Dynamic Programming

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[partially based on slides by Prof. Welch]

Dynamic Programming

- Optimal substructure
  - An optimal solution to the problem contains within it optimal solutions to subproblems.
- Overlapping subproblems
  - The space of subproblem is "small" so that the recursive algorithm has to solve the same problems over and over.

Giving Optimal Change

Motivation

We have discussed a greedy algorithm for giving change.

However, the greedy algorithm is not optimal for all denominations.

Can we design an algorithm that will give the minimum number of coins as change for any given amount?

Answer: Yes, using dynamic programming.

Dynamic Programming Task

For dynamic programming, we have to find some subproblems that might help in solving the coin-change problem.

Idea:

- Vary amount
- Restrict the available coins

Initial Set Up

Suppose we want to compute the minimum number of coins with values

v[1]>v[2]>…>v[n]=1

to give change for an amount C.

Let us call the (i,j)-problem the problem of computing minimum number of coins with values

v[i]>v[i+1]>…>v[n]=1

to give change for an amount 1<= j <= C.

The original problem is the (1,C)-problem.

Tabulation

Let m[i][j] denote the solution to the (i,j)-problem.

Thus, m[i][j] denotes the minimum number of coins to make change for the amount j using coins with values v[i],…,v[n].

Tabulation Example

Denomination v[1]=10, v[2]=6, v[3]=1

Table of m[i][j] values:

		j											
		0	1	2	3	4	5	6	7	8	9	10	11	12
	1	0	1	2	3	4	5	1	2	3	4	1	2	2
i	2	0	1	2	3	4	5	1	2	3	4	5	6	2
	3	0	1	2	3	4	5	6	7	8	9	10	11	12

A Simple Observation

In calculating m[i][j], notice that:

a) Suppose we do not use the coin with value v[i] in the solution of the (i,j)-problem, then m[i][j] = m[i+1][j]

b) Suppose we use the coin with value v[i] in the solution of the (i,j)-problem, then m[i][j] = 1 + m[i][j-v[i]]

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i	2	0	1	2	3	4	5	1	2	3	4	5	6	2
	3	0	1	2	3	4	5	6	7	8	9	10	11	12

Recurrence

We either use a coin with value v[i] in the solution or we don't.

$$m[i][j] = \begin{cases} m[i+1][j] & \text{if } v[i] > j \\ \\ \min\{\, m[i+1][j],\ 1+m[i][\, j-v[i]\,]\,\} & \text{otherwise} \end{cases}$$

DP Coin-Change Algorithm

```
Dynamic_Coin_Change(C,v,n)
    allocate array m[1..n][0..C];
    for(i = 0; i<=C, i++)
        m[n][i] = i;  // make change for amount i using coins of value v[n]=1.
    for(i=n-1; i=>1; i--) { // successively allow a larger number coin values
        for(j=0; j<=C; j++) { // calc values of the array.
            if( v[i]>j || m[i+1][j]<1+m[i][j – v[i]] )
                m[i][j] = m[i+1][j]; // large coin does not help
            else m[i][j] = 1+m[i][j –v[i]]; //
        }
    }
    return &m;
```

Question

The previous algorithm allows us to find the minimum number of coins.

How can you modify the algorithm to actually compute the change (i.e., the multiplicities of the coins)?

Matrix Chain Algorithm

Matrices

An n x m matrix A over the real numbers is a rectangular array of nm real numbers that are arranged in n rows and m columns.

