

students of CSCE 222 have a birthday.
Pigeonhole Principle: Example 2

Ten points are given within a square of unit size. Then there are two points that are closer to each other than 0.48.

Proof: Let us partition the square into nine squares of side length 1/3, see Figure (a) below.

By the pigeonhole principle, one square must contain at least 2 points, see Figure (b). The distance of two points within a square of side length 1/3 is at most \((2/9)^{1/2}\) by Pythagoras’ theorem. The claim follows, since \((2/9)^{1/2} < 0.471405 < 0.48.\)
Counting in Two Different Ways Rule

When two different formulas enumerate the same set, then they must be the same.

[In other words, you count the elements of the set in two different ways.]
Double Counting: Example 1

Take an array of \((n+1) \times (n+1)\) dots. Thus, it contains \((n+1)^2\) dots.

Counting the points on the main diagonal, the upper diagonals, and the lower diagonals, we get

\[
(n + 1)^2 = (n + 1) + \sum_{i=1}^{n} i + \sum_{i=1}^{n} i
\]

\(\implies n(n + 1) = (n + 1)^2 - (n + 1) = 2 \sum_{i=1}^{n} i\)

\(\implies \frac{n(n + 1)}{2} = \sum_{i=1}^{n} i\)
Permutations and Combinations
Permutations

Let $S$ be a set with $n$ elements. An ordered arrangement of $r$ elements of $S$ is called an $r$-permutation of $S$. A permutation of $S$ is an $n$-permutation.

The number of $r$-permutations of a set with $n$ elements is denoted by $P(n,r)$.

Example: $S = \{1,2,3,4\}$. Then $(2,4,3)$ and $(4,3,2)$ are two distinct 3-permutations of $S$. Order matters here!
Number of r-Permutations.

**Theorem:** Let \( n \) and \( r \) be positive integers, \( r \leq n \). Then \( P(n,r) = n(n-1) \ldots (n-r+1) \).

**Proof:** Let \( S \) be a set with \( n \) elements. The first element of the permutation can be chosen in \( n \) ways, the second in \( n-1 \) ways, ..., the \( r \)-th element can be chosen in \( n-r+1 \) ways. The claim follows by the product rule.
**Corollary**: Let $n$ be a positive integer, and $r$ an integer in the range $0 \leq r \leq n$.

Then $P(n,r) = n!/(n-r)!$

Proof: For $r$ in the range $1 \leq r \leq n$, this follows from the previous theorem and the fact that $n!/(n-r)! = n(n-1) \ldots (n-r+1)$.

For $r=0$, we have $P(n,0)=1$, which equals $n!/(n-0)! = n!/n!=1$. 
Permutation Example

How many permutations of the letters ABCDEFGH contain the string ABC?

Let us regard ABC, D, E, F, G, and H as blocks. Any permutation of these six blocks will yield a valid permutation containing ABC, and there are no others. Therefore, we have $6! = 720$ permutations of the letters ABCDEFGH that contain ABC as a block.
Combinations

Let $S$ be a set of $n$ elements. An $r$-combination of $S$ is a subset of $r$ elements from $S$.

The number of $r$-combinations of a set $S$ with $n$ elements is denoted by

$C(n,r)$ or $\binom{n}{r}$. 
Number of r-Combinations

**Theorem**: The number of r-combinations of a set with n elements is given by

\[
\binom{n}{r} = \frac{n!}{(n-r)!r!}
\]

**Proof.** We can form all r-permutations of a set with n elements by first choosing an r-combination and then ordering the r elements in all possible ways. Thus, \( P(n,r) = C(n,r)P(r,r) \).

Hence, \( C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)!r!} \).

Since \((r-r)! = 0! = 1\), this yields our claim.
Binomial Coefficient Identity

Corollary: We have \[ \binom{n}{r} = \binom{n}{n - r} \]

Proof: Let \( S \) be a set with \( n \) elements. Each subset \( A \) of \( S \) is determined by its complement \( A^c \), which specifies the elements of \( S \) that are not contained in \( A \). Therefore, we can use double counting:

The number \( C(n, r) \) of subsets of cardinality \( r \) of \( S \) corresponds to the number of “complements of subsets of cardinality \( r \) in \( S \)”. Since \( |A| = r \) iff \( |A^c| = n - r \), the complements of subsets of cardinality \( r \) of \( S \) correspond to subsets of cardinality \( n - r \) of \( S \).

