Undecidability

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[based on slides by Prof. Welch]

Sources

- Theory of Computing, A Gentle Introduction, by E. Kinber and C. Smith, Prentice-Hall, 2001
- Automata Theory, Languages and Computation, 3rd Ed., by J. Hopcroft, R. Motwani, and J. Ullman, 2007

Understanding Limits of Computing

- So far, we have studied how efficiently various problems can be solved.
- There has been no question as to whether it is possible to solve the problem
- If we want to explore the boundary between what can and what cannot be computed, we need a model of computation

Models of Computation

- Need a way to clearly and unambiguously specify how computation takes place
- Many different mathematical models have been proposed:
  - Turing Machines
  - Random Access Machines
  - …
- They have all been found to be equivalent!

Church-Turing Thesis

- Conjecture: Anything we reasonably think of as an algorithm can be computed by a Turing Machine (specific formal model).
- So we might as well think in our favorite programming language, or in pseudocode.
- Frees us from the tedium of having to provide boring details
    - in principle, pseudocode descriptions can be converted into some appropriate formal model

Short Review of some Basic Set Theory Concepts

Some Notation

If A and B are sets, then the set of all functions from A to B is denoted by B^A.

If A is a set, then P(A) denotes the power set, i.e., P(A) is the set of all subsets of A.

Cardinality

Two sets A and B are said to have the same cardinality if and only if there exists a bijective function from A onto B.

[A function is bijective if it is one-to-one and onto]

We write |A|=|B| if A and B have the same cardinality.

[Note that |A|=|B| says that A and B have the same number of elements, even if we do not yet know about numbers!]

How Set Theorists Count

Set theorists count

- 0 = {}   // the empty set exists by axiom This set contains no elements
- 1 = {0} = {{}}   // form the set containing {}   This set contains one element
- 2 = {0,1} = { {}, {{}} }   This set contains two elements
- Keep including all previously created sets as elements of the next set.

Example

Theorem: $|P(X)| = |2^X|$

Proof: The bijection from $P(X)$ onto 2^X is given by the characteristic function. q.e.d.

Example: $X = \{a,b\}$

$\emptyset$	corresponds to	$f(a)=0$, $f(b)=0$
$\{a\}$	corresponds to	$f(a)=1$, $f(b)=0$
$\{b\}$	corresponds to	$f(a)=0$, $f(b)=1$
$\{a,b\}$	corresponds to	$f(a)=1$, $f(b)=1$

More About Cardinality

Let A and B be sets.

We write |A|<=|B| if and only if there exists an injective function from A to B.

We write |A|< |B| if and only if there exist an injective function from A to B, but no bijection exists from A to B.

Cardinality

Cantor's Theorem: Let S be any set. Then |S|<|P(S)|.

Proof: Since the function i from S to P(S) given by i(s)= {s} is injective, we have |S| <= |P(S)|.

Claim: There does not exist any function f from S to P(S) that is surjective.

Indeed, T = $\{ s \in S: s \notin f(s) \}$ is not contained in f(S).

An element s in S is either contained in T or not.

- If $s \in T$, then $s \notin f(s)$ by definition of T. Thus, $T \neq f(s)$.
- If $s \notin T$, then $s \in f(s)$ by definition of T. Thus, $T \neq f(s)$.

Therefore, f is not surjective. This proves the claim.

Uncountable Sets and Uncomputable Functions

Countable Sets

Let N be the set of natural numbers.

A set X is called countable if and only if there exists a surjective function from N onto X.

Thus, finite sets are countable, N is countable, but the set of real numbers is not countable.

An Uncountable Set

Theorem: The set $N^N = \{ f \mid f:N \to N \}$ is not countable.

Proof: We have $|N|<|P(N)|$ by Cantor's theorem. Since $|P(N)|=|2^N|$ and 2^N is a subset of N^N we can conclude that

$$|N| < |P(N)| = | 2^N | \leq | N^N |.$$ q.e.d.

Alternate Proof: The Set N^N is Uncountable

Seeking a contradiction, we assume that the set of functions from N to N is countable.

Let the functions in the set be f_0, f_1, f_2, …

We will obtain our contradiction by defining a function f^d (using "diagonalization") that should be in the set but is not equal to any of the f_i's.

Diagonalization

	0	1	2	3	4	5	6
f_0	4	14	34	6	0	1	2
f_1	55	32	777	3	21	12	8
f_2	90	2	5	21	66	901	2
f_3	4	44	4	7	8	34	28
f_4	80	56	32	12	3	6	7
f_5	43	345	12	7	3	1	0
f_6	0	3	6	9	12	15	18

Diagonalization

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  - $f^d(4) = 3 + 1 = 4$, so $f^d \neq f_4$

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  - $f^d(4) = 3 + 1 = 4$, so $f^d \neq f_4$
  - etc.

Uncomputable Functions Exist!

Consider all programs (in our favorite model) that compute functions in N^N.

The set N^N is uncountable, hence cannot be enumerated.

However, the set of all programs can be enumerated (i.e., is countable).

Thus there must exist some functions in N^N that cannot be computed by a program.

Set of All Programs is Countable

- Fix your computational model (e.g., programming language).
- Every program is finite in length.
- For every integer n, there is a finite number of programs of length n.
- Enumerate programs of length 1, then programs of length 2, then programs of length 3, etc.

Uncomputable Functions

- Previous proof just showed there must exist uncomputable functions
- Did not exhibit any particular uncomputable function
- Maybe the functions that are uncomputable are uninteresting…
- But actually there are some VERY interesting functions (problems) that are uncomputable

The Halting Problem

The Function Halt

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  - in the compiler example, a compiler can be run on its own code

