NP Completeness

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[partially based on slides by Jennifer Welch]

Overview

We already know the following examples of NPC problems:

- SAT
- 3SAT

We are going to show that the following are NP complete:

- Vertex Cover
- Clique
- Independent Set

Vertex Cover

Vertex Cover of a Graph

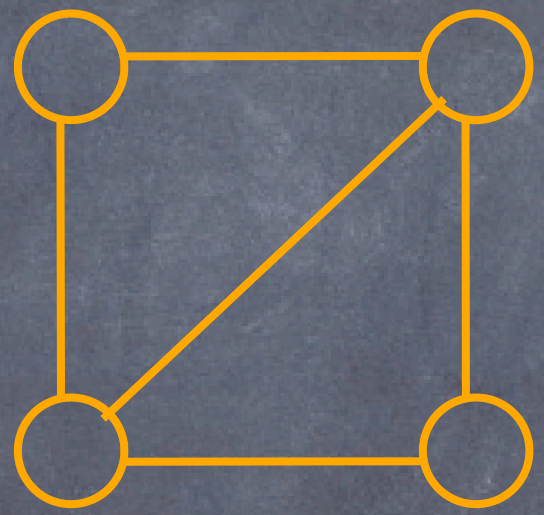
Given: An undirected graph G = (V,E)

A subset C of V is called a vertex cover if every edge in E is incident with C.

Easy: Find a big vertex cover.

Difficult: Find a small vertex cover.

Vertex Cover Example



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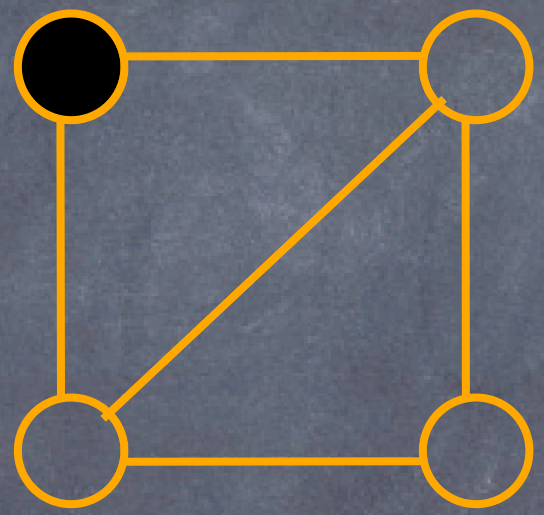

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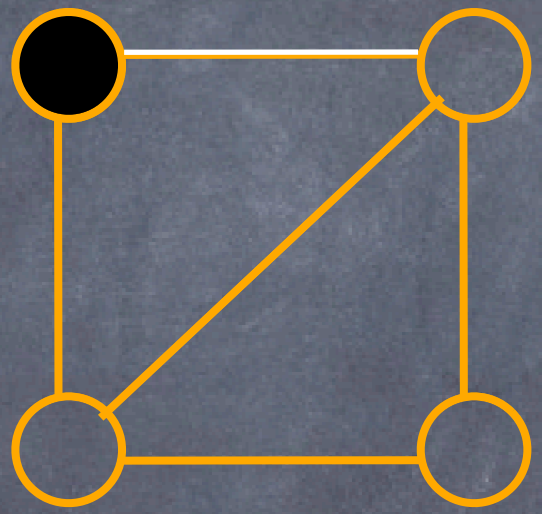

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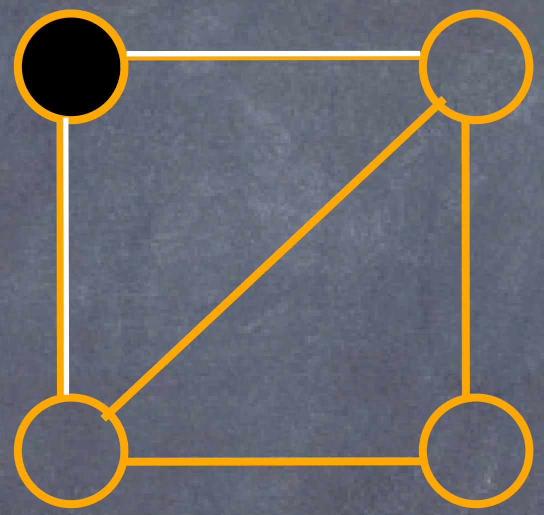

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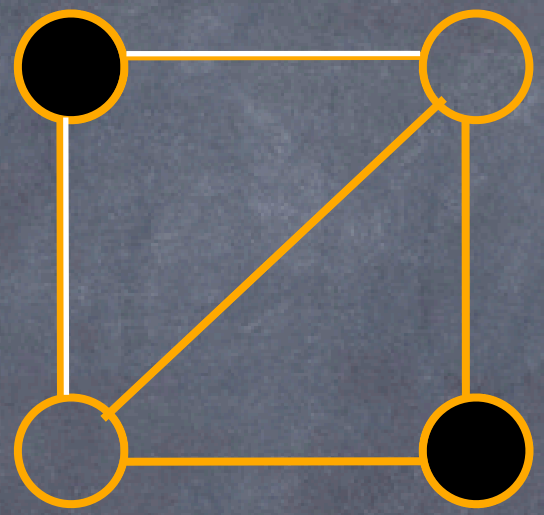

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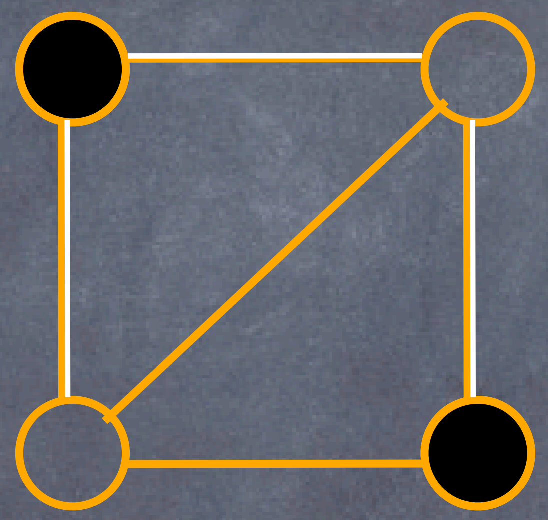

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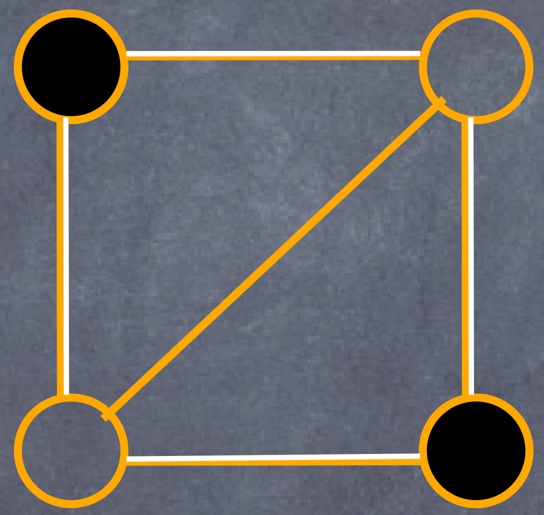

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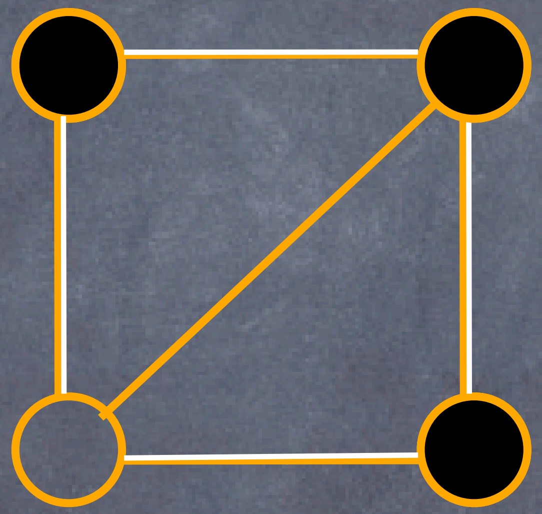

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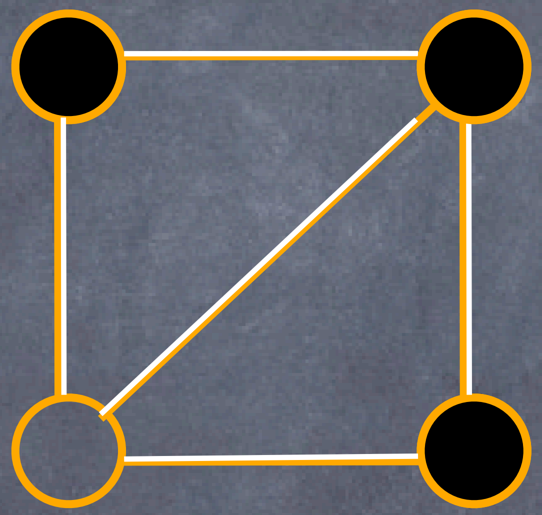

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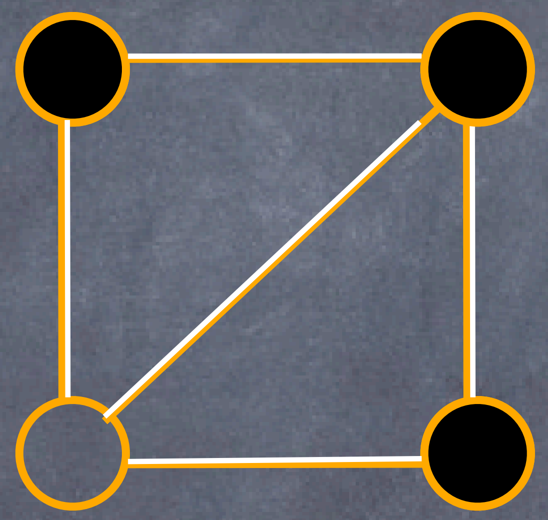

vertex cover
of size 3

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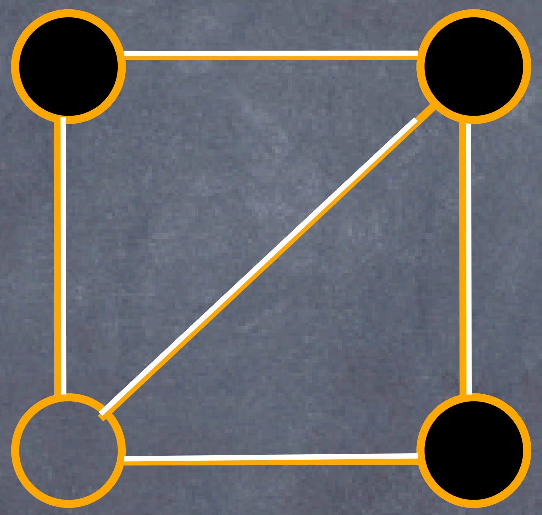

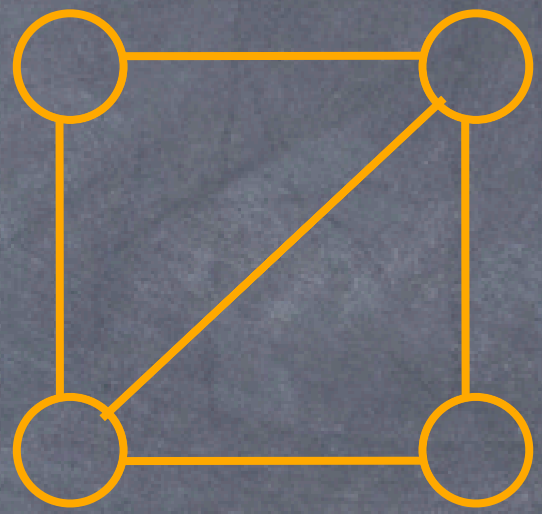

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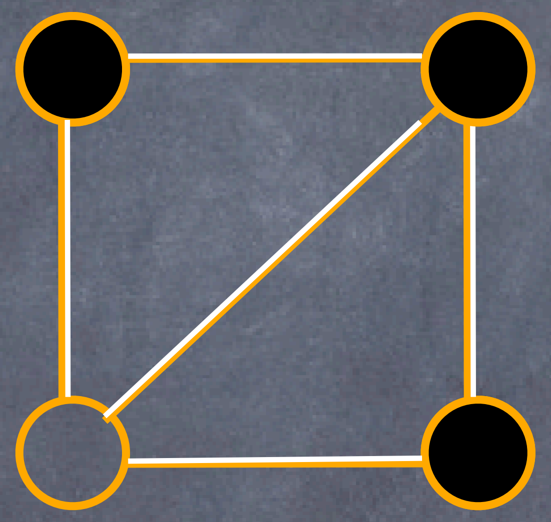

