Finding the Second Largest Element

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Problem

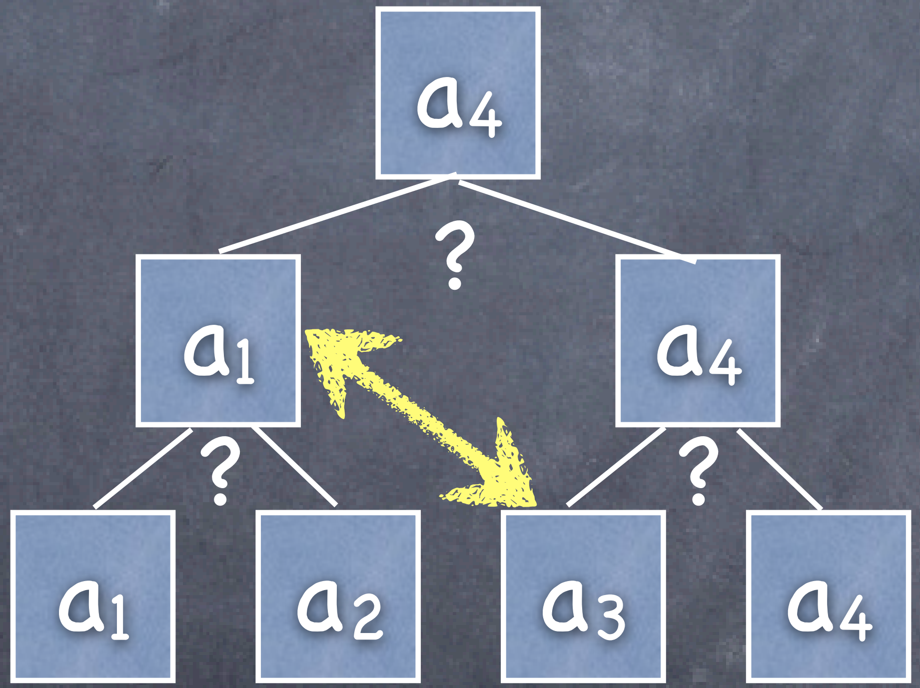
Given a set of n elements from a totally ordered domain, our goal is to find the second largest element m_2.

How many queries are needed to determine m_2?

Upper Bound

We can compare the elements pairwise in a tournament style.

If x < y, then we say that y wins.



- If an element was never compared to the largest element, then it cannot be second largest (e.g. a_2).
- Find second largest among the one's who have lost to the largest.
- So <= $(n-1) + \lceil \lg n \rceil - 1$ comparisons

Lower Bound

Any algorithm to determine the second largest element of a totally ordered set n elements needs at least $(n-2) + \lceil \lg n \rceil$ comparisons in the worst case.

Lower Bound

Let m_1 be the largest element and m_2 the second largest element.

An algorithm to determine m_2 needs to find the largest element m_1 for otherwise an adversary would be able to exchange m_1 for m_2.

Furthermore, the n-2 elements below m_2 must be identified by the algorithm, meaning that they must have lost in comparison to m_2 or some element below m_2. This means that there are n-2 comparisons that do not involve m_1.

It remains to show that an adversary can force any algorithm to do at least $\lceil \lg n \rceil$ comparisons with the largest element m_1.

Adversary 1

Our goal is to show that an algorithm Z needs to make lg n or more comparisons with the largest element.

We construct an adversary that answers comparisons "Is a <= b?" consistent with a total order of the n elements.

For each element x, we let K(x) denote the set of elements y known to Z that satisfy y <= x. Initially K(x) = {x}.

The adversary uses previous query history of Z and K(a) and K(b) to create answer for questions such as "Is a <= b".

Adversary 2

The adversary behaves as follows:

- If "Is $a \leq b$?" was asked before, give same answer.
- If "Is $a \leq b$?" was not asked before, then answer
  - yes if $|K(a)| \leq |K(b)|$. Update $K(b) := K(a) \cup K(b)$
  - no, if $|K(a)| > |K(b)|$. Update $K(a) := K(a) \cup K(b)$

Adversary 3

Let S be the totally ordered domain of n elements.

- At the beginning |K(a)| = 1 holds for all a in S.

- For each query involving a, |K(a)| can at most double.

- Since Z needs to determine largest element, $|K(m_1)| = n$ must hold at the end.

- The number k of queries involving m_1 satisfies $2^k >= n$, so k >= lg n

and since k must be an integer, we have k >= $\lceil \lg n \rceil$.

Conclusions

Any algorithm to determine the second largest element of a totally ordered set n elements needs at least $(n-2) + \lceil \lg n \rceil$ comparisons in the worst case.

We have given an optimal algorithm that attains this lower bound.