Calculus of Finite Differences

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Motivation

When we analyze the runtime of algorithms, we simply count the number of operations. For example, the following loop

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for k = 1 to n do

  square(k);
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where square(k) is a function that has running time $T_2 k^2$. Then the total number of instructions is given by

$$T_1(n+1) + \sum_{k=1}^{n} T_2 k^2$$

where T_1 is the time for loop increment and comparison.

Motivation

The question is how to find closed form representations of sums such as

$$\sum_{k=1}^{n} k^2$$

Of course, you can look up this particular sum. Perhaps you can even guess the solution and prove it by induction. However, neither of these "methods" are entirely satisfactory.

Motivation

The sum

$$\sum_{k=a}^{b} g(k)$$

may be regarded as a discrete analogue of the integral

$$\int_a^b g(x)dx$$

We can evaluate the integral by finding a function $f(x)$ such that $\frac{d}{dx}f(x) = g(x)$, since the fundamental theorem of calculus yields

$$\int_a^b g(x)dx = f(b) - f(a).$$

Motivation

- We would like to find a result that is analogous to the fundamental theorem of calculus for sums. The calculus of finite differences will allow us to find such a result.
- Some benefits:
  - Closed form evaluation of certain sums.
  - The calculus of finite differences will explain the real meaning of the Harmonic numbers (and why they occur so often in the analysis of algorithms).

Difference Operator

The discrete version of the differential operator

Difference Operator

Given a function $g(n)$, we define the difference operator Δ as

$$\Delta g(n) = g(n+1) - g(n)$$

Let E denote the shift operator $Eg(n) = g(n+1)$, and I the identity operator. Then

$$\Delta = E - I$$

Examples

a) Let $f(n) = n$. Then

$$\Delta f(n) = n + 1 - n = 1.$$

b) Let $f(n) = n^2$. Then

$$\Delta f(n) = (n+1)^2 - n^2 = 2n + 1.$$

c) Let $f(n) = n^3$.Then

$$\Delta f(n) = (n+1)^3 - n^3 = 3n^2 + 3n + 1.$$

Falling Power

We define the m-th falling power of n as

$$n^{\underline{m}} = n(n-1)\cdots(n-m+1)$$

for $m \geq 0$. We have

$$\Delta n^{\underline{m}} = m\, n^{\underline{m-1}}.$$

Falling Power

Theorem. We have

$$\Delta n^{\underline{m}} = m\, n^{\underline{m-1}}.$$

Proof. By definition,

$$\begin{aligned}\Delta n^{\underline{m}} = (n+1)n\cdots(n-m+2)& \\ -n\cdots(n-m+2)(n-m+1)& \\ = \qquad mn\cdots(n-m+2)&\end{aligned}$$

Negative Falling Powers

Since

$$n^{\underline{m}}/n^{\underline{m-1}} = (n-m+1),$$

we have

$$n^{\underline{2}}/n^{\underline{1}} = n(n-1)/n = (n-1),$$

$$n^{\underline{1}}/n^{\underline{0}} = n/1 = n$$

so we expect that

$$n^{\underline{0}}/n^{\underline{-1}} = n+1$$

holds, which implies that

$$n^{\underline{-1}} = 1/(n+1).$$

Negative Falling Powers

Similarly, we want

$$n^{\underline{-1}}/n^{\underline{-2}} = n+2$$

so

$$n^{\underline{-2}} = \frac{1}{(n+1)(n+2)}$$

We define

$$n^{\underline{-m}} = \frac{1}{(n+1)(n+2)\cdots(n+m)}$$

Exercise

Show that for $m \geq 0$, we have

$$\Delta n^{\underline{-m}} = -mn^{\underline{-m-1}}$$

Exponentials

Let $c \neq 1$ be a fixed real number. Then

$$\Delta c^n = c^{n+1} - c^n = (c-1)c^n.$$

In particular,

$$\Delta 2^n = 2^n.$$

Antidifference Operator

The discrete version of an indefinite integral

Antidifference

A function $f(n)$ with the property that

$$\Delta f(n) = g(n)$$

is called the antidifference of the function $g(n)$.