For example, a 3 x 2 matrix A has 6 entries

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

where each of the entries a_{ij} is a real

Definition of Matrix Multiplication

Let A be an n x m matrix

B an m x p matrix

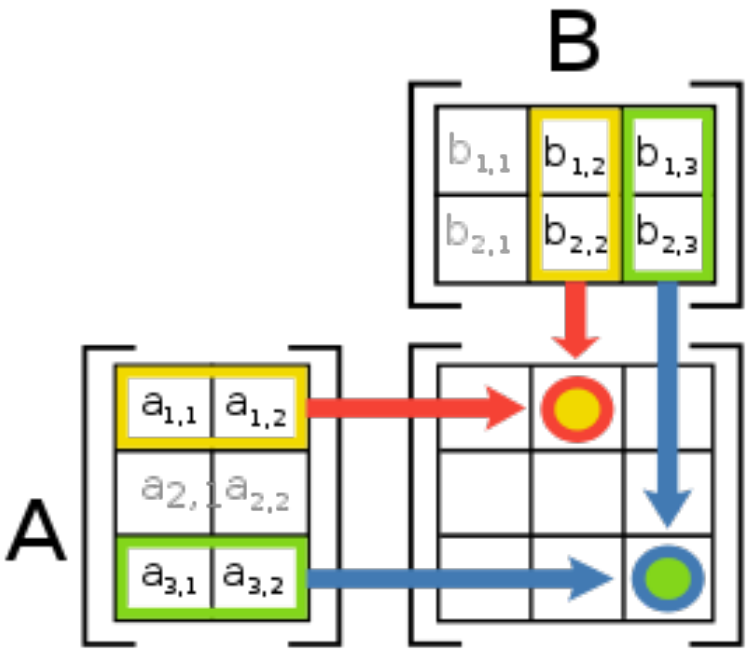
The product of A and B is n x p matrix AB whose (i,j)-th entry is

$$\sum_{k=1}^{m} a_{ik} b_{kj}$$

In other words, we multiply the entries of the i-th row of A with the entries of the j-th column of B and add them up.

Matrix Multiplication

$$
\begin{aligned}
x_{1,2} &= (a_{1,1}, a_{1,2}) \cdot (b_{1,2}, b_{2,2}) \\
&= a_{1,1}b_{1,2} + a_{1,2}b_{2,2} \\
x_{3,3} &= (a_{3,1}, a_{3,2}) \cdot (b_{1,3}, b_{2,3}) \\
&= a_{3,1}b_{1,3} + a_{3,2}b_{2,3}.
\end{aligned}
$$



[Images courtesy of Wikipedia]

Complexity of Naïve Matrix

- Multiplying non-square matrices:
  - A is $n \times m$,
  - B is $m \times p$
  - AB is $n \times p$ matrix

    [whose (i,j) entry is $\sum a_{ik}\, b_{kj}$]
- Computing the product AB takes
  - nmp scalar multiplications
  - $n(m-1)p$ scalar additions

  if we take basic matrix multiplication algorithm.

Matrix Chain Order Problem

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Therefore, we have a choice of forming the product of several matrices.

What is the least expensive way to form the product of several matrices if the naïve matrix multiplication algorithm is used?

[Use number of scalar multiplications as cost.]

Why Order Matters

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- Suppose we have 4 matrices:
  - A: 30 x 1
  - B: 1 x 40
  - C: 40 x 10
  - D: 10 x 25

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  - A: 30 x 1
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  - D: 10 x 25
- ((AB)(CD)) : requires 41,200 mults.
- (A((BC)D)) : requires 1400 mults.

Matrix Chain Order Problem

Given matrices $A_1, A_2, \ldots, A_n$,

where A_i is a $d_{i-1} \times d_i$ matrix.

[1] What is minimum number of scalar multiplications required to compute the product $A_1 \cdot A_2 \cdot \ldots \cdot A_n$?

[2] What order of matrix multiplications achieves this minimum?

We focus on question [1];

We will briefly sketch an answer to [2].

A Possible Solution

- Try all possibilities and choose the best one.
- Drawback: There are too many of them (exponential in the number of matrices to be multiplied)
- Need to be more clever - try dynamic programming!

Step 1: Develop a Recursive

- Define M(i,j) to be the minimum number of multiplications needed to compute

$$A_i \cdot A_{i+1} \cdot \ldots \cdot A_j$$

- Goal: Find M(1,n).
- Basis: M(i,i) = 0.
- Recursion: How can one define M(i,j) recursively?