Thus, \( C(n, r) = \# \) of \( r \)-subsets of \( S \)

\[ = \# \) of complements of \( r \)-subsets of \( S = C(n, n-r) \), as claimed.\]
Counting Subset Identity

**Theorem:** For any nonnegative integer $n$, we have

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]

**Proof:** Let $S$ be a set with $n$ elements. The number of subsets of $S$ is $2^n$. The number of subsets with 0, 1, 2, ..., $n$ elements is given by

\[
\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}
\]

Since subsets of $S$ need to have between 0 and $n$ elements, the claim of the theorem follows.
Binomial Theorem

Theorem: Let $x$ and $y$ be variables. Let $n$ be a nonnegative integer. Then

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof: Let us expand the left hand side. The terms of the product in expanded form are $x^k y^{n-k}$ for $0 \leq k \leq n$.

To obtain the term $x^k y^{n-k}$ one must choose $k$ $x$’s from the $n$ $(x+y)$ terms. There are $\binom{n}{k}$ ways to do that.
Another Binomial Coefficient Identity

Corollary: Let \( n \) be a positive integer. Then

\[
\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} = 0
\]

Proof: We have \( 0 = 0^n = (-1+1)^n \). Expanding the right hand side with the help of the binomial theorem, we obtain the claim.

[This also says that the number of subsets with an even number of elements is equal to the number of subsets with an odd number of elements.]
Theorem: Let $n$ and $k$ be positive integers with $n \geq k$. Then

\[
\binom{n + 1}{k} = \binom{n}{k - 1} + \binom{n}{k}
\]

Proof: We are going to prove this by counting the number of subsets with $k$ elements of a set $T$ with $n+1$ elements in two different ways.

First way of counting:

The set $T$ clearly contains $\binom{n + 1}{k}$ subsets of size $k$. 
Pascal's Identity (2)

Second way of counting:

Recall that \( T \) is a set with \( n+1 \) elements. Let us consider an element \( t \) of \( T \). We will count the subsets of \( T \) of size \( k \) that (a) contain the element \( t \), and (b) do not contain the element \( t \).

\[
\text{(a) There are } \binom{n}{k-1} \text{ subsets of } T \text{ that contain } t, \text{ since } t \text{ is already chosen, but the remaining } k-1 \text{ elements need to be chosen from } T-\{t\}, \text{ a set of size } n.
\]

\[
\text{(b) There are } \binom{n}{k} \text{ subset of } T \text{ not containing } t, \text{ since one can choose any } k \text{ elements from the set } T-\{t\} \text{ with } n \text{ elements.}
\]
Pascal’s Identity (3)

The first way of counting yields the LHS, and the second way of counting (using the sum rule) yields the RHS of the following formula:

\[
\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}
\]
Vandermonde's Identity

Let $m$, $n$, and $r$ be nonnegative integers, $r \leq \min(n, m)$. Then

$$\binom{m + n}{r} = \sum_{k=0}^{r} \binom{m}{r - k} \binom{n}{k}.$$

In particular, when choosing $m = n = r$, we get

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$
Vandermonde's Identity (2)

Proof: We will prove this by counting in two different ways. Let $S$ and $T$ be two disjoint sets with $m = |S|$ and $n = |T|$.

**Counting in the first way:**
We can choose $r$ elements from $S \cup T$ in $\binom{n+m}{r}$ ways.

**Counting in the second way:** We can pick $r$ elements from $S \cup T$ by picking $r - k$ elements from $S$ and $k$ elements from $T$, where $0 \leq k \leq r$. By the product rule, this can be done in $\binom{m}{r-k}\binom{n}{k}$ ways. Hence the total number of ways to pick $r$ elements from $S \cup T$ is

$$
\sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.
$$
Permutations and Combinations with Repetitions
Motivation

So far, we assumed that (a) the elements are clearly distinguishable and (b) each element is chosen at most once for a permutation and combinations.