Example. The antidifference of the function $g(n) = n^{\underline{m}}$ is given by

$$f(n) = \frac{1}{m+1} n^{\underline{m+1}}.$$

Antidifference

Example. The antidifference of the function $g(n) = c^n$ is given by

$$f(n) = \frac{1}{c-1} c^n.$$

Indeed,

$$\Delta f(n) = \frac{1}{c-1}(c^{n+1} - c^n) = c^n.$$

Fundamental Theorem of FDC

Theorem. Let $f(n)$ be an antiderivative of $g(n)$. Then

$$\sum_{n=a}^{b} g(n) = f(b+1) - f(a).$$

Proof. We have

$$\begin{aligned}
\sum_{n=a}^{b} g(n) &= \sum_{n=a}^{b} \Delta f(n) \\
&= \sum_{n=a}^{b} (f(n+1) - f(n)) \\
&= \sum_{n=a+1}^{b+1} f(n) - \sum_{n=a}^{b} f(n) = f(b+1) - f(a).
\end{aligned}$$

Example 1

Suppose we want to find a closed form for the sum

$$\sum_{n=5}^{64} c^n.$$

An antiderivative of c^n is $\frac{1}{c-1}c^n$. Therefore, by the fundamental theorem of finite difference, we have

$$\sum_{n=5}^{64} c^n = \frac{1}{c-1}c^n\Big|_5^{65} = \frac{c^{65}-c^5}{c-1}$$

Antidifference

We are going to denote an antidifference of a function $f(n)$ by

$$\sum f(n)\,\delta n.$$

The δn plays the same role as the dx term in integration.

For example,

$$\sum n^{\underline{m}}\,\delta n = \frac{1}{m+1} n^{\underline{m+1}}$$

when $m \neq -1$. What about $m = -1$?

Harmonic Numbers = Discrete ln

We have

$$\sum n^{\underline{-1}}\,\delta n = H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Indeed,

$$\Delta H_n = H_{n+1} - H_n = \frac{1}{n+1} = n^{\underline{-1}}.$$

Thus, the antidifference of $n^{\underline{-1}}$ is H_n.

Linearity

Let $f(n)$ and $g(n)$ be two sequences and a and b two constants. Then

$$\Delta(af(n) + bg(n)) = a\,\Delta f(n) + b\,\Delta g(n).$$

Consequently, the antidifferences are linear as well:

$$\sum (af(n) + bg(n))\,\delta n = a \sum f(n)\,\delta n + b \sum g(n)\,\delta n$$

Example

To solve our motivating example, we need to find a closed form for the sum

$$\sum_{k=1}^{n} k^2.$$

Since $k^2 = k^{\underline{2}} + k^{\underline{1}}$, an antiderivative of k^2 is given by

$$\sum k^2\,\delta k = \sum (k^{\underline{2}} + k^{\underline{1}})\delta k = \frac{1}{3}k^{\underline{3}} + \frac{1}{2}k^{\underline{2}}.$$

Thus, the sum

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}k^{\underline{3}}\Big|_1^{n+1} + \frac{1}{2}k^{\underline{2}}\Big|_1^{n+1} = \ldots = \frac{n(2n+1)(n+1)}{6}.$$

Binomial Coefficients

By Pascal's rule for binomial coefficients, we have

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

Therefore,

$$\Delta\binom{n}{k+1} = \binom{n}{k}.$$

In other words,

$$\sum \binom{n}{k}\,\delta n = \binom{n}{k+1}.$$

For example, this shows that

$$\sum_{n=0}^{m} \binom{n}{k} = \binom{m+1}{k+1} - \binom{0}{k+1} = \binom{m+1}{k+1}.$$

Partial Summation

Example

$$\sum_{k=0}^{n} k2^k = ?$$

Product Rule

$$\begin{aligned}
\Delta(f(n)g(n)) &= f(n+1)g(n+1) - f(n)g(n) \\
&= f(n+1)g(n+1) - f(n)g(n+1) \\
&\qquad + f(n)g(n+1) - f(n)g(n) \\
&= (\Delta f(n))g(n+1) + f(n)(\Delta g(n)) \\
&= (\Delta f(n))(Eg(n)) + f(n)(\Delta g(n))
\end{aligned}$$

Partial Summation

$$\sum f(n)(\Delta g(n))\,\delta n = f(n)g(n) - \sum (\Delta f(n))(Eg(n))$$

Example

We are now going to find a closed form for the sum

$$\sum_{k=0}^{n} k2^k$$

Set $f(k) = k$ and $\Delta g(k) = 2^k$, so that $g(k) = 2^k$.
Then by partial summation, we have

$$\begin{aligned}
\sum_{k=0}^{n} k2^k &= \sum_{k=0}^{n} f(k)\,\Delta g(k) \\
&= f(k)g(k)\Big|_{0}^{n+1} - \sum_{k=0}^{n}(\Delta f(k))E(g(k)) \\
&= k2^k\Big|_{0}^{n+1} - \sum_{k=0}^{n} 1 \cdot 2^{k+1} \\
&= k2^k + 2^{k+1}\Big|_{0}^{n+1} \\
&= (n+1)2^{n+1} - 2^{n+2} - (0 \cdot 2^0 - 2)
\end{aligned}$$

References

- D. Gleich: Finite Calculus: A Tutorial for Solving Nasty Sums
- Graham, Knuth, Patashnik: Concrete Mathematics, Addison Wesley,
- Ch. Jordan: Calculus of finite differences, AMS Chelsea, 1965.