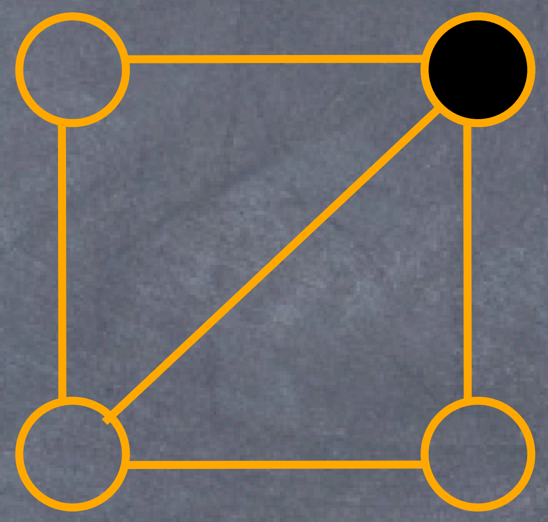

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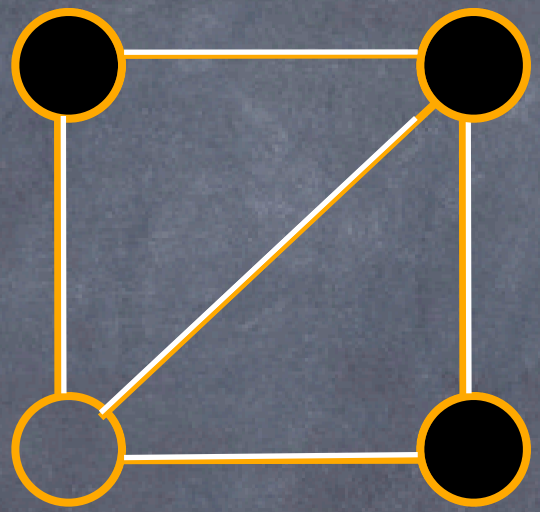

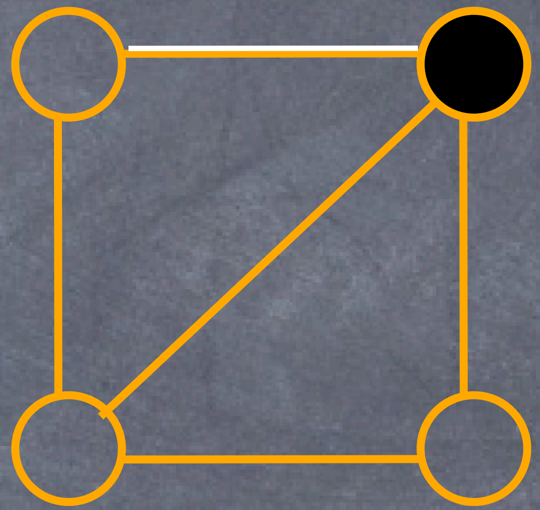

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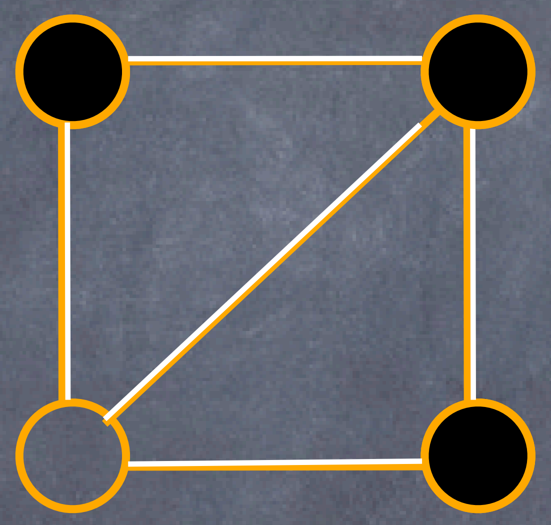

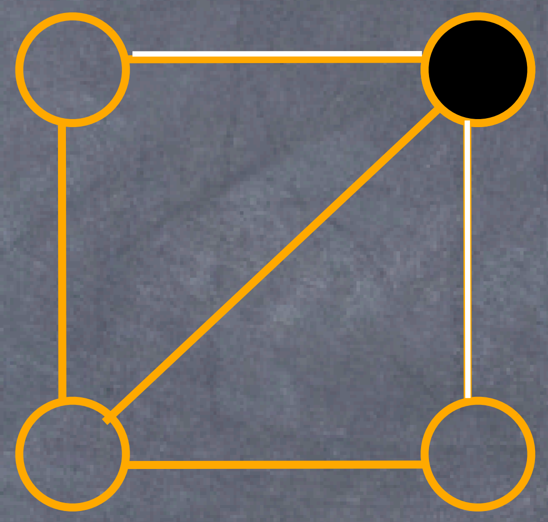

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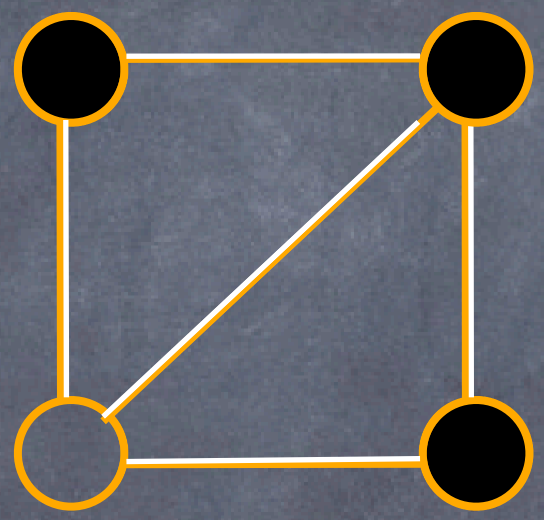

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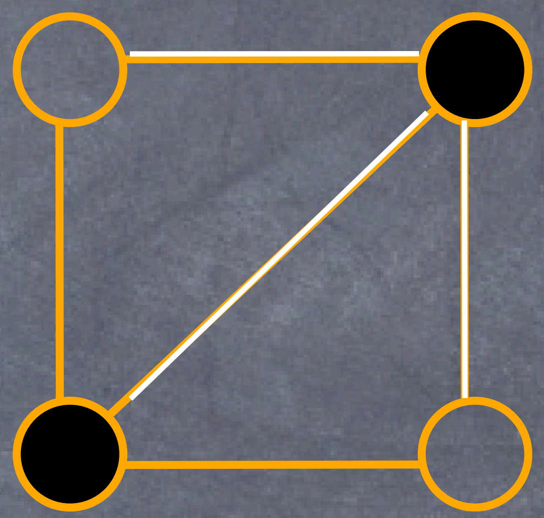

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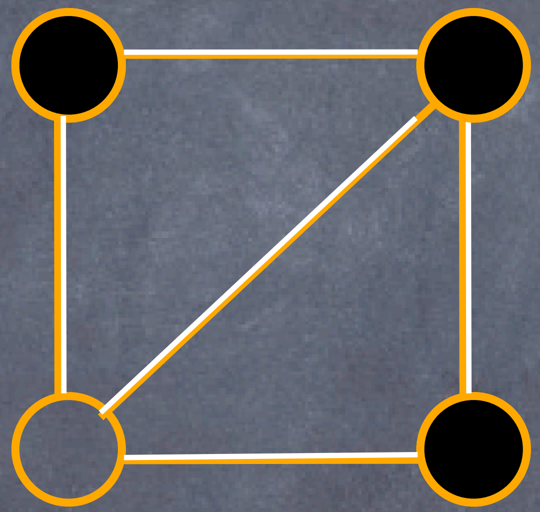

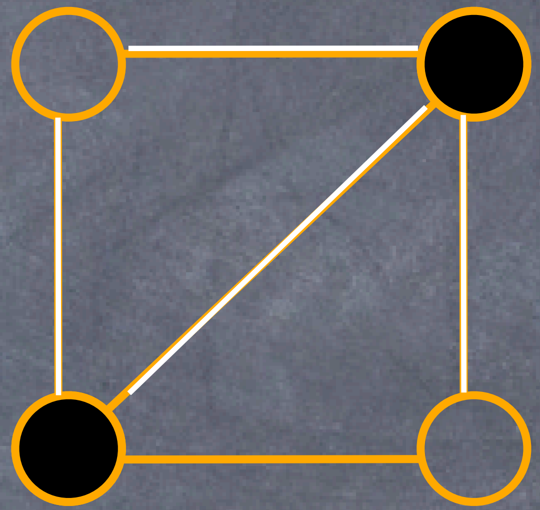

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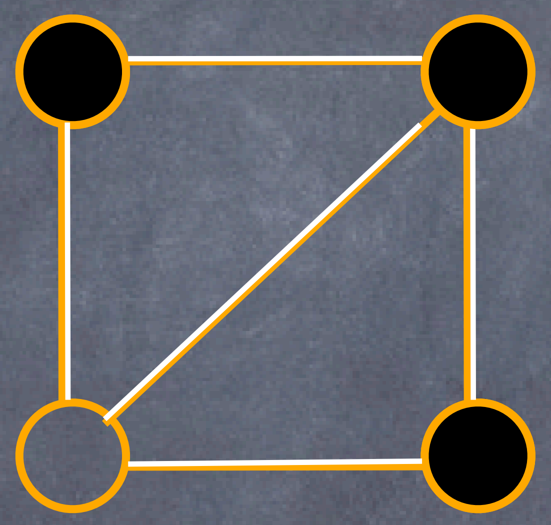

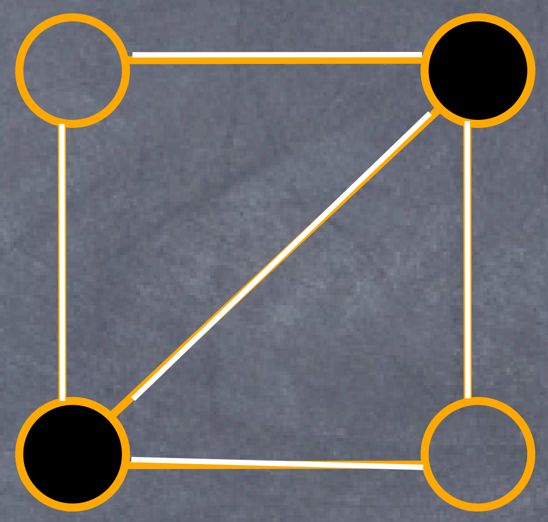

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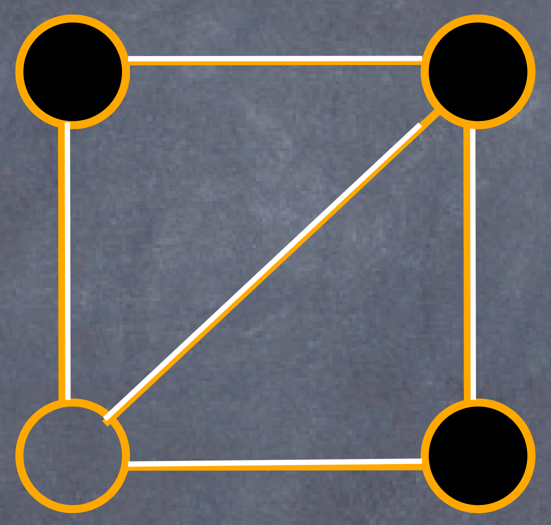

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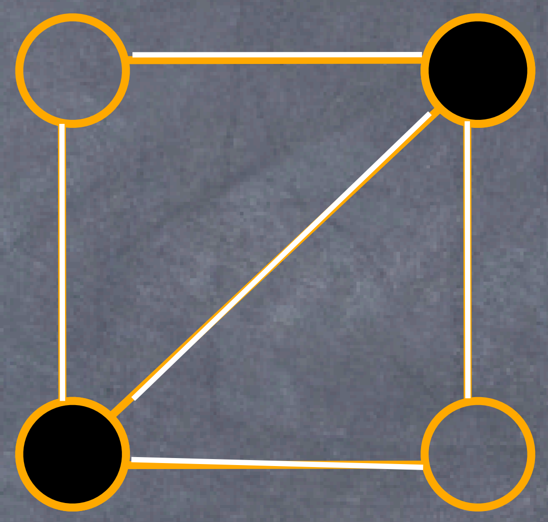

vertex cover
of size 2

Vertex Cover Decision Problem

The Vertex Cover Problem: Given a graph G and an integer K, does G have a vertex cover of size at most K?

Theorem: VC is NP-complete.

Proof: Notice that vertex cover is in NP. Indeed, given a candidate solution (a subset C of the nodes), one can check in polynomial time if $|C| \leq K$ and whether every edge has at least one endpoint in C.

VC is NP-Complete

We need to show that some known NP-complete problem reduces to Vertex Cover.

We have two choices: SAT and 3SAT.

We will show that 3SAT $\leqslant_p$ Vertex Cover.

Reducing 3SAT to VC

- Let $c_1 \wedge \ldots \wedge c_m$ be any 3SAT input over set over variables $U = \{u_1,\ldots,u_n\}$.
- We construct a graph G as follows:
  - two nodes for each variable, u_i and $\neg u_i$, with an edge between them ("literal" nodes)
  - three nodes for each clause c_j, as "placeholders" for the three literals in the clause: a^1_j, a^2_j, a^3_j, with edges making a triangle
  - edges connecting each placeholder node in a triangle to the corresponding literal node
- Set K to be n + 2m.