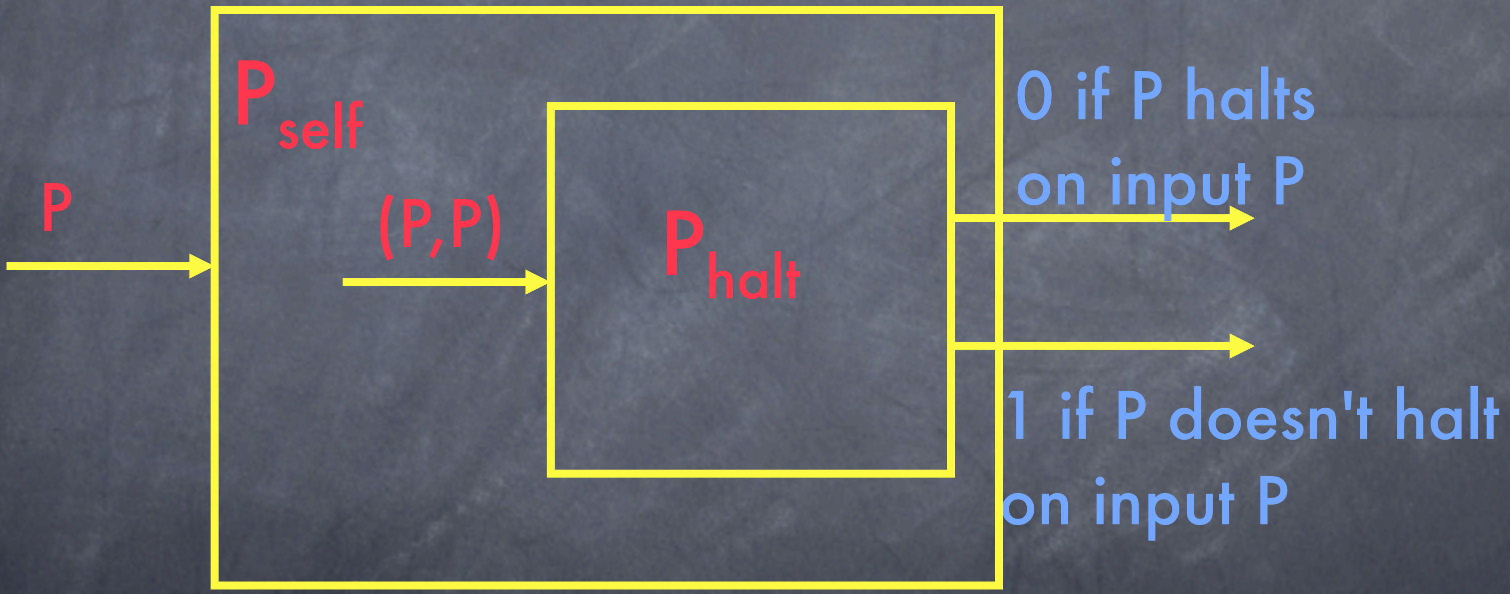
The Function Halt

- We can view Halt as a function from N to N:
  - P and X can be represented in ASCII, which is a string of bits.
  - This string of bits can also be interpreted as a natural number.
- The function Halt would be a useful diagnostic tool in debugging programs

Halt is Uncomputable

- Suppose in contradiction there is a program P_{halt} that computes Halt.
- Use P_{halt} as a subroutine in another program, P_{self}.
- Description of P_{self}:
  - input: code for any program P
  - constructs pair (P,P) and calls P_{halt} on (P,P)
  - returns same answer as P_{halt}

P_{self}

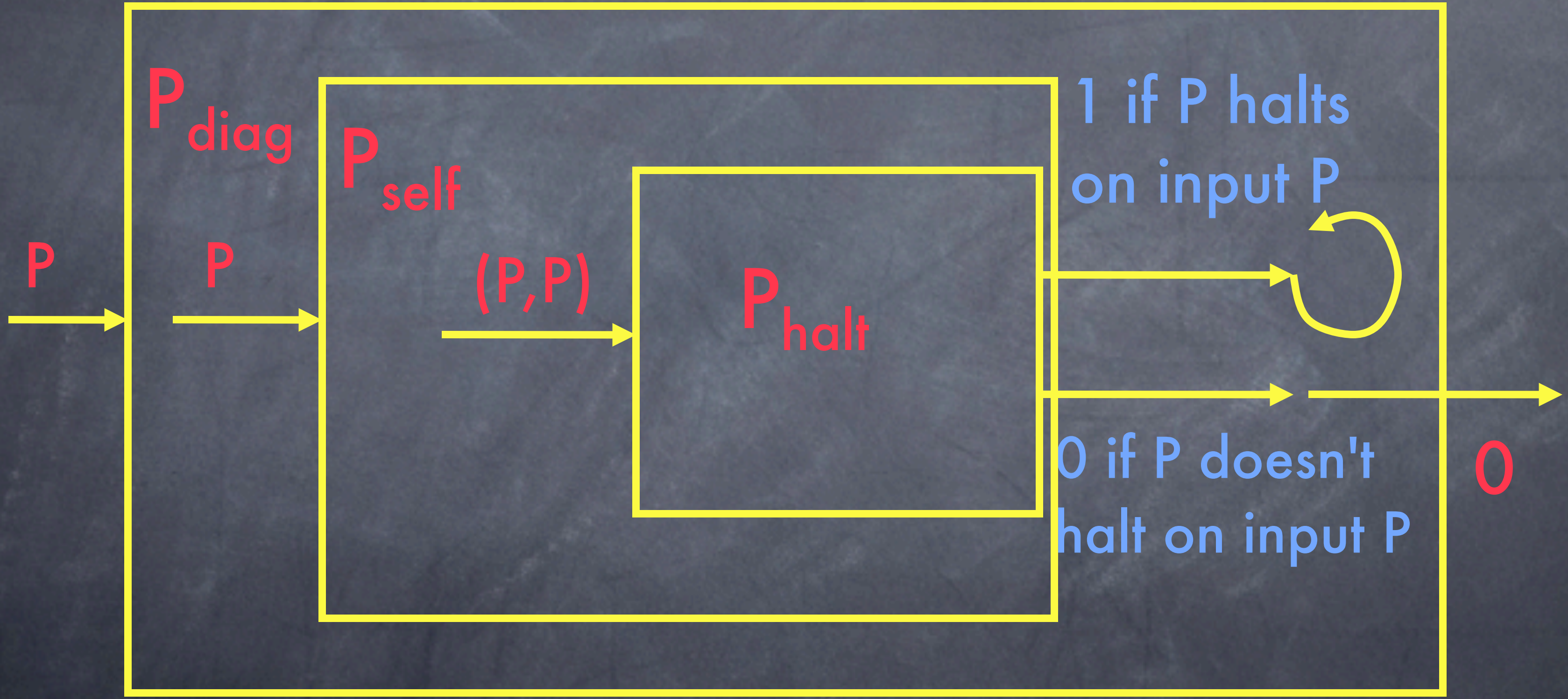


P_{self}
P
(P,P)
P_{halt}
0 if P halts on input P
1 if P doesn't halt on input P

Halt is Uncomputable

- Now use P_{self} as a subroutine inside another program P_{diag}.
- Description of P_{diag}:
  - input: code for any program P
  - call P_{self} on input P
  - if P_{self} returns 1 then go into an infinite loop
  - if P_{self} returns 0 then output 0
- P_{diag} on input P does the opposite of what program P does on input P