Defining M(i,j) Recursively

- Consider all possible ways to split A_i through A_j into two pieces.
- Compare the costs of all these splits:
  - best case cost for computing the product of the two pieces
  - plus the cost of multiplying the two products
- Take the best one
- $M(i,j) = \min_k(M(i,k) + M(k+1,j) + d_{i-1}d_kd_j)$

Defining M(i,j) Recursively

$$\underbrace{(A_i \cdot \ldots \cdot A_k)}_{P_1} \cdot \underbrace{(A_{k+1} \cdot \ldots \cdot A_j)}_{P_2}$$

- minimum cost to compute P_1 is $M(i,k)$
- minimum cost to compute P_2 is $M(k+1,j)$
- cost to compute $P_1 \cdot P_2$ is $d_{i-1} d_k d_j$

Step 2: Find Dependencies

M:

	1	2	3	4	5
1	0				
2	n/a	0			
3	n/a	n/	0		
4	n/a	n/	n/	0	
5	n/a	n/	n/	n/	0

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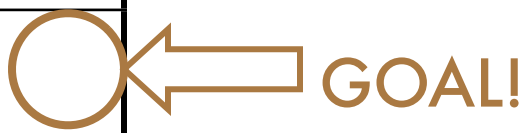

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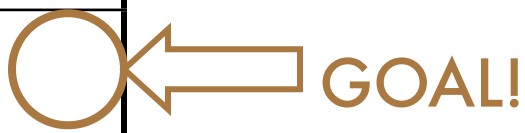

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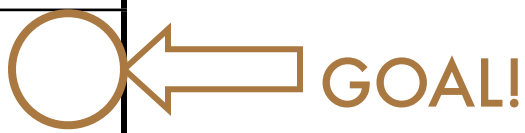

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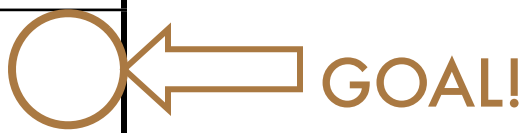

computing the pink square requires the purple ones: to the left and below.

Defining the Dependencies

- Computing M(i,j) uses
  - everything in same row to the left:

    M(i,i), M(i,i+1), …, M(i,j-1)
  - and everything in same column below:

    M(i,j), M(i+1,j),…,M(j,j)

Step 3: Identify Order for

- Recall the dependencies between subproblems just found
- Solve the subproblems (i.e., fill in the table entries) this way:
  - go along the diagonal
  - start just above the main diagonal
  - end in the upper right corner (goal)

Order for Solving Subproblems

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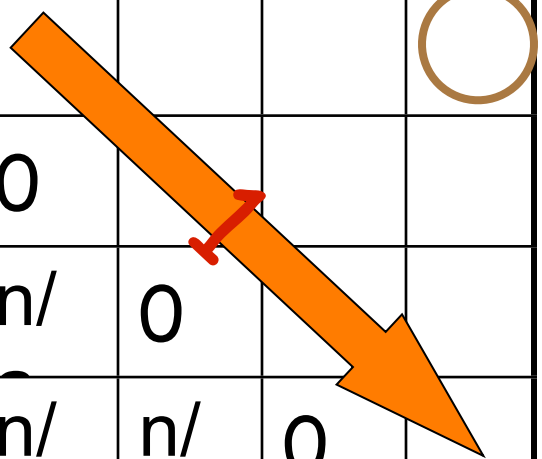

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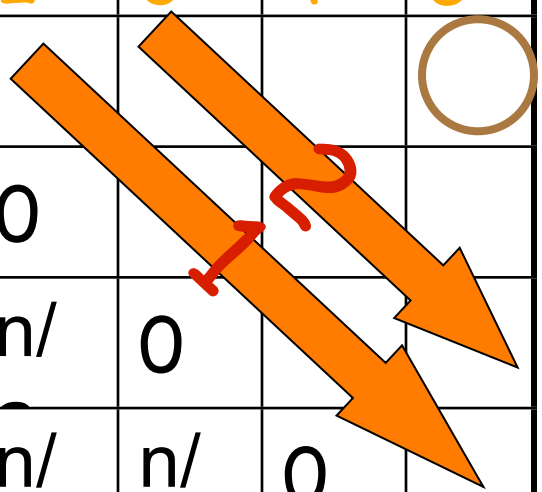