Example of Reduction

- 3SAT input has variables u1, u2, u3, u4 and clauses $(u_1 \vee \neg u_3 \vee \neg u_4) \wedge (\neg u_1 \vee u_2 \vee \neg u_4)$.
- K = 4 + 2*2 = 8

Example of Reduction

- 3SAT input has variables u1, u2, u3, u4 and clauses $(u_1 \vee \neg u_3 \vee \neg u_4) \wedge (\neg u_1 \vee u_2 \vee \neg u_4)$.
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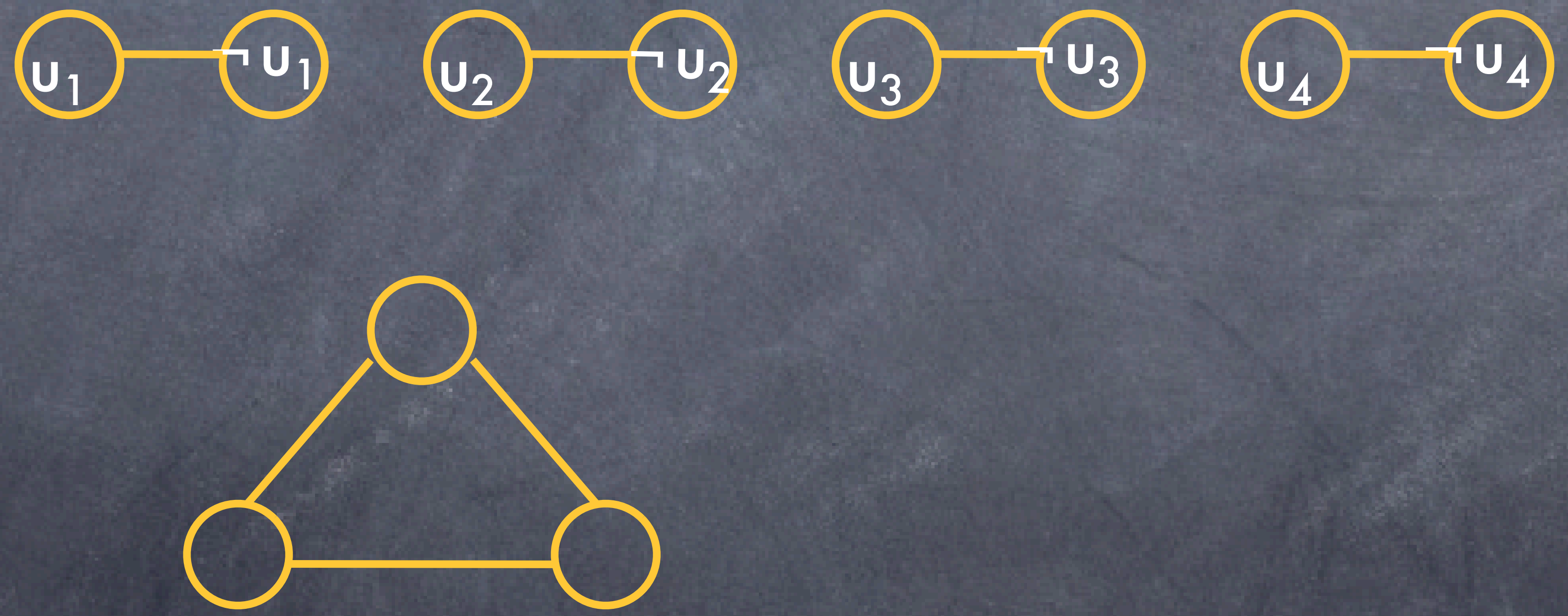

Example of Reduction

- 3SAT input has variables u1, u2, u3, u4 and clauses $(u_1 \lor \neg u_3 \lor \neg u_4) \land (\neg u_1 \lor u_2 \lor \neg u_4)$.
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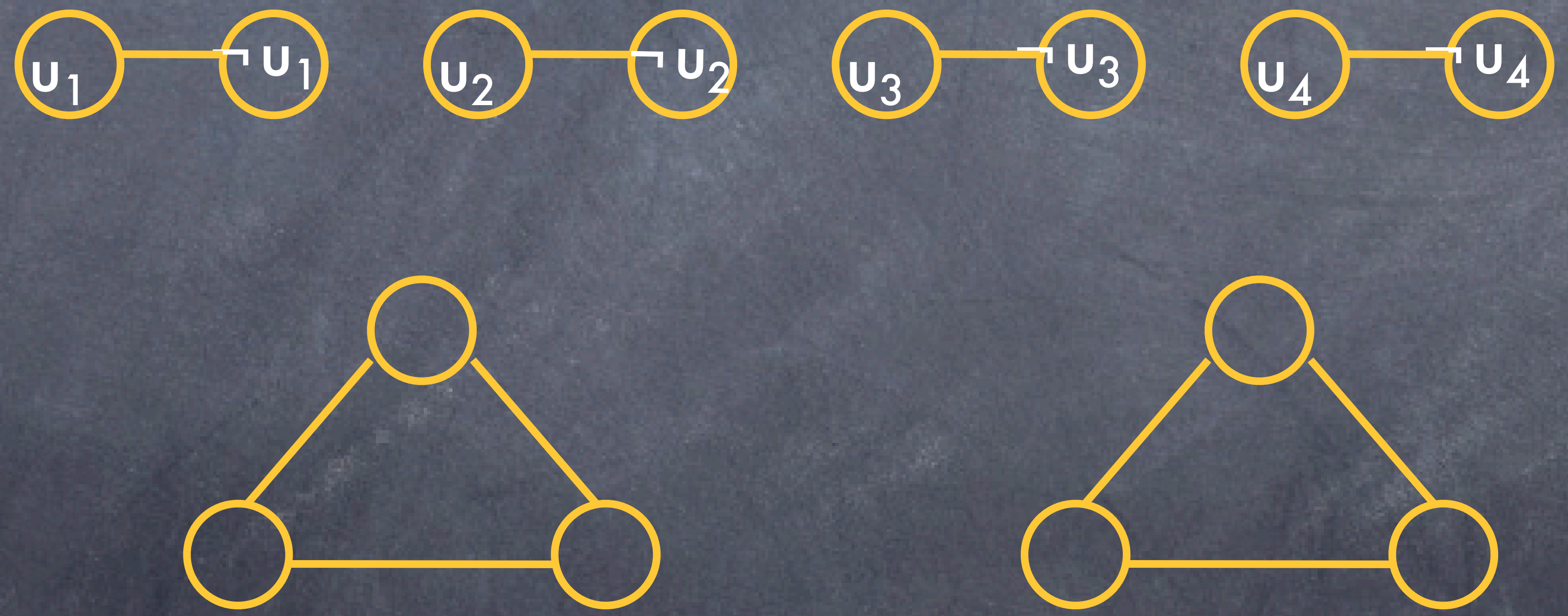

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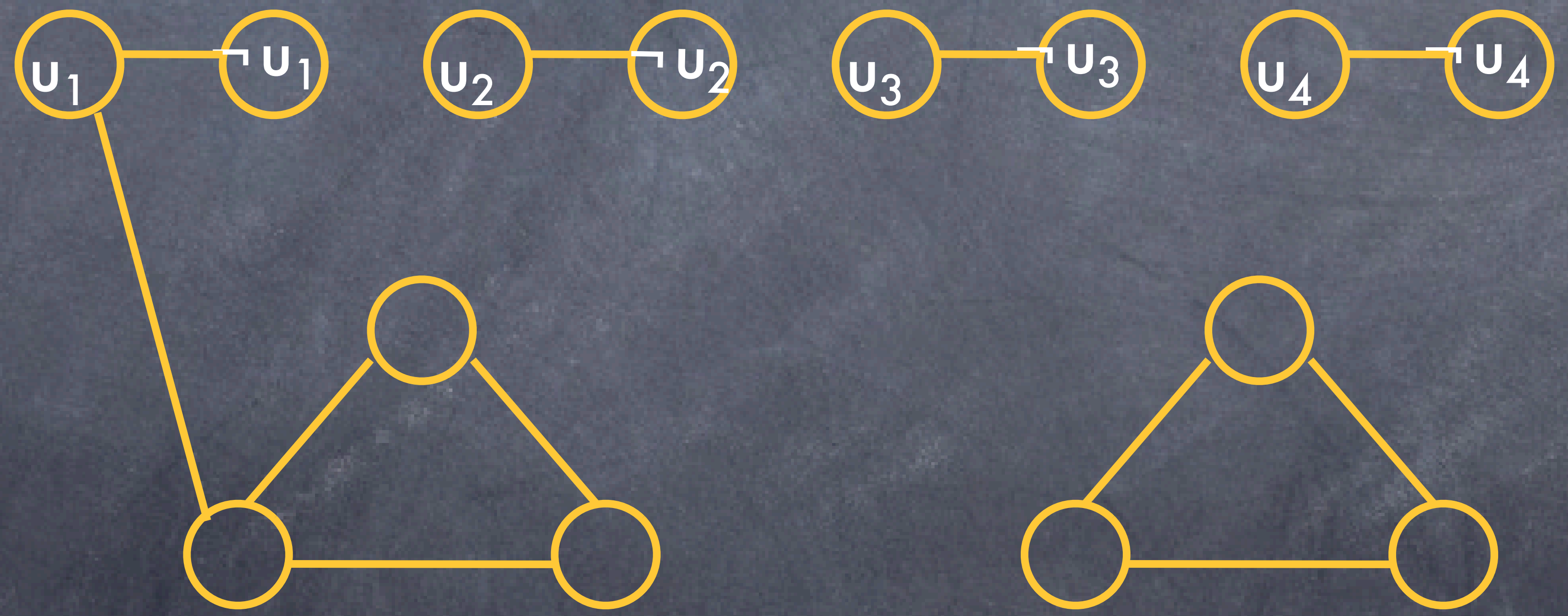

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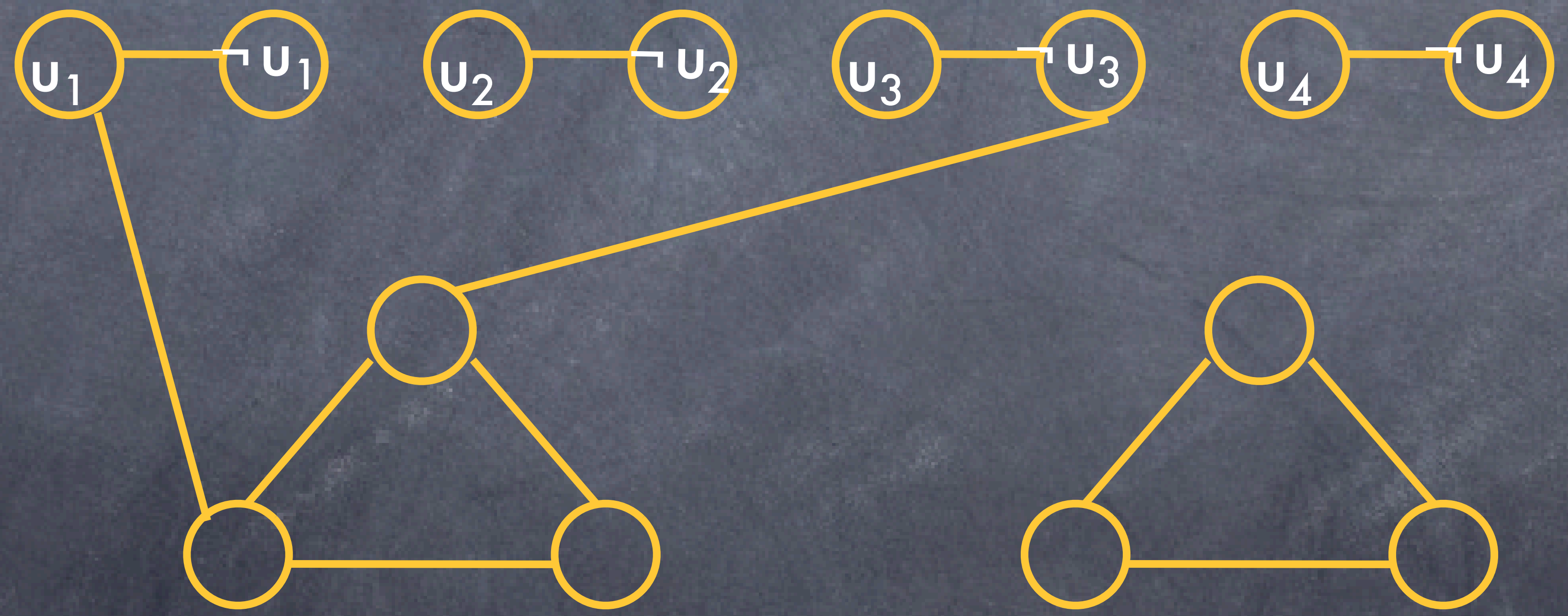