We will still keep the assumption (a) that the elements from which we choose are clearly distinguishable. However, we will now assume (b’) that each element can be chosen repeatedly in permutations and combinations.
Multisets

A multiset is a generalization of the notion of a set in which elements are allowed to appear more than once.

For example, {{a,a,b,c,c,d,d,d}} contains the elements

- a with multiplicity 2,
- b with multiplicity 1,
- c with multiplicity 2,
- d with multiplicity 3.
Let $S$ be a set with $n$ elements. An \textit{$r$-combination with repetition} of $S$ is a multisubset with $r$ elements of the set $S$.

The \textit{number of $r$-combinations with repetitions of a set $S$ with $n$ elements} is denoted by

\[
\binom{n}{r}.
\]
**Theorem.** The number of \( r \)-combination with repetition of a set with \( n \) elements is given by

\[
\binom{n}{r} = \binom{n + r - 1}{r}.
\]

Proof. Let \( S \) be a set with \( n \) elements. Order the elements of this set.

We can specify an \( r \)-multisubset \( T \) of \( S \) by specifying the multiplicity of the first element, the multiplicity of the second element, and so on.
Proof (continued)

Let us use \( n \) bins and \( m_i \) balls in the \( i \)th bin to specify that the \( i \)th element of \( S \) occurs with multiplicity \( m_i \) in \( T \). Then \( r \) balls distributed in the \( n \) bins determine an \( r \)-multisubset of \( S \).

Therefore, it suffices to count the number of ways we can distribute \( r \) balls into \( n \) bins.

This number corresponds to the number of strings consisting of \( r \) bullets \( \bullet \) (representing the balls), and \( n - 1 \) bars \( | \) (representing the walls between the bins. Since the positions of the \( r \) balls among the \( n - 1 + r \) balls and bars suffice to specify such a string, we can conclude that there are \( \binom{n-1+r}{r} \) strings consisting of \( n - 1 \) bars and \( r \) balls. By construction, this coincides with the number multisubsets with \( r \) elements of a set with \( n \) elements, so the theorem is proved.
Example

Let $S = \{a, b, c, d\}$ be a set with four elements. Then the number of 3-combinations with repetitions from $S$ is given by

$$\binom{\binom{4}{3}}{3} = \binom{4 + 3 - 1}{3} = 20$$
Permutations with Repetition

Let $S$ be a set with $n$ elements. An $r$-permutation with repetition of $S$ is an arrangement of $r$ elements of $S$ with repetitions allowed.

**Theorem:** The number of $r$-permutations with repetition of a set with $n$ elements is given by $n^r$.

**Proof.** This follows directly from the product rule.

[Wow, EASY!!!]
Permutations under Indistinguishability Constraints
So far, we assumed that (a) the elements are clearly distinguishable and (b) each element is chosen at most once for a permutation and combinations or (b’) that each element can be chosen repeatedly in permutations and combinations.

Now we will allow that (a’) the elements from which we are allowed to choose are not necessarily distinguishable.

[For example, we could choose from a multiset of $m$ red balls and $n$ blue balls.]
Permutations with Indistinguishable Objects

In how many ways can we reorder SUCCESS?

We can choose 3 locations for the Ss in $C(7,3)$ ways.

Then we can choose 1 location for the U in $C(4,1)$ way.

Then we can choose 2 locations for the Cs in $C(3,2)$ ways.

Finally, we can choose 1 location for the E in $C(1,1)$ way.

$$C(7,3)C(4,1)C(3,2)C(1,1) = \frac{7!}{3!4!} \cdot \frac{4!}{1!3!} \cdot \frac{3!}{2!1!} \cdot \frac{1!}{1!0!} = \frac{7!}{3!2!1!1!} = 420$$
Theorem. Suppose we have \( n \) different objects that are of \( k \) different types, where the objects of the same type are indistinguishable. If there are \( n_i \) indistinguishable objects of type \( i \) for \( 1 \leq i \leq k \), then the total number of permutations is

\[
\frac{n!}{n_1!n_2! \cdots n_k!}.
\]