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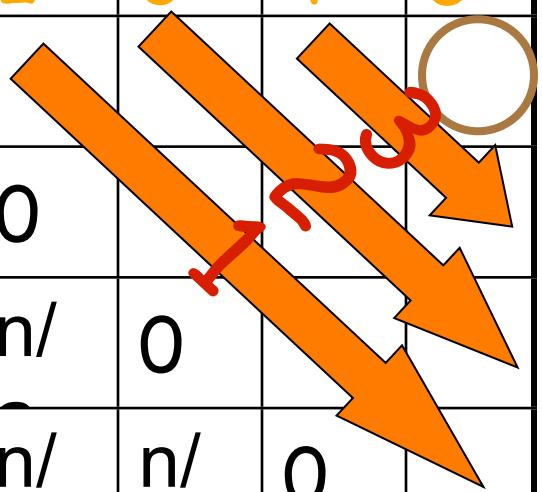

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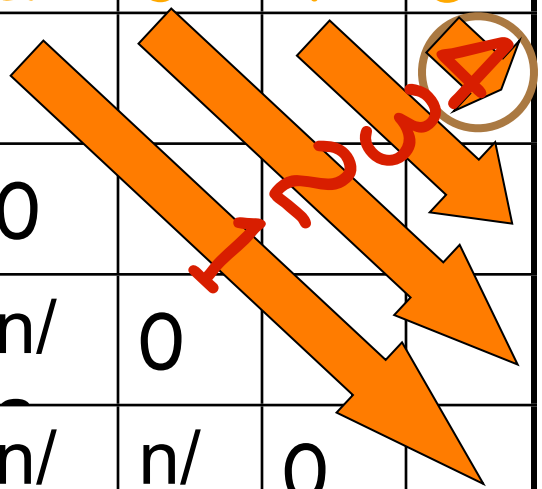

Pseudocode

```
for i := 1 to n do M[i,i] := 0
for d := 1 to n-1 do  // diagonals
   for i := 1 to n-d to // rows w/ an entry on d-th diagonal
      j := i + d           // column corresp. to row i on d-th diagonal
      M[i,j] := infinity
      for k := i to j-1 to
         M[i,j] := min(M[i,j], M[i,k]+M[k+1,j]+d_{i-1}d_k d_j)
      endfor
   endfor
endfor
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running time $O(n^3)$

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      endfor
   endfor
endfor
```

pay attention here to remember actual sequence of mults.

running time $O(n^3)$

Example

M:

	1	2	3	4
1	0	1200	700	1400
2	n/a	0	400	650
3	n/a	n/a	0	10,000
4	n/a	n/a	n/a	0

1: A is 30x1
2: B is 1x40
3: C is 40x10
4: D is 10x25

Keeping Track of the Order

- It's fine to know the cost of the cheapest order, but what is that cheapest order?
- Keep another array S and update it when computing the minimum cost in the inner loop
- After M and S have been filled in, then call a recursive algorithm on S to print out the actual order

Modified Pseudocode

```
for i := 1 to n do M[i,i] := 0
for d := 1 to n-1 do  // diagonals
   for i := 1 to n-d to // rows w/ an entry on d-th diagonal
      j := i + d          // column corresponding to row i on d-th diagonal
      M[i,j] := infinity
      for k := i to j-1 to
        M[i,j] := min(M[i,j], M[i,k]+M[k+1,j]+d_{i-1}d_k d_j)
          if previous line changed value of M[i,j] then S[i,j] := k
      endfor
   endfor
endfor
```

keep track of cheapest split point found so far: between A_k and A_{k+1}

Example

M:

S:

	1	2	3	4
1	0	1200 1	700 1	1400 1
2	n/a	0	400 2	650 3
3	n/a	n/a	0	10,000 3
4	n/a	n/a	n/a	0

1: A is 30x1
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Using S to Print Best Ordering

Call Print(S,1,n) to get the entire ordering.

```
Print(S,i,j):
   if i = j then output "A" + i   //+ is string concat
   else
      k := S[i,j]
      output "(" + Print(S,i,k) + Print(S,k+1,j) + ")"
```