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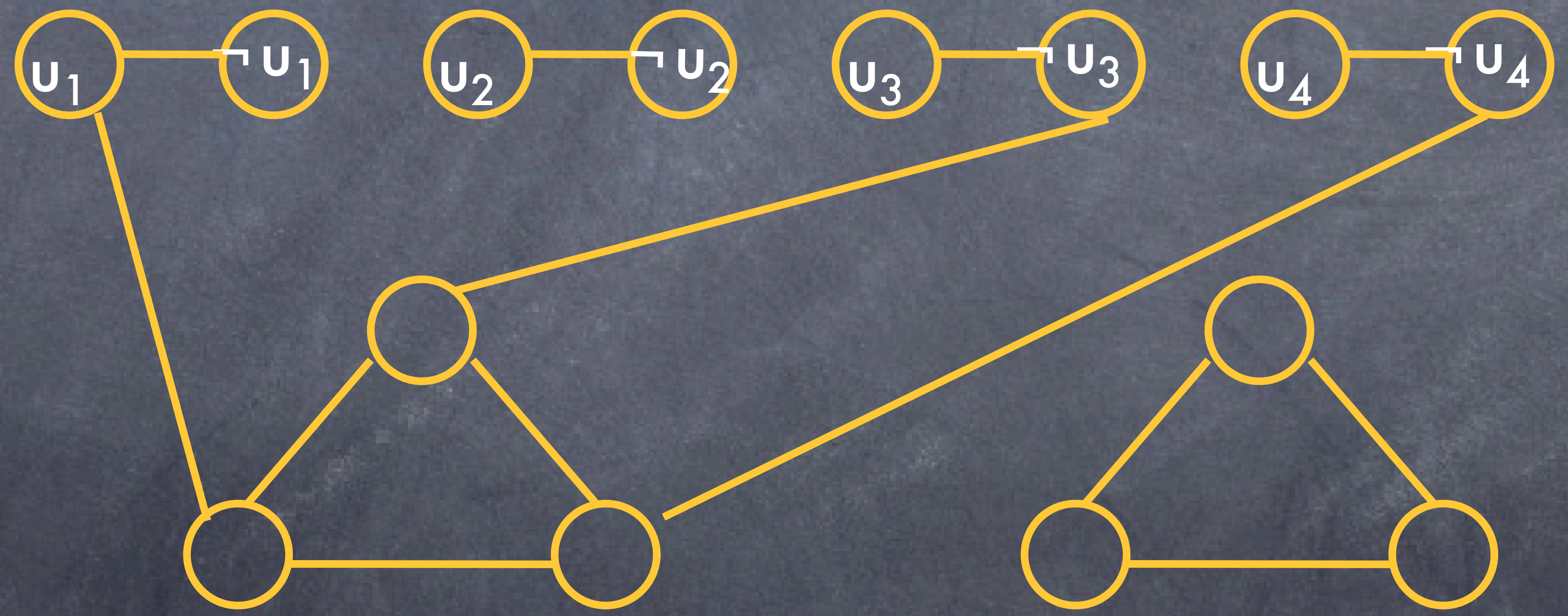

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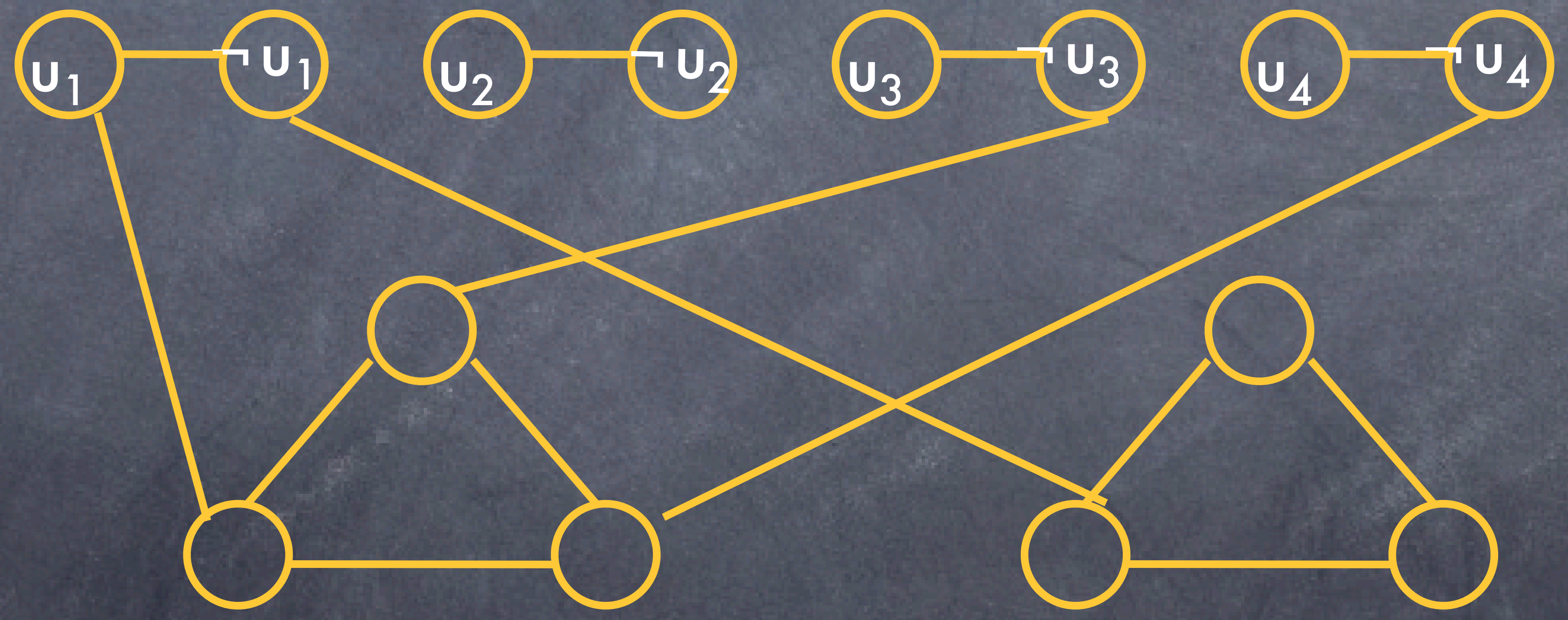

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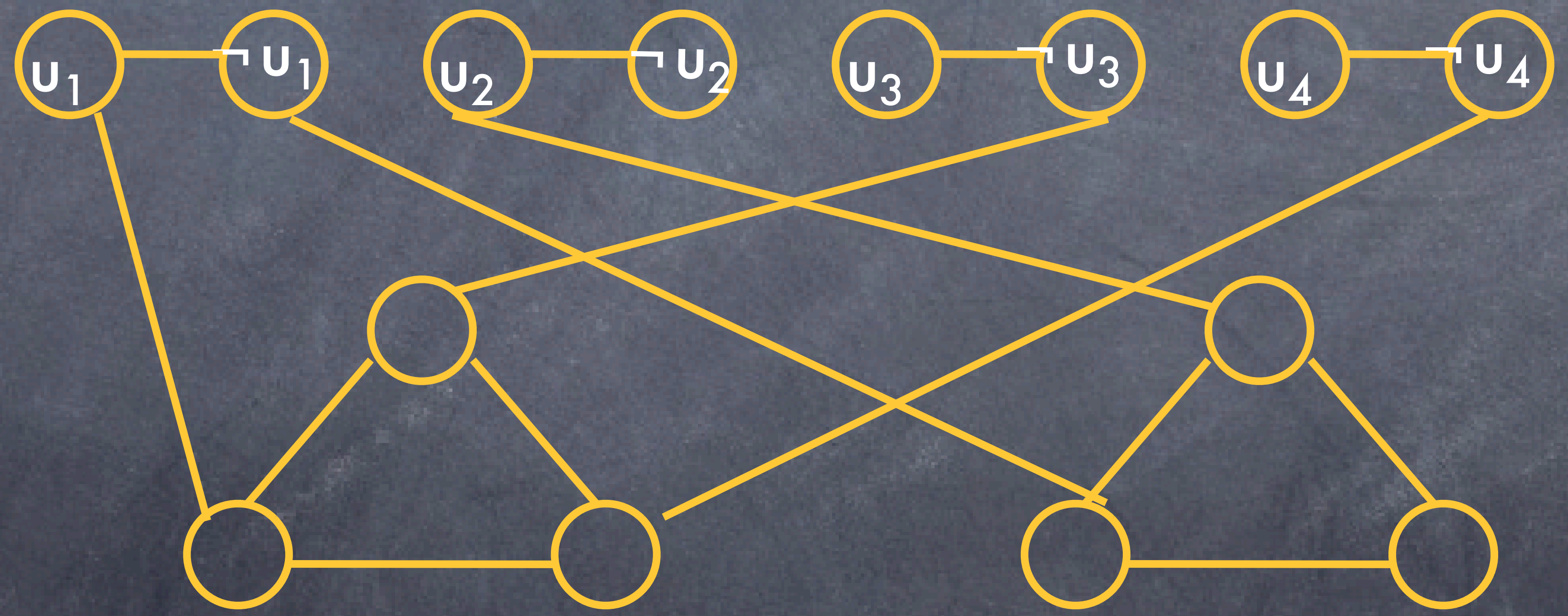

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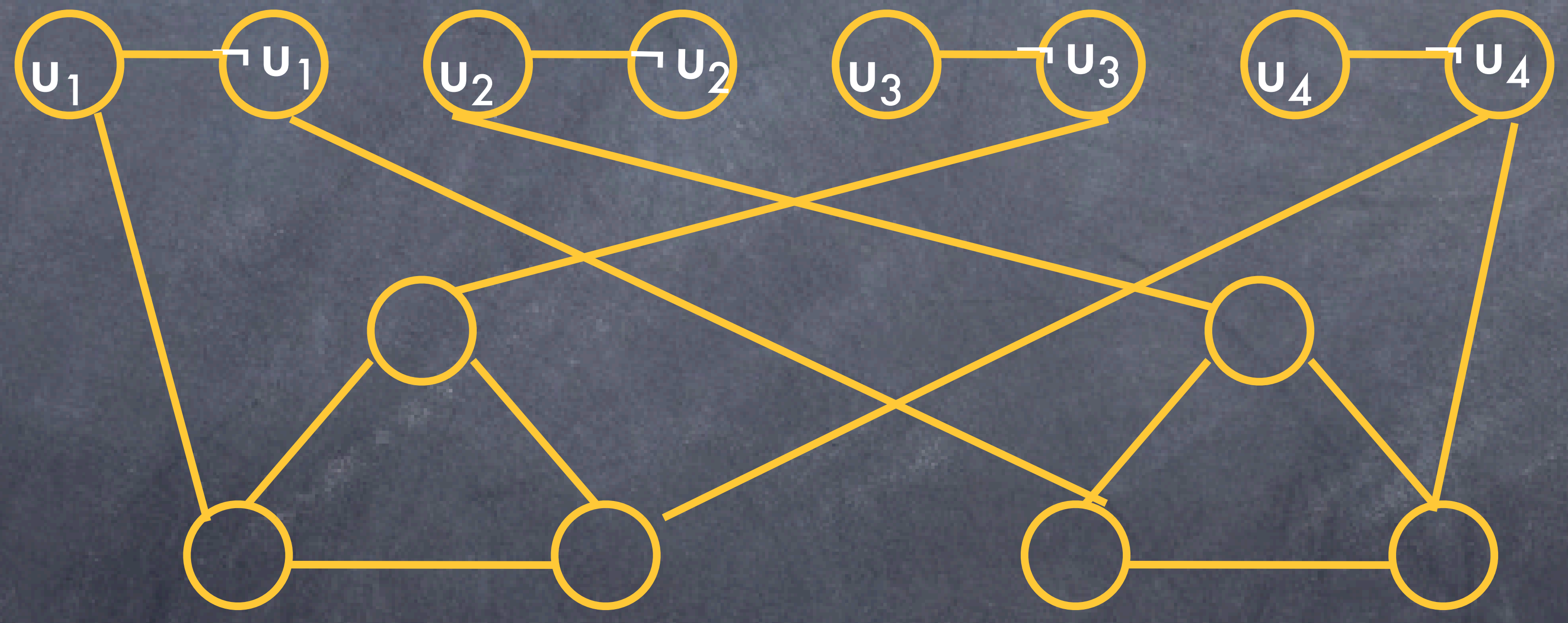

Correctness of Reduction

Suppose the 3SAT input (with m clauses over n variables) has a satisfying truth assignment. Then there is a VC of G of size n + 2m. Indeed,

- pick the node in each pair corresponding to the true literal w.r.t. the satisfying truth assignment
- pick two of the nodes in each triangle such that the excluded node is connected to a true literal