P_{diag}


P_{diag}
P_{self}
P_{halt}
P
P
(P,P)
1 if P halts on input P
0 if P doesn't halt on input P
0

Halt is Uncomputable

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Contradiction

Halt is Uncomputable

- What went wrong?
- Our assumption that there is an algorithm to compute Halt was incorrect.
- So there is no algorithm that can correctly determine if an arbitrary program halts on an arbitrary input.

Undecidability

Undecidability

- The analog of an uncomputable function is an undecidable set.
- The theory of what can and cannot be computed focuses on identifying sets of strings:
  - an algorithm is required to "decide" if a given input string is in the set of interest
  - similar to deciding if the input to some NP-complete problem is a YES or NO instance

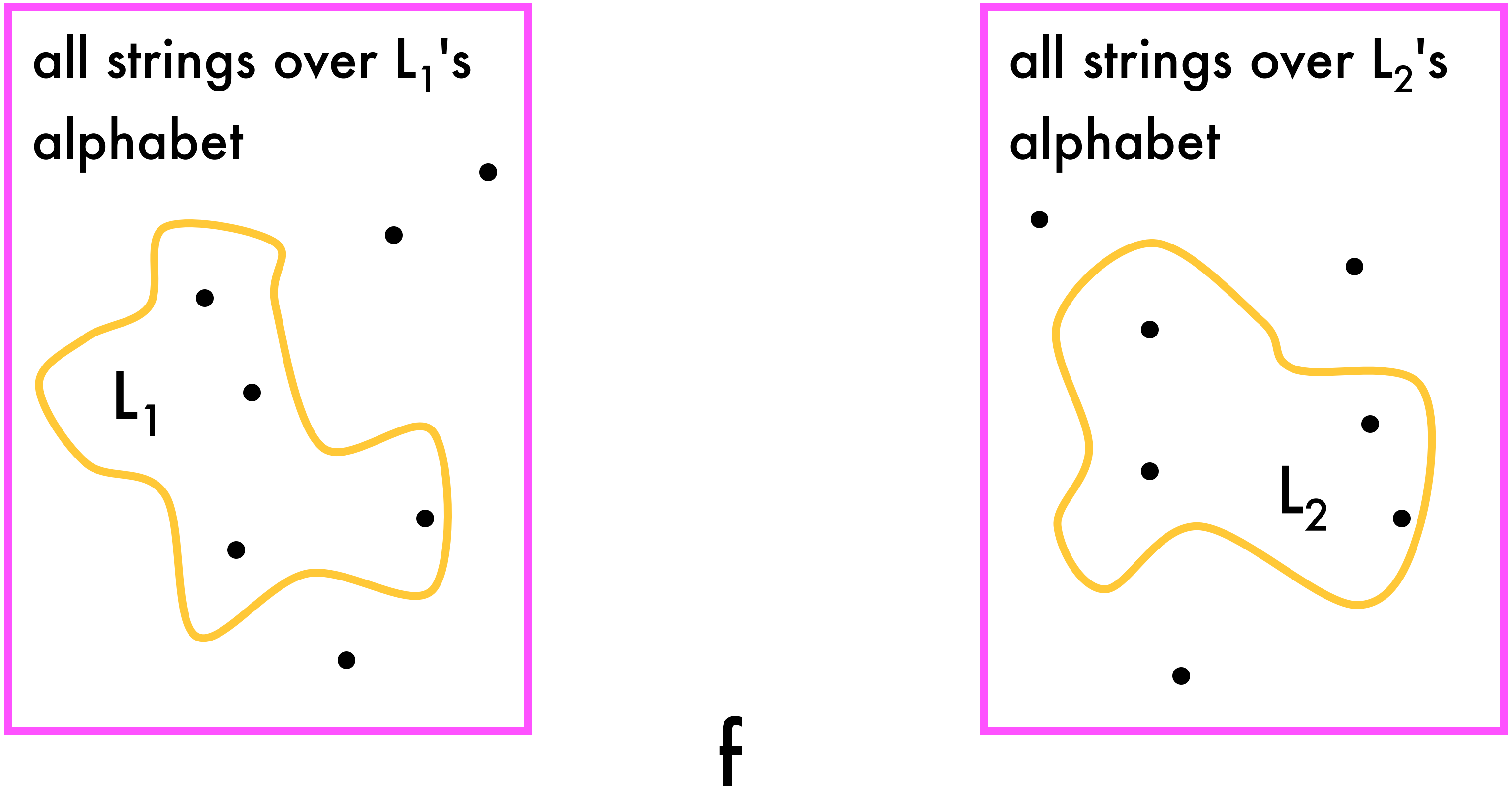
Undecidability

- Recall that a (formal) language is a set of strings, assuming some encoding.
- Analogous to the function Halt is the set H of all strings that encode a program P and an input X such that P halts when executed on X.
- There is no algorithm that can correctly identify for every string whether it belongs to H or not.

