Calculus of Finite Differences

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Motivation

When we analyze the runtime of algorithms, we simply count the number of operations. For example, the following loop

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for k = 1 to n do
  square(k);
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where square(k) is a function that has running time T_2k^2. Then the total number of instructions is given by

$$T_1(n+1) + \sum_{k=1}^{n} T_2 k^2$$

where T_1 is the time for loop increment and comparison.

Motivation

The question is how to find closed form representations of sums such as

$$\sum_{k=1}^{n} k^2$$

Of course, you can look up this particular sum. Perhaps you can even guess the solution and prove it by induction. However, neither of these “methods” are entirely satisfactory.

Motivation

The sum

$$\sum_{k=a}^{b} g(k)$$

may be regarded as a discrete analogue of the integral

$$\int_a^b g(x)dx$$

We can evaluate the integral by finding a function $f(x)$ such that $\frac{d}{dx}f(x) = g(x)$, since the fundamental theorem of calculus yields

$$\int_a^b g(x)dx = f(b) - f(a).$$

Motivation

We would like to find a result that is analogous to the fundamental theorem of calculus for sums. The calculus of finite differences will allow us to find such a result.

Some benefits:

- Closed form evaluation of certain sums.

- The calculus of finite differences will explain the real meaning of the Harmonic numbers (and why they occur so often in the analysis of algorithms).

Difference Operator

Given a function $g(n)$, we define the difference operator Δ as

$$\Delta g(n) = g(n+1) - g(n)$$

Let E denote the shift operator $Eg(n) = g(n+1)$, and I the identity operator. Then

$$\Delta = E - I$$

Examples

a) Let $f(n) = n$. Then

$$\Delta f(n) = n + 1 - n = 1.$$

b) Let $f(n) = n^2$. Then

$$\Delta f(n) = (n+1)^2 - n^2 = 2n + 1.$$

c) Let $f(n) = n^3$. Then

$$\Delta f(n) = (n+1)^3 - n^3 = 3n^2 + 3n + 1.$$

Falling Power

We define the m-th falling power of n as

$$n^{\underline{m}} = n(n-1)\cdots(n-m+1)$$

for $m \geq 0$. We have

$$\Delta n^{\underline{m}} = m\, n^{\underline{m-1}}.$$

Falling Power

Theorem. We have

$$\Delta n^{\underline{m}} = m\, n^{\underline{m-1}}.$$

Proof. By definition,

$$\begin{aligned}\Delta n^{\underline{m}} = (n+1)n\cdots(n-m+2) \\ -n\cdots(n-m+2)(n-m+1) \\ = \quad mn\cdots(n-m+2)\end{aligned}$$

Negative Falling Powers

Since

$$n^{\underline{m}}/n^{\underline{m-1}} = (n - m + 1),$$

we have

$$n^{\underline{2}}/n^{\underline{1}} = n(n-1)/n = (n-1),$$

$$n^{\underline{1}}/n^{\underline{0}} = n/1 = n$$

so we expect that

$$n^{\underline{0}}/n^{\underline{-1}} = n + 1$$

holds, which implies that

$$n^{\underline{-1}} = 1/(n+1).$$

Negative

Similarly, we want

$$n^{\underline{-1}}/n^{\underline{-2}} = n + 2$$

so

$$n^{\underline{-2}} = \frac{1}{(n+1)(n+2)}$$

We define

$$n^{\underline{-m}} = \frac{1}{(n+1)(n+2)\cdots(n+m)}$$

Exercise

Show that for $m \geq 0$, we have

$$\Delta n^{\underline{-m}} = -mn^{\underline{-m-1}}$$

Exponentials

Let $c \neq 1$ be a fixed real number. Then

$$\Delta c^n = c^{n+1} - c^n = (c-1)c^n.$$

In particular,

$$\Delta 2^n = 2^n.$$

Antidifference

A function $f(n)$ with the property that

$$\Delta f(n) = g(n)$$

is called the antidifference of the function $g(n)$.

Example. The antidifference of the function $g(n) = n^{\underline{m}}$ is given by

$$f(n) = \frac{1}{m+1} n^{\underline{m+1}}.$$

Antidifference

Example. The antidifference of the function $g(n) = c^n$ is given by

$$f(n) = \frac{1}{c-1} c^n.$$

Indeed,

$$\Delta f(n) = \frac{1}{c-1}(c^{n+1} - c^n) = c^n.$$

Fundamental Theorem of FDC

Theorem. Let $f(n)$ be an antiderivative of $g(n)$. Then

$$\sum_{n=a}^{b} g(n) = f(b+1) - f(a).$$

Proof. We have

$$\begin{aligned}
\sum_{n=a}^{b} g(n) &= \sum_{n=a}^{b} \Delta f(n) \\
&= \sum_{n=a}^{b} (f(n+1) - f(n)) \\
&= \sum_{n=a+1}^{b+1} f(n) - \sum_{n=a}^{b} f(n) = f(b+1) - f(a).
\end{aligned}$$

Example 1

Suppose we want to find a closed form for the sum

$$\sum_{n=5}^{64} c^n.$$

An antiderivative of c^n is $\frac{1}{c-1}c^n$. Therefore, by the fundamental theorem of finite difference, we have

$$\sum_{n=5}^{64} c^n = \frac{1}{c-1}c^n \Big|_5^{65} = \frac{c^{65} - c^5}{c - 1}$$

Antidifference

We are going to denote an antidifference of a function $f(n)$ by

$$\sum f(n)\,\delta n.$$

The δn plays the same role as the dx term in integration.

For example,

$$\sum n^{\underline{m}}\,\delta n = \frac{1}{m+1} n^{\underline{m+1}}$$

when $m \neq -1$. What about $m = -1$?

Harmonic Numbers = Discrete ln

We have

$$\sum n^{\underline{-1}}\,\delta n = H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Indeed,

$$\Delta H_n = H_{n+1} - H_n = \frac{1}{n+1} = n^{\underline{-1}}.$$

Thus, the antidifference of $n^{\underline{-1}}$ is H_n.

Linearity

Let $f(n)$ and $g(n)$ be two sequences and a and b two constants. Then

$$\Delta(af(n) + bg(n)) = a\,\Delta f(n) + b\,\Delta g(n).$$

Consequently, the antidifferences are linear as well:

$$\sum(af(n) + bg(n))\,\delta n = a\sum f(n)\,\delta n + b\sum g(n)\,\delta n$$

Example

To solve our motivating example, we need to find a closed form for the sum

$$\sum_{k=1}^{n} k^2.$$

Since $k^2 = k^{\underline{2}} + k^{\underline{1}}$, an antiderivative of k^2 is given by

$$\sum k^2\,\delta k = \sum (k^{\underline{2}} + k^{\underline{1}})\delta k = \frac{1}{3}k^{\underline{3}} + \frac{1}{2}k^{\underline{2}}.$$

Thus, the sum

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}k^{\underline{3}}\Big|_{1}^{n+1} + \frac{1}{2}k^{\underline{2}}\Big|_{1}^{n+1} = \ldots = \frac{n(2n+1)(n+1)}{6}.$$

Binomial Coefficients

By Pascal's rule for binomial coefficients, we have

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

Therefore,

$$\Delta \binom{n}{k+1} = \binom{n}{k}.$$

In other words,

$$\sum \binom{n}{k} \delta n = \binom{n}{k+1}.$$

For example, this shows that

$$\sum_{n=0}^{m} \binom{n}{k} = \binom{m+1}{k+1} - \binom{0}{k+1} = \binom{m+1}{k+1}.$$