Example of Reduction

$(u_1 \vee \neg u_3 \vee \neg u_4) \wedge (\neg u_1 \vee u_2 \vee \neg u_4)$

$u_1 = T$
$u_2 = F$
$u_3 = T$
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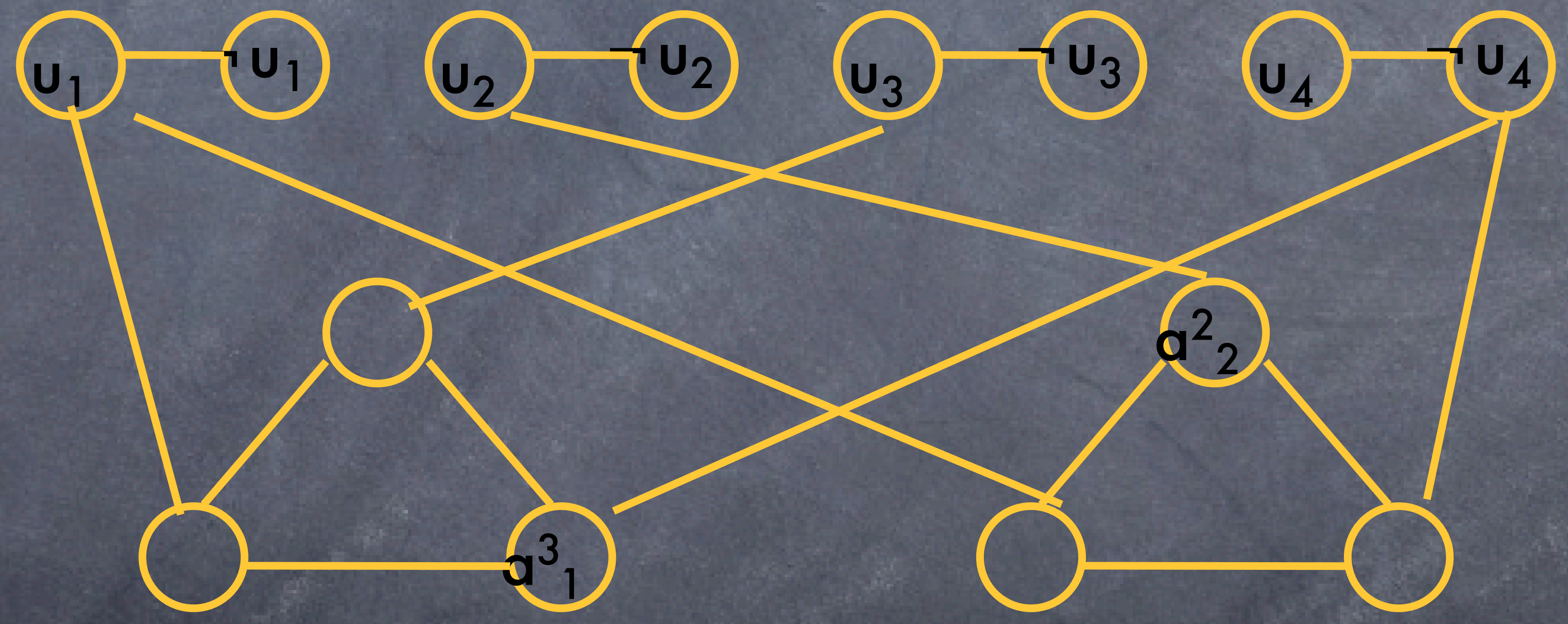

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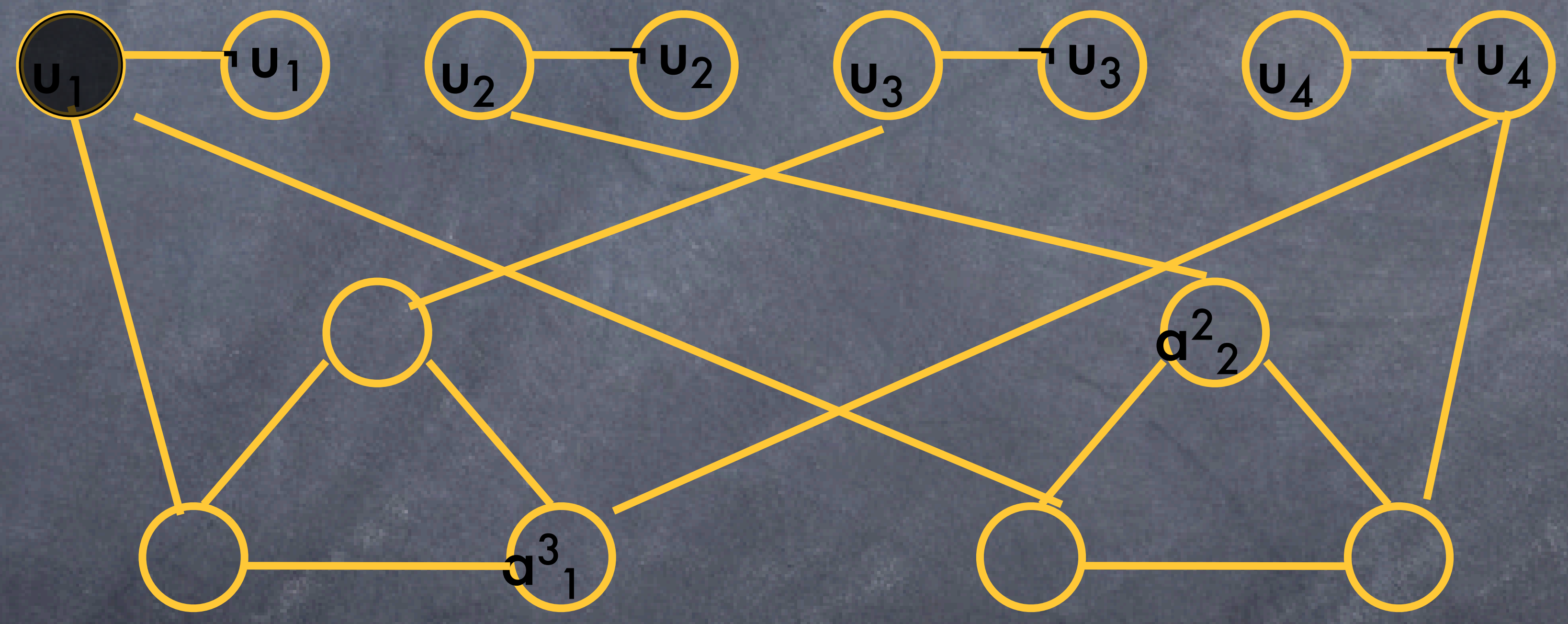

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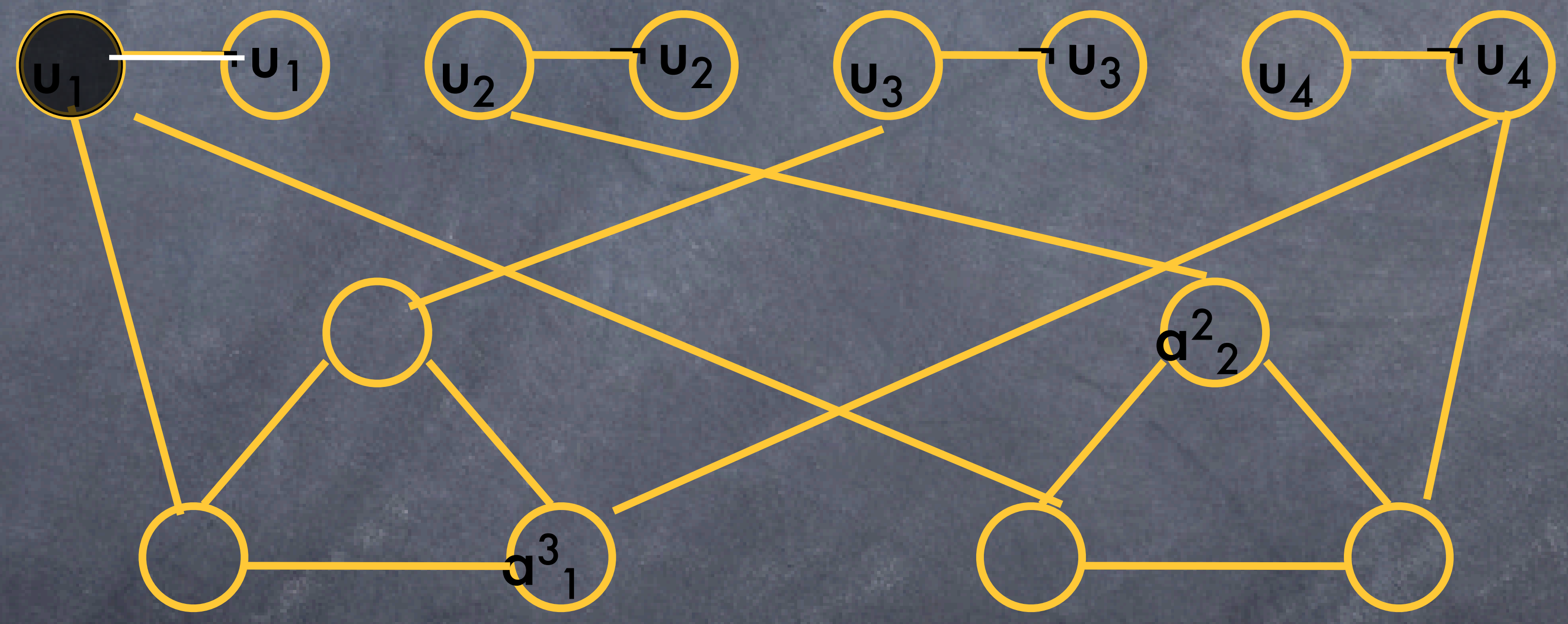

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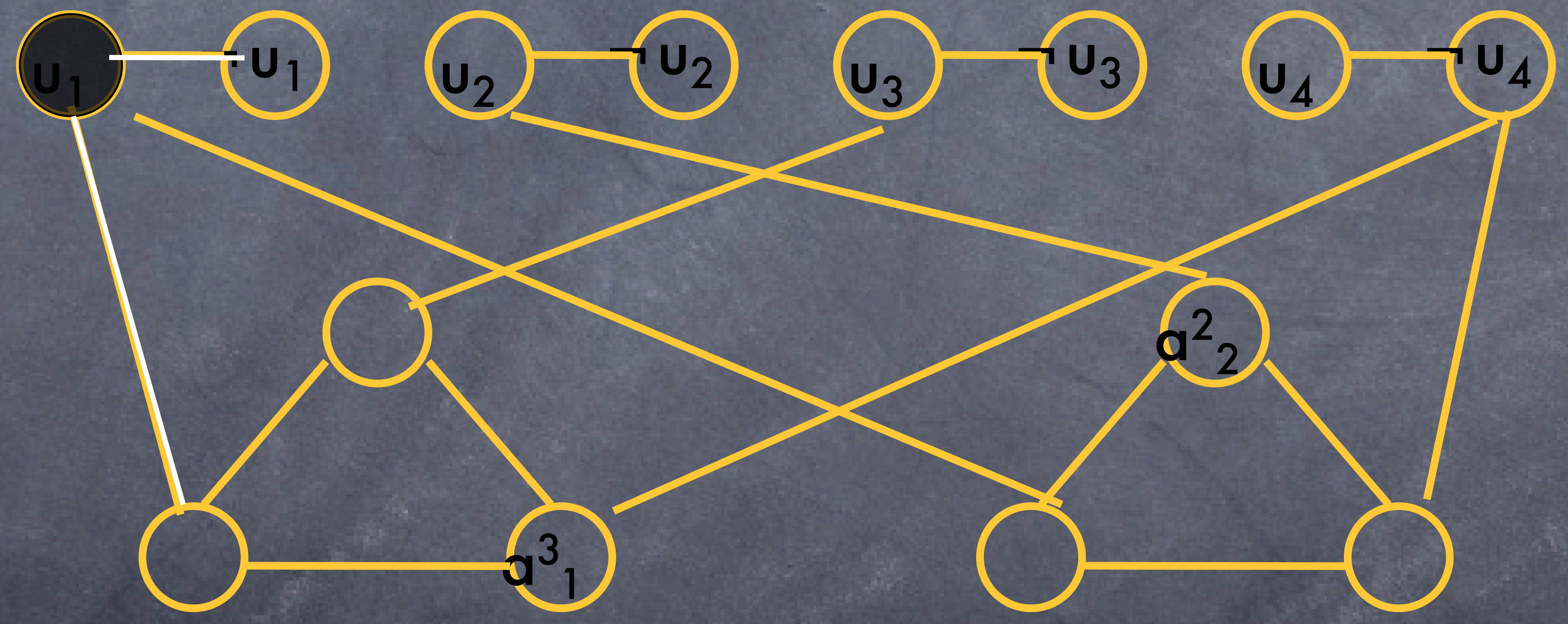