More Reductions

- For NP-completeness, we were concerned with (time) complexity of probems:
  - reduction from P1 to P2 had to be fast (polynomial time)
- Now we are concerned with computability of problems:
  - reduction from P1 to P2 just needs to be computable, don't care how slow it is

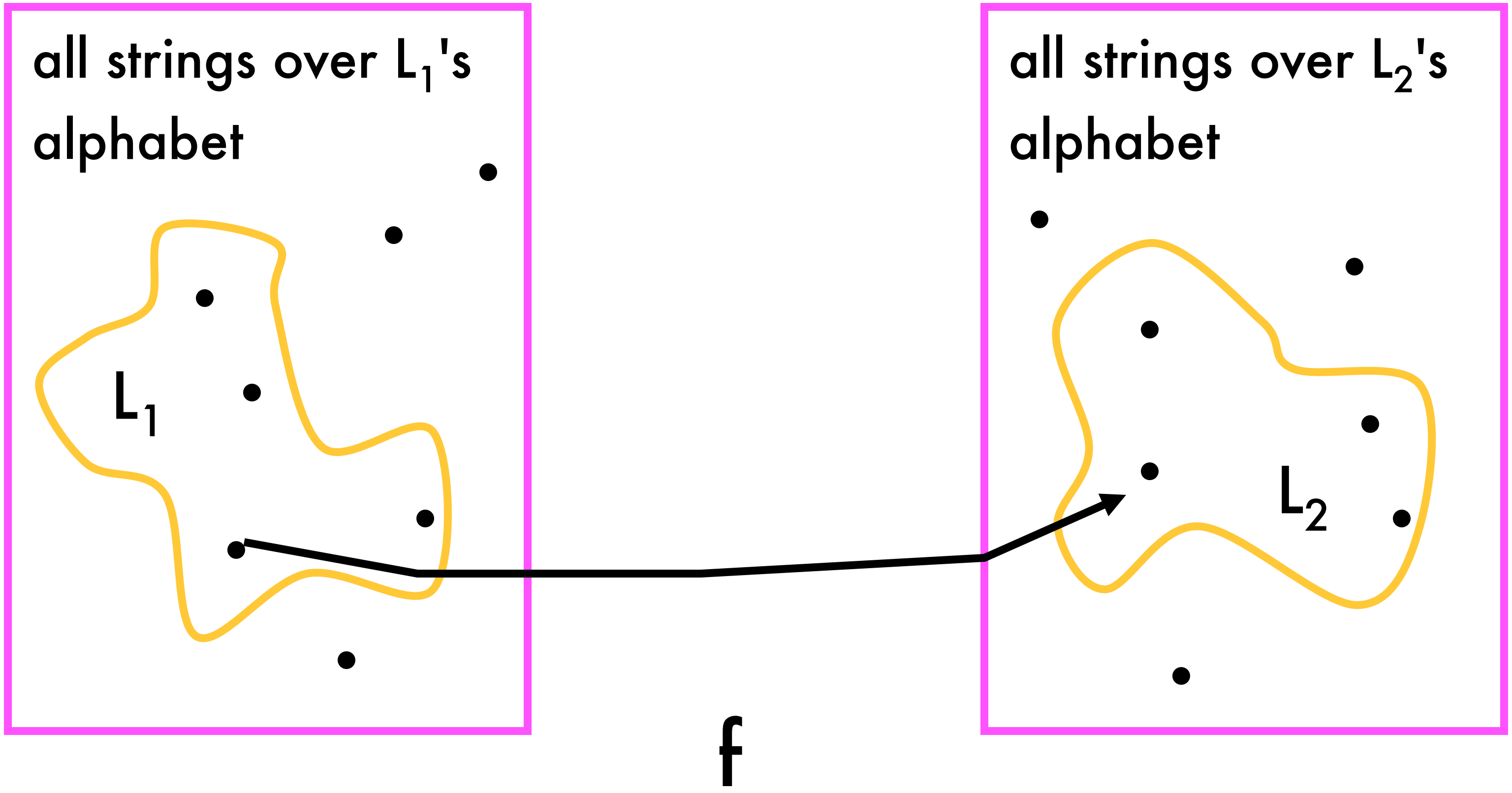
Many-One Reduction



all strings over L_1's alphabet

all strings over L_2's alphabet

Many-One Reduction



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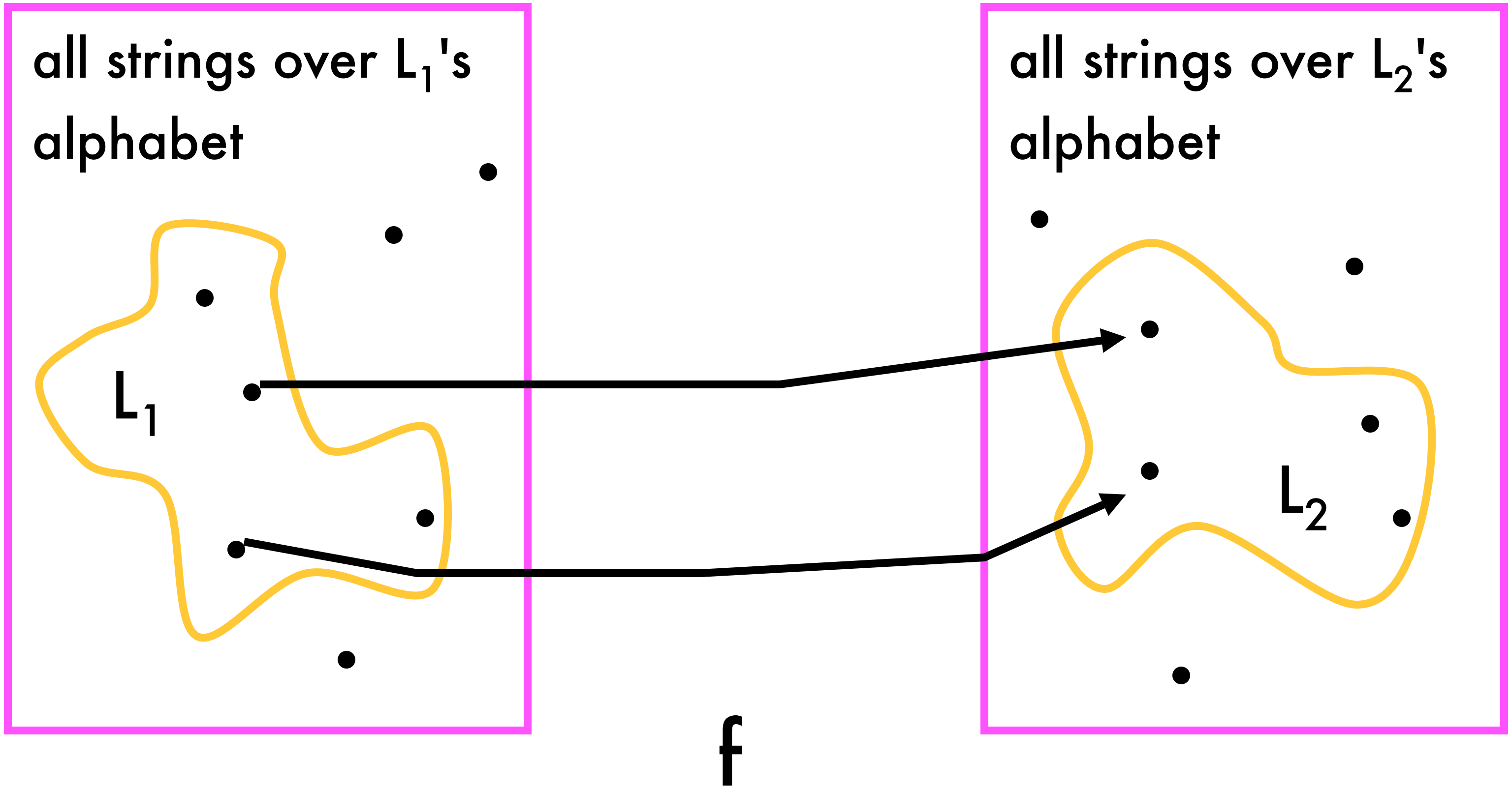