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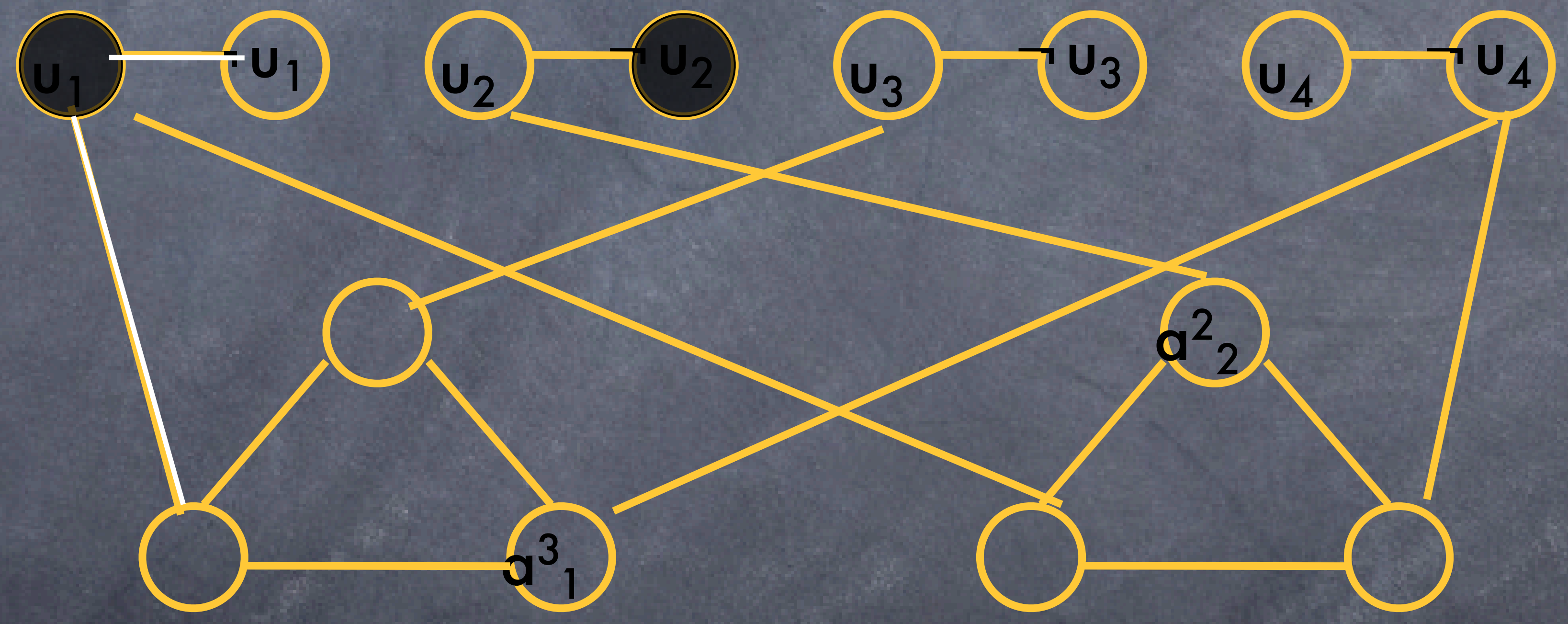

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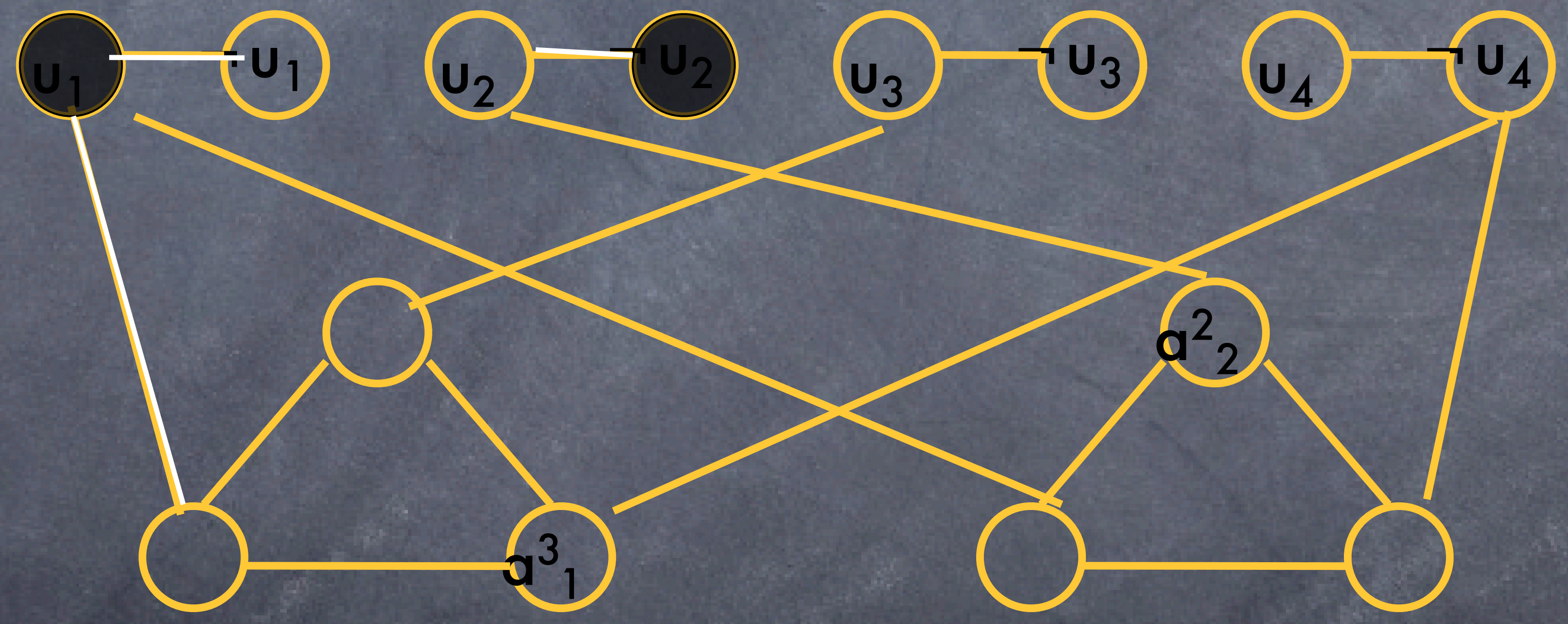

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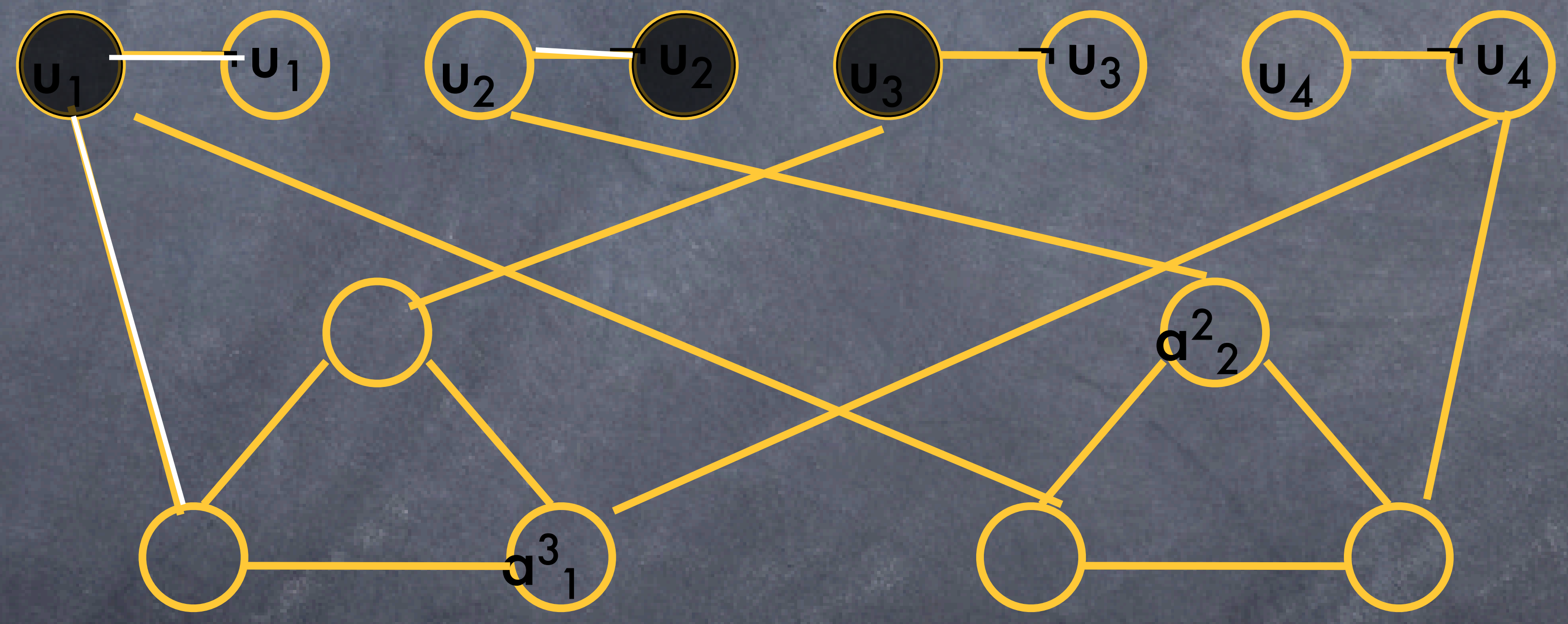

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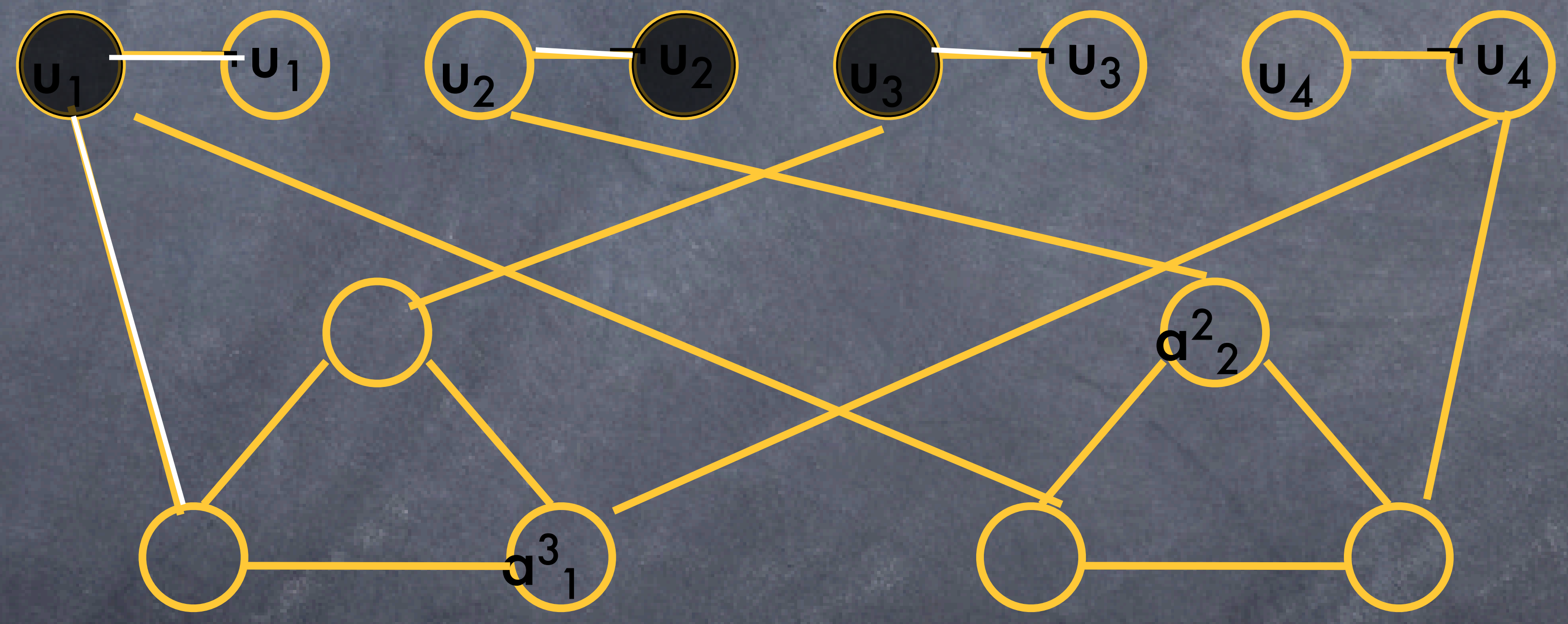

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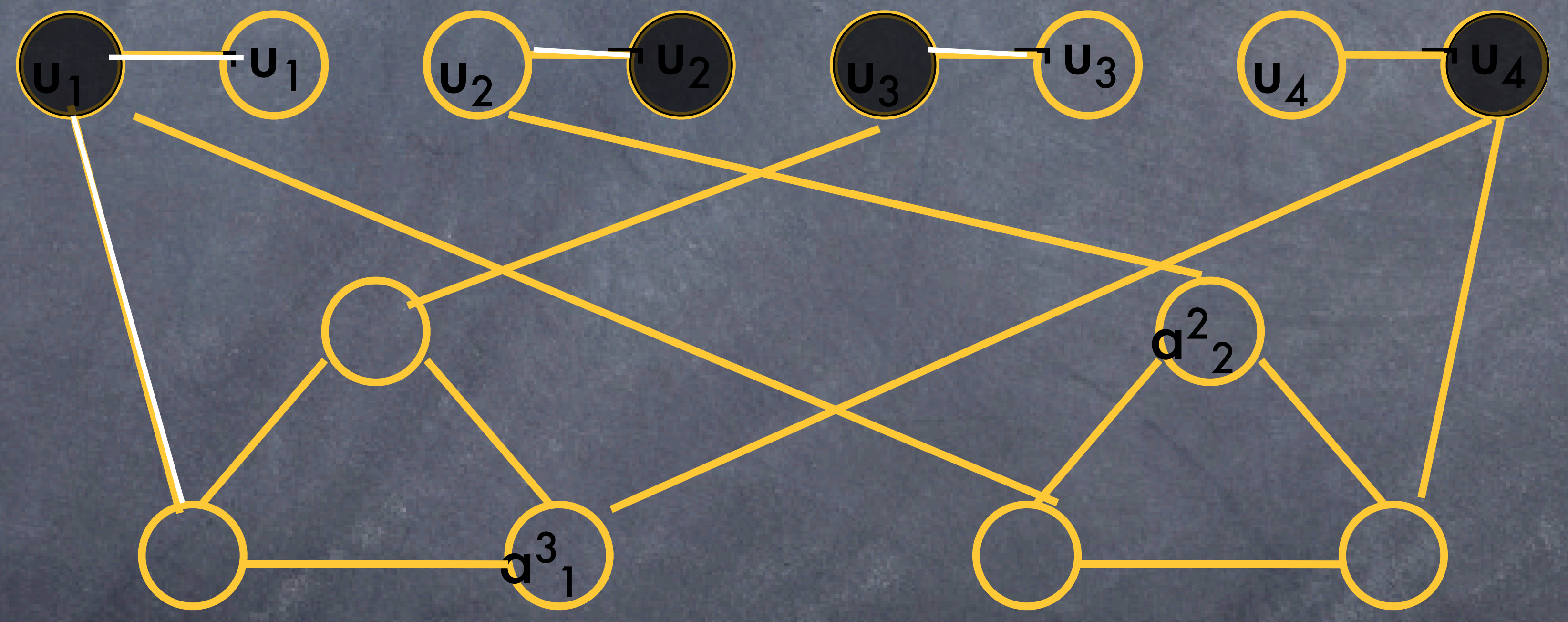