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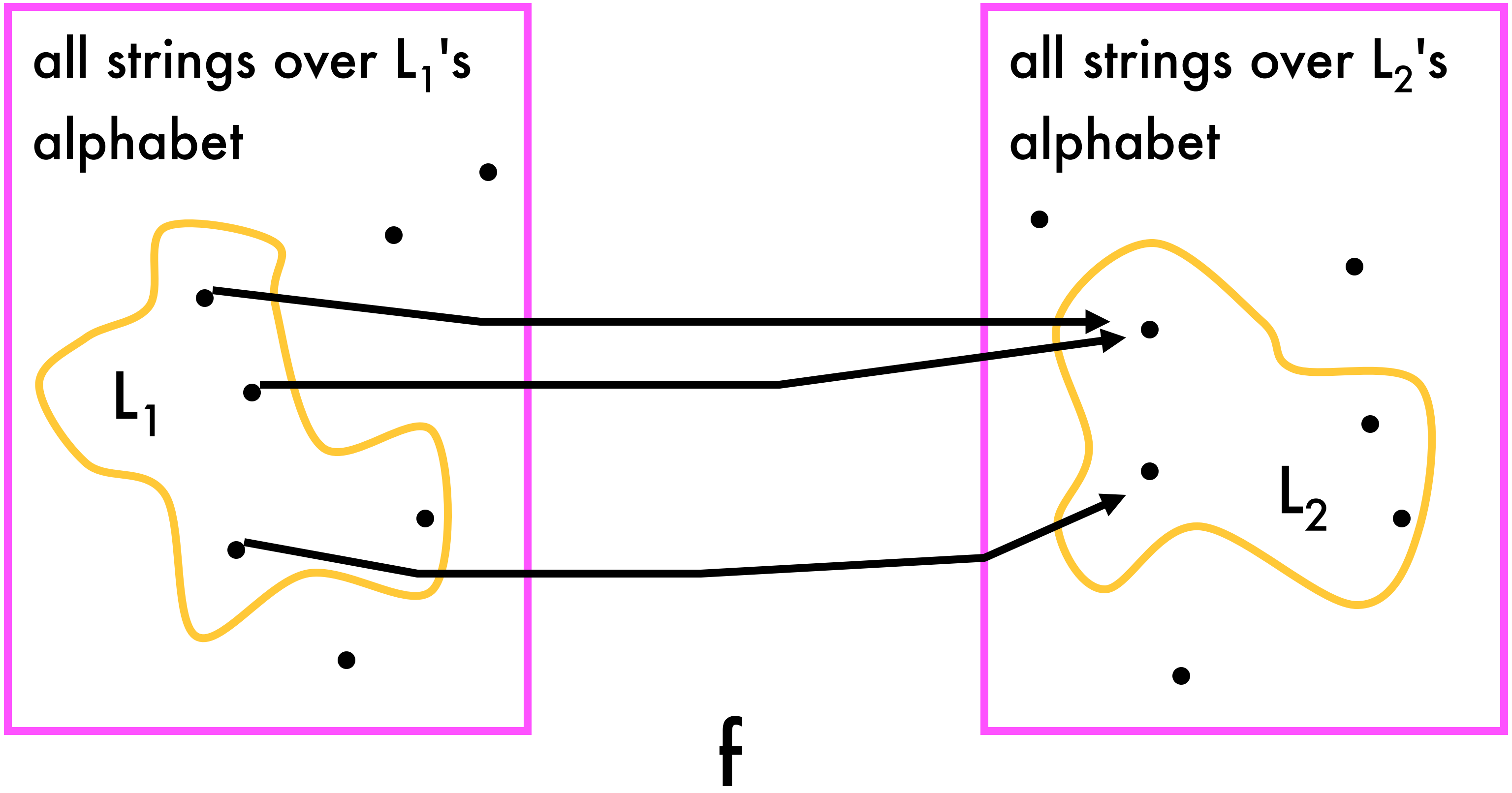