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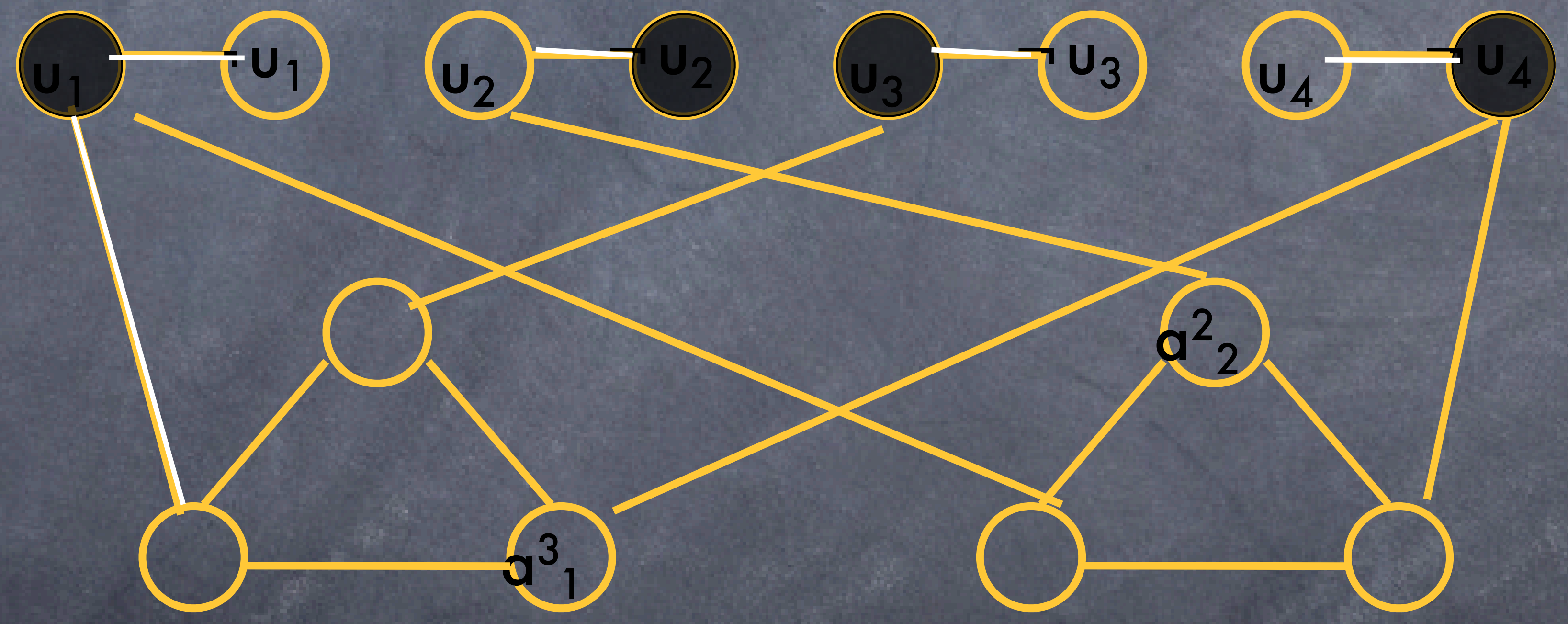

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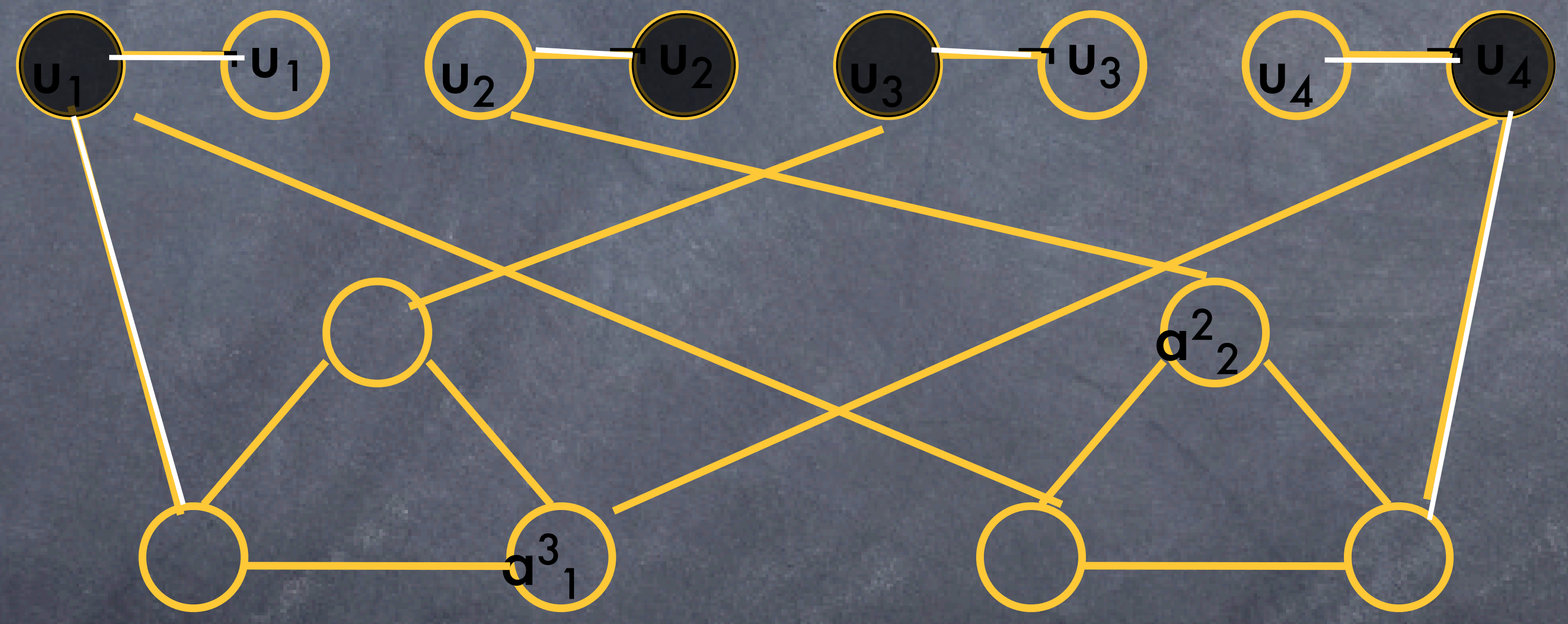

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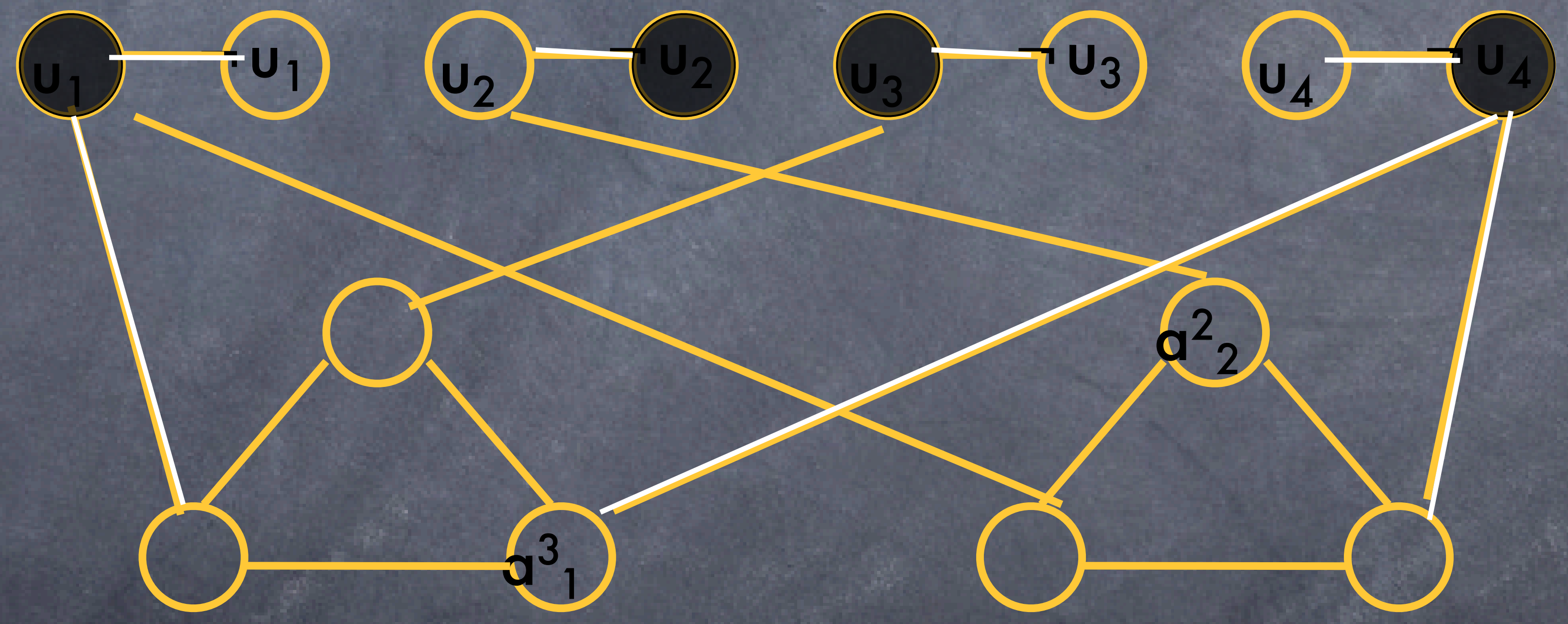

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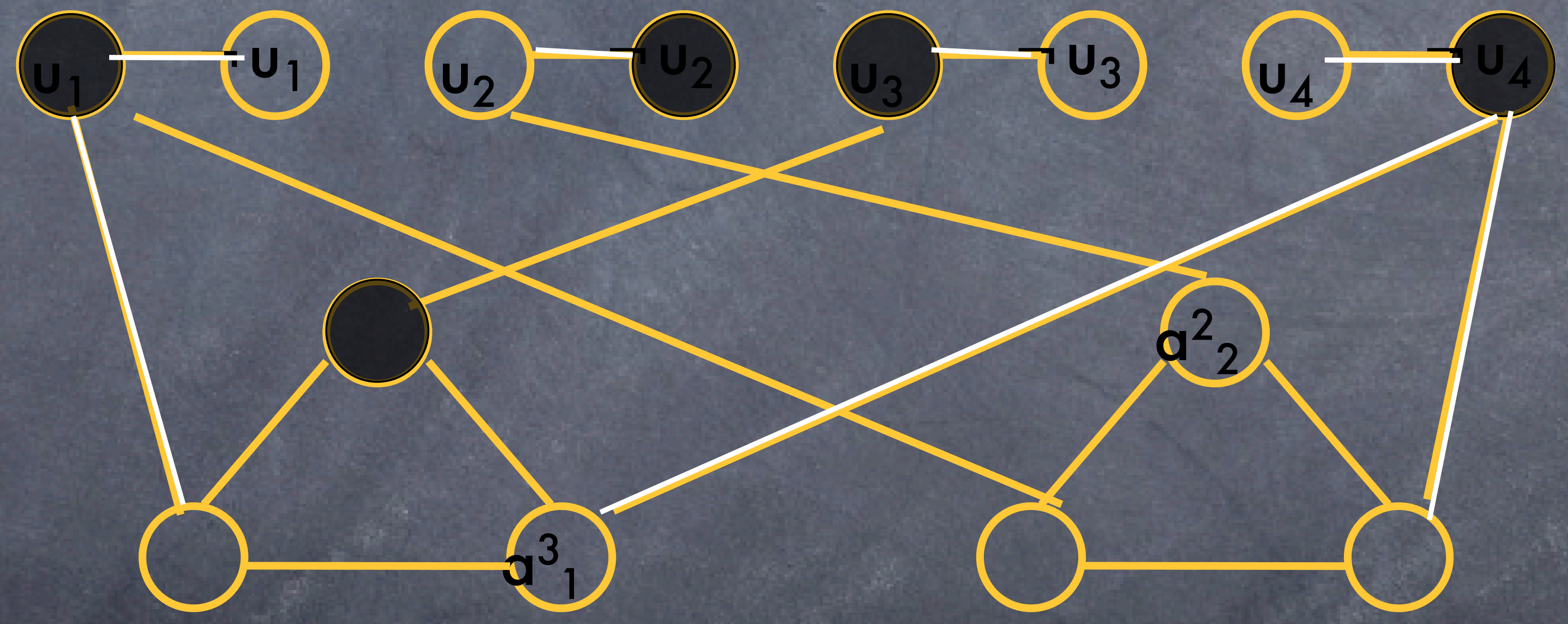