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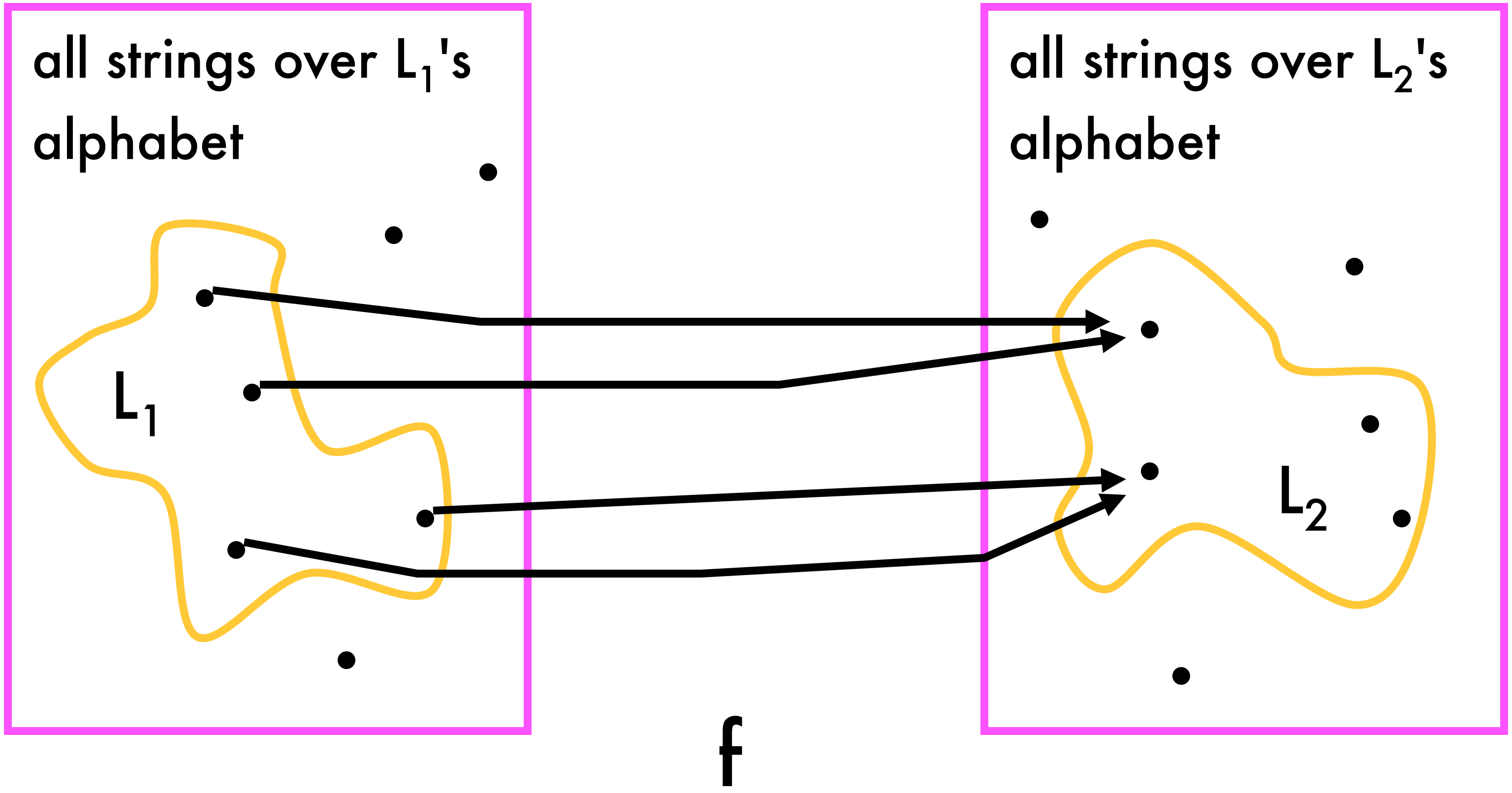

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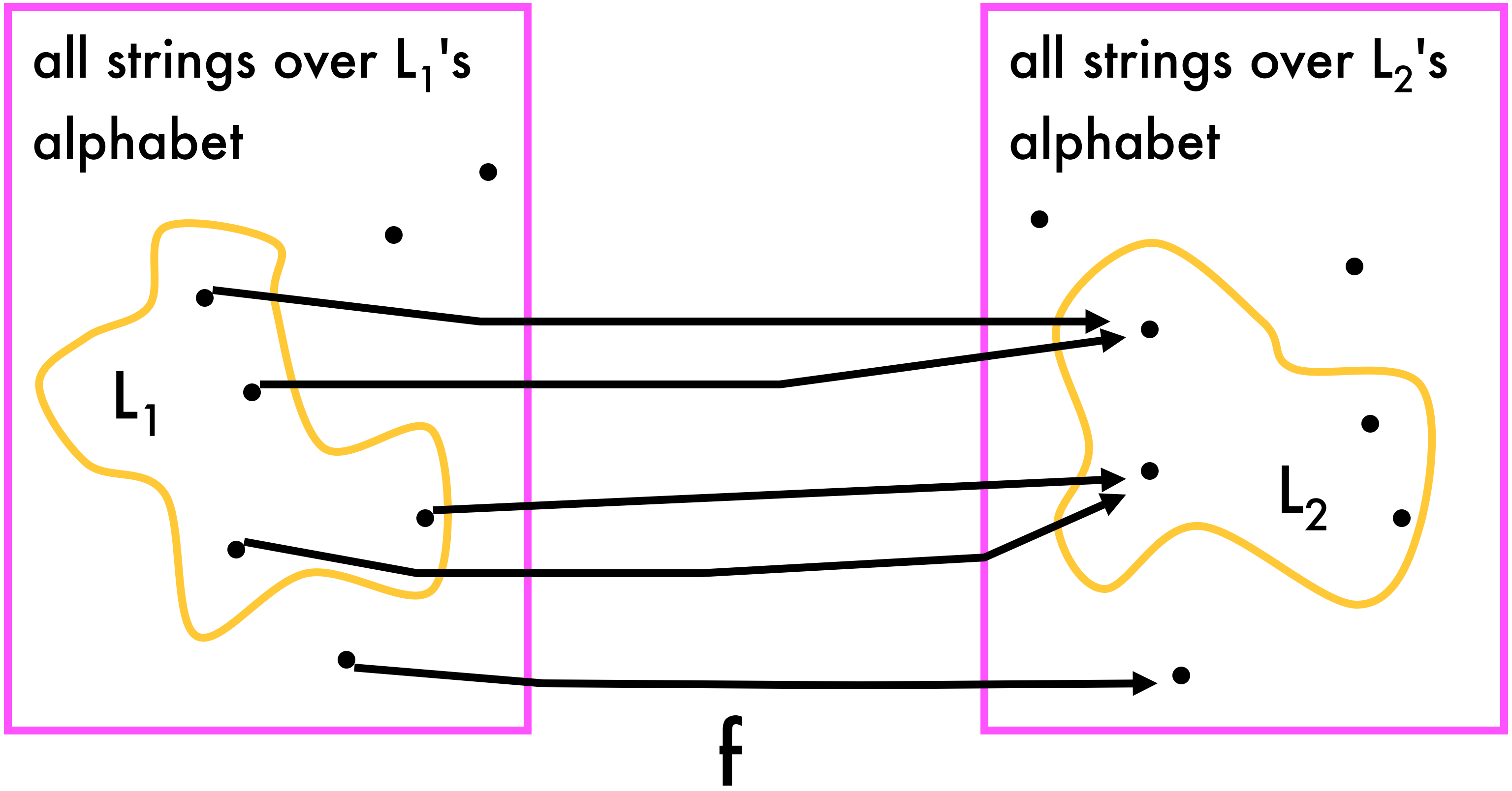

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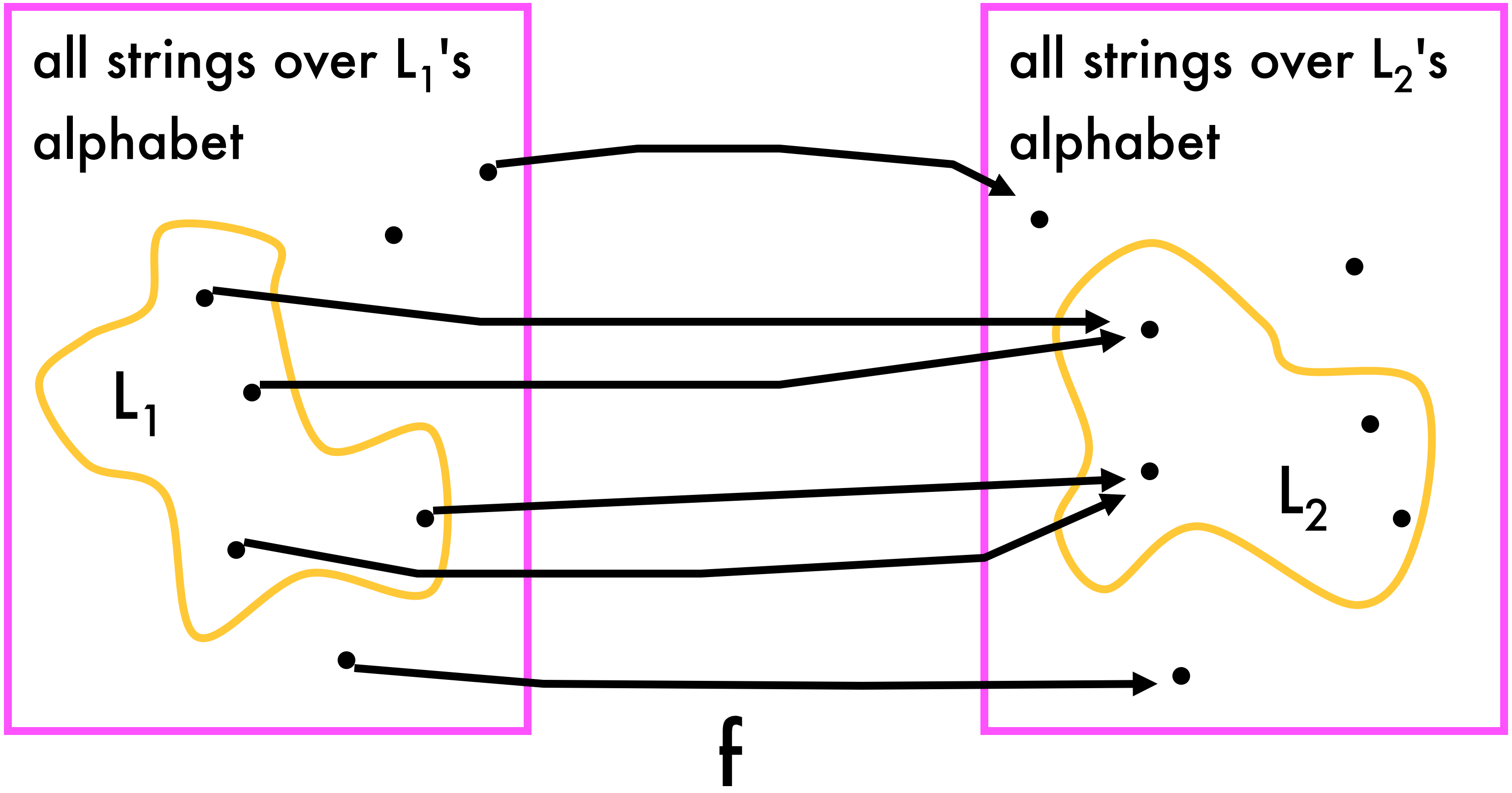

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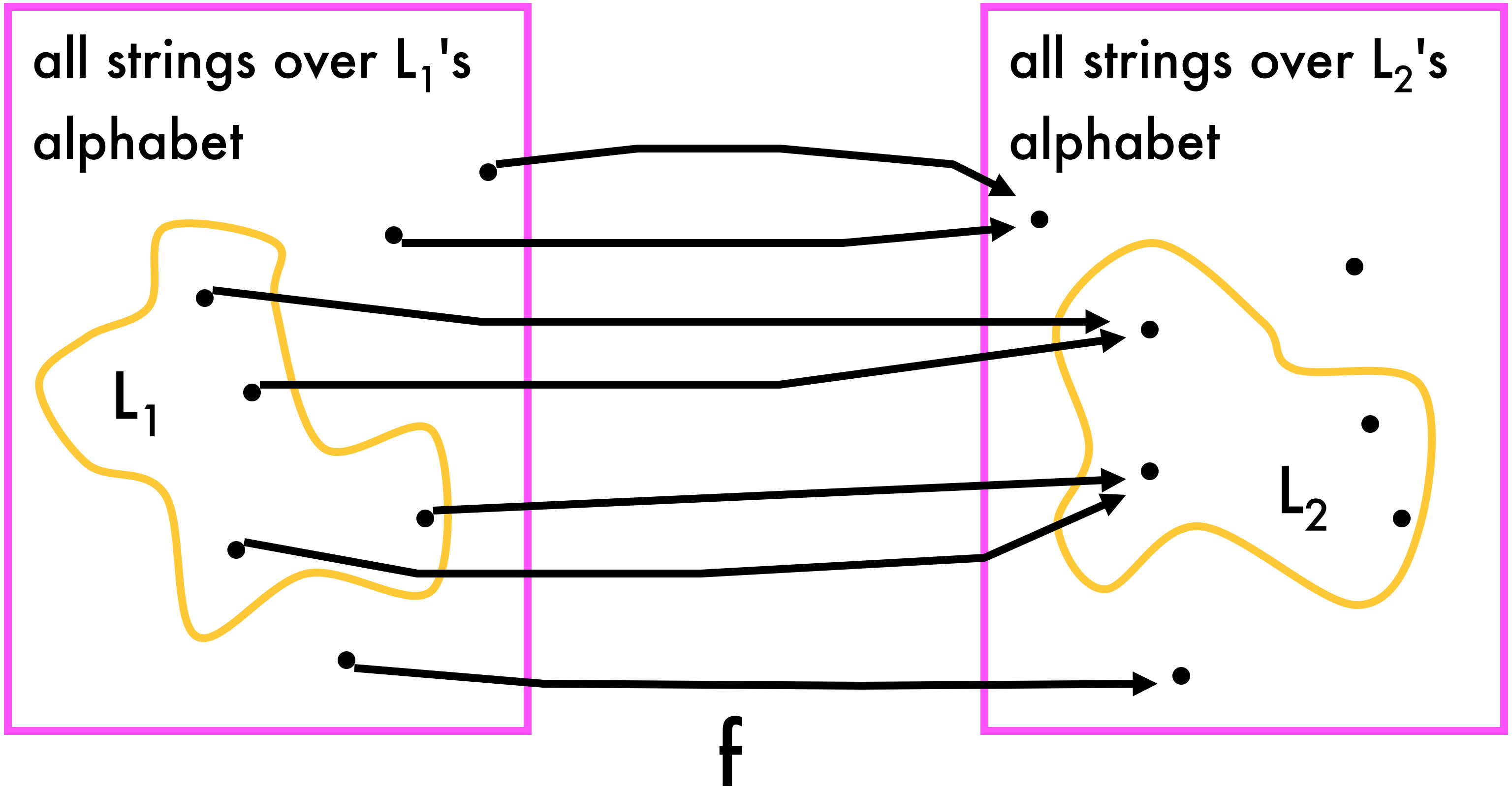

Many-One Reduction

- YES instances map to YES instances
- NO instances map to NO instances
- computable (doesn't matter how slow)
- Notation: $L_1 \leq_m L_2$
- Think: L_2 is at least as hard to compute as L_1

Many-One Reduction Theorem

Theorem: If $L_1 \leq_m L_2$ and L_2 is computable, then L_1 is computable.

Proof: Let f be the many-one reduction from L_1 to L_2. Let A_2 be an algorithm for L_2. Here is an algorithm A_1 for L_1.

- input: x
- compute f(x)
- run A_2 on input f(x)

Implication

- If there is no algorithm for L_1, then there is no algorithm for L_2.
- In other words, if L_1 is undecidable, then L_2 is also undecidable.
- Pay attention to the direction!

Example of a Reduction

- Consider the language L_{NE} consisting of all strings that encode a program that halts (does not go into an infinite loop) on at least one input.
- Use a reduction to show that L_{NE} is not decidable:
  - Show some known undecidable language $\leq_m L_{NE}$.
  - Our only choice for the known undecidable language is H (the language corresponding to the halting problem)
  - So show $H \leq_m L_{NE}$.

Example of a Reduction

- Given an arbitrary H input (encoding of a program P and an input X for P), compute an L_{NE} input (encoding of a program P')
  - such that P halts on input X if and only if P' halts on at least one input.
- Construction consists of writing code to describe P'.
- What should P' do? It's allowed to use P and X

Example of a Reduction

- The code for P' does this:
  - input X':
  - ignore X'
  - call program P on input X
  - if P halts on input X then return whatever P returns
- How does P' behave?
  - If P halts on X, then P' halts on every input
  - If P does not halt on X, then P' does not halt on any input

Example of a Reduction

- Thus if (P,X) is a YES input for H (meaning P halts on input X), then P' is a YES input for L_{NE} (meaning P' halts on at least one input).
- Similarly, if (P,X) is NO input for H (meaning P does not halt on input X), then P' is a NO input for L_{NE} (meaning P' does not halt on even one input)
- Since H is undecidable, and we showed $H \le_m L_{NE}$, L_{NE} is also undecidable.

Generalizing Such Reductions

- There is a way to generalize the reduction we just did, to show that lots of other languages that describe properties of programs are also undecidable.
- Focus just on programs that accept languages (sets of strings):
  - I.e., programs that say YES or NO about their inputs
  - Ex: a compiler tells you YES or NO whether its input is syntactically correct

Properties About Programs

- Define a property about programs to be a set of strings that encode some programs.
  - The "property" corresponds to whatever it is that all the programs have in common
- Example:
  - Program terminates in 10 steps on input y
  - Program never goes into an infinite loop
  - Program accepts a finite number of strings
  - Program contains 15 variables
  - Program accepts 0 or more inputs

Functional Properties

- A property about programs is called functional if it just refers to the language accepted by the program and not about the specific code of the program
  - Program terminates in 10 steps on input y (n.f.)
  - Program never goes into an infinite loop (f.)
  - Program accepts a finite number of strings (f.)
  - Program contains 15 variables (n.f.)

Nontrivial Properties

- A functional property about programs is nontrivial if some programs have the property and some do not
- Example of nontrivial programs:
  - Program never goes into an infinite loop
  - Program accepts a finite number of strings
- Example of a trivial program:
  - Program accepts 0 or more inputs

Rice's Theorem

- Every nontrivial (functional) property about programs is undecidable.
- The proof is a generalization of the reduction shown earlier.
- Very powerful and useful theorem:
  - To show that some property is undecidable, only need to show that is nontrivial and functional, then appeal to Rice's Theorem

Applying Rice's Theorem

- Consider the property "program accepts a finite number of strings".
- This property is functional:
  - it is about the language accepted by the program and not the details of the code of the program
- This property is nontrivial:
  - Some programs accept a finite number of strings (for instance, the program that accepts no input)
  - some accept an infinite number (for instance, the program that accepts every input)
- By Rice's theorem, the property is undecidable.

Implications of Undecidable Program Property

- It is not possible to design an algorithm (write a program) that can analyze any input program and decide whether the input program satisfies the property!
- Essentially all you can do is simulate the input program and see how it behaves
  - but this leaves you vulnerable to an infinite loop