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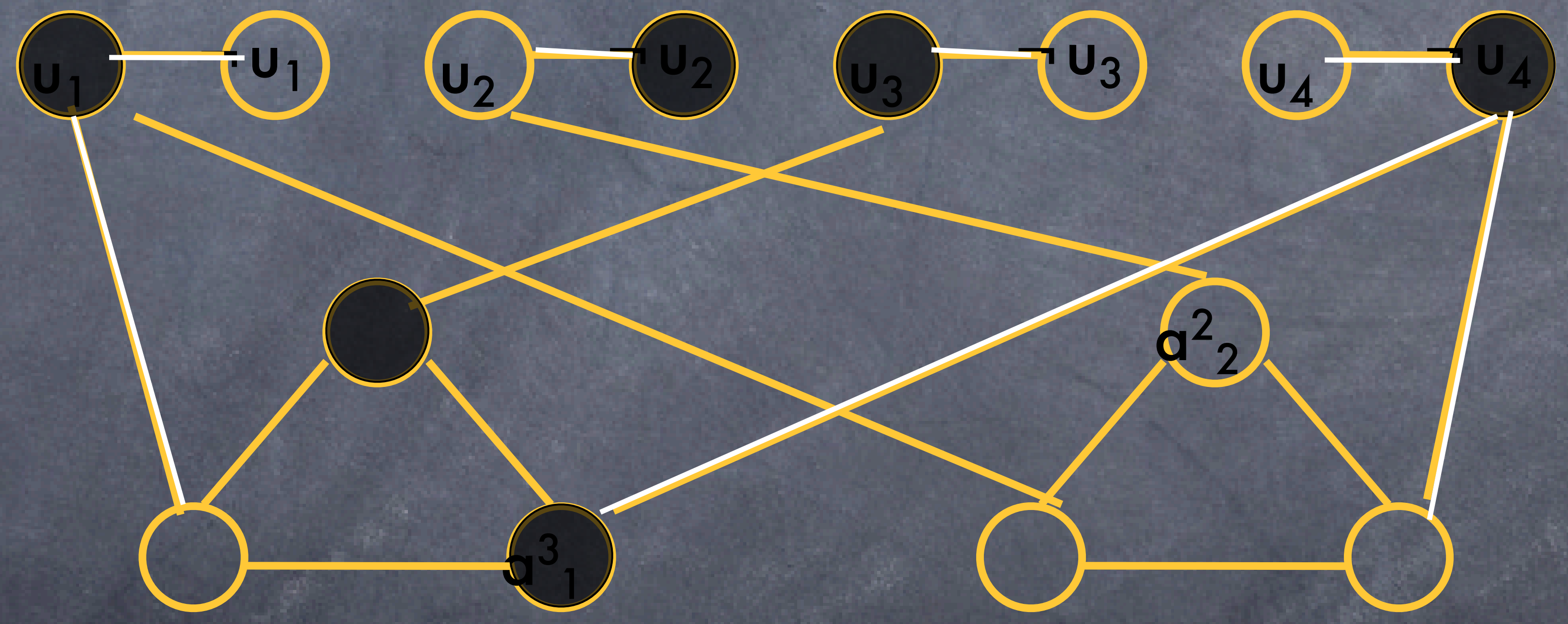

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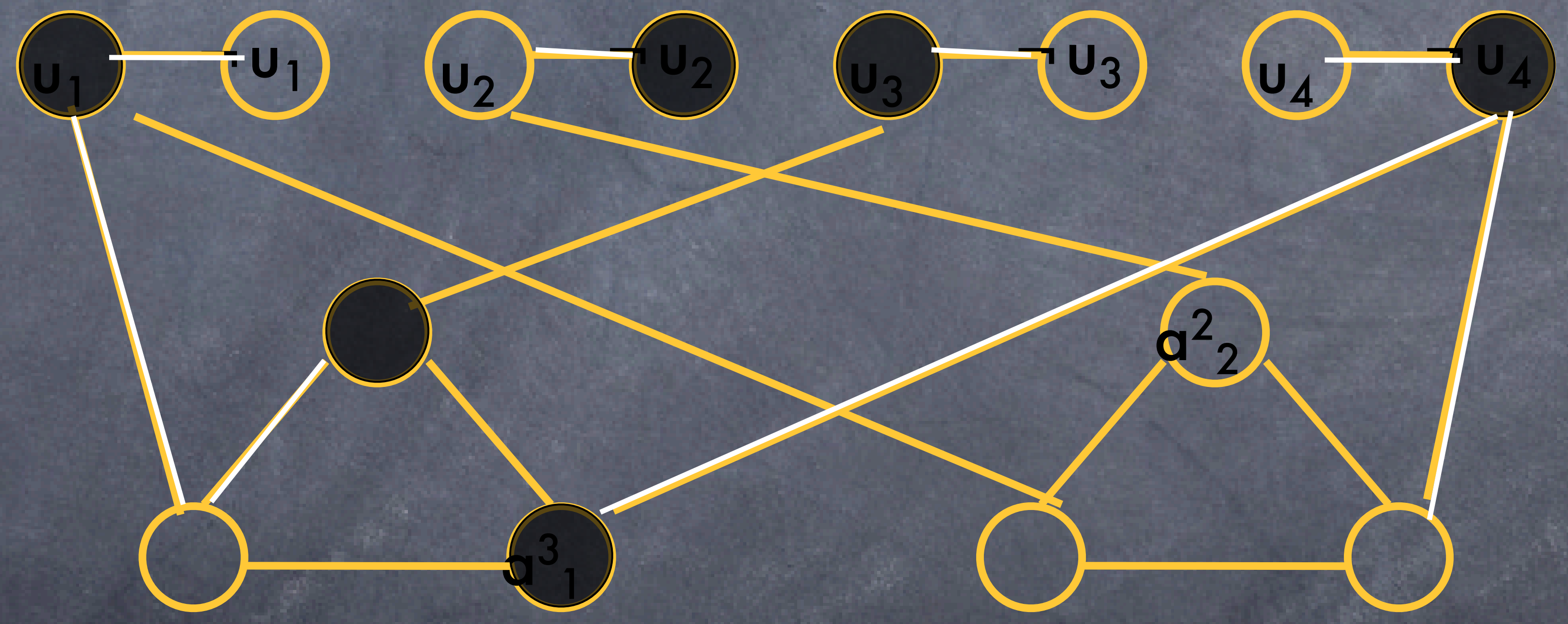

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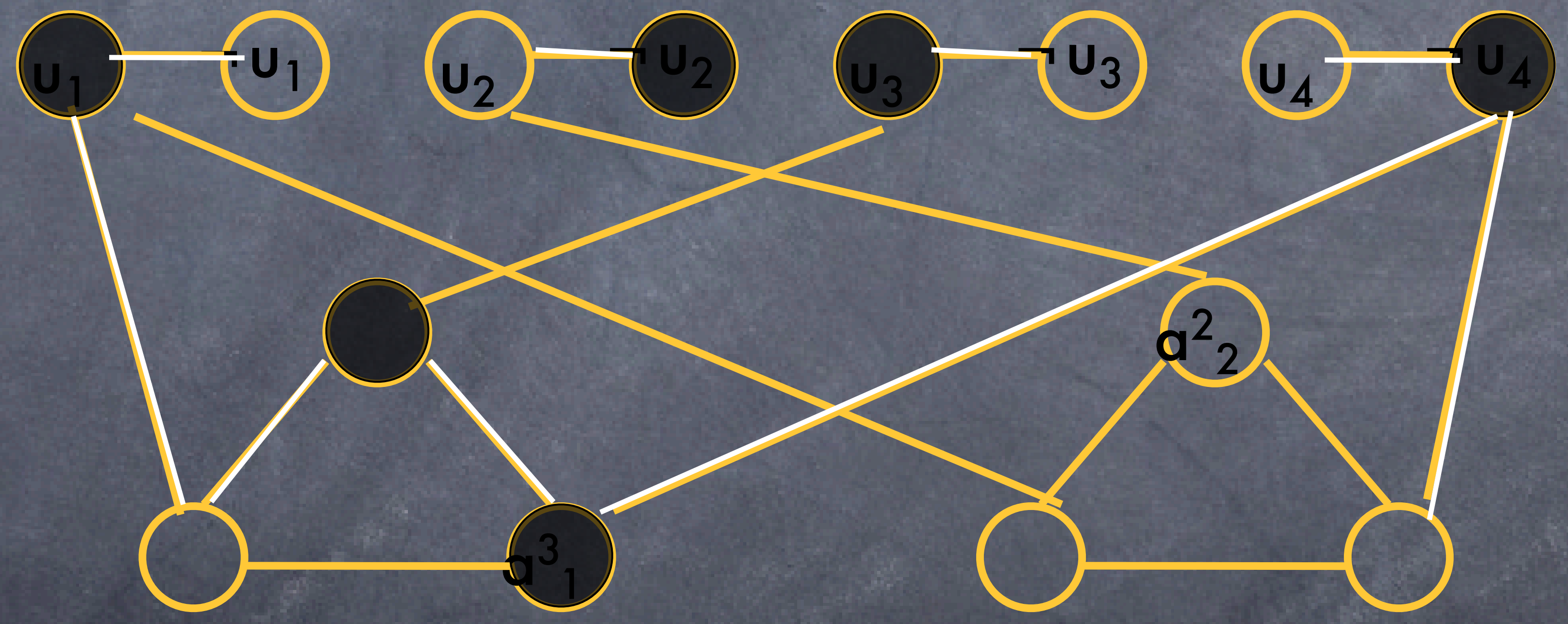

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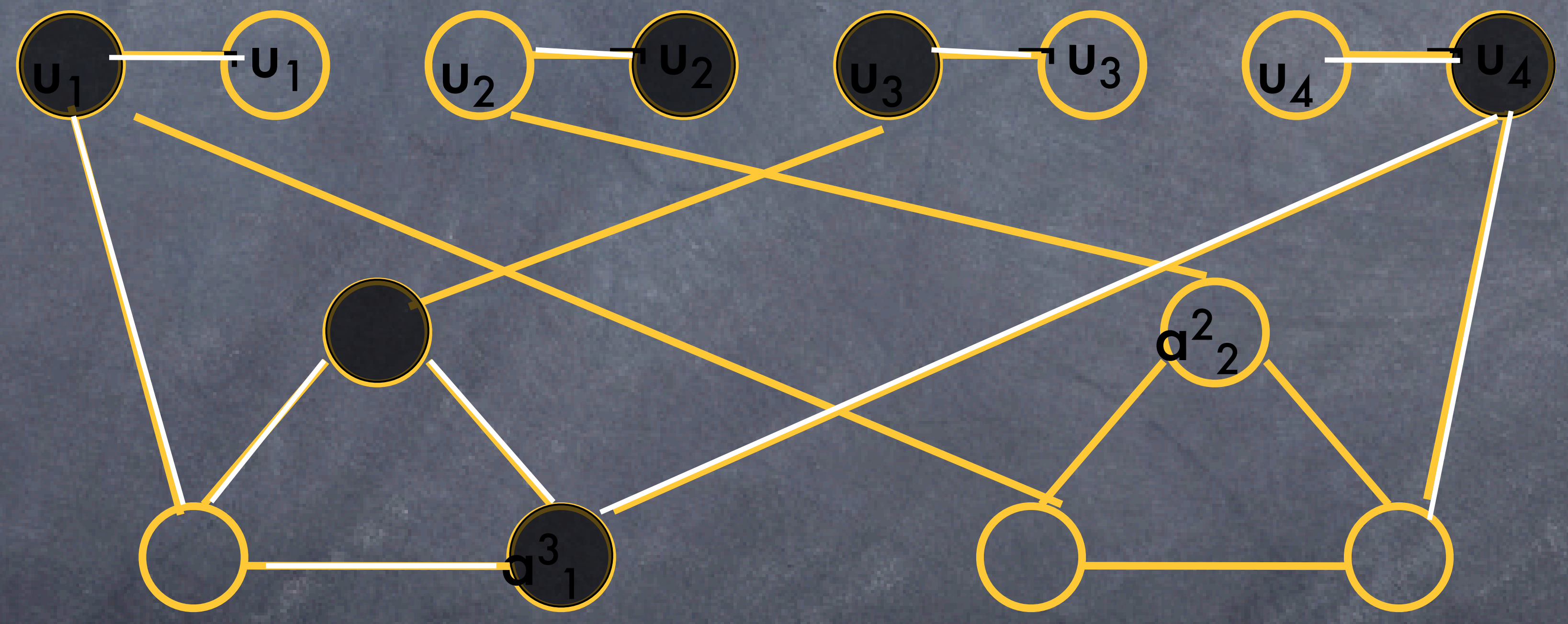


Example of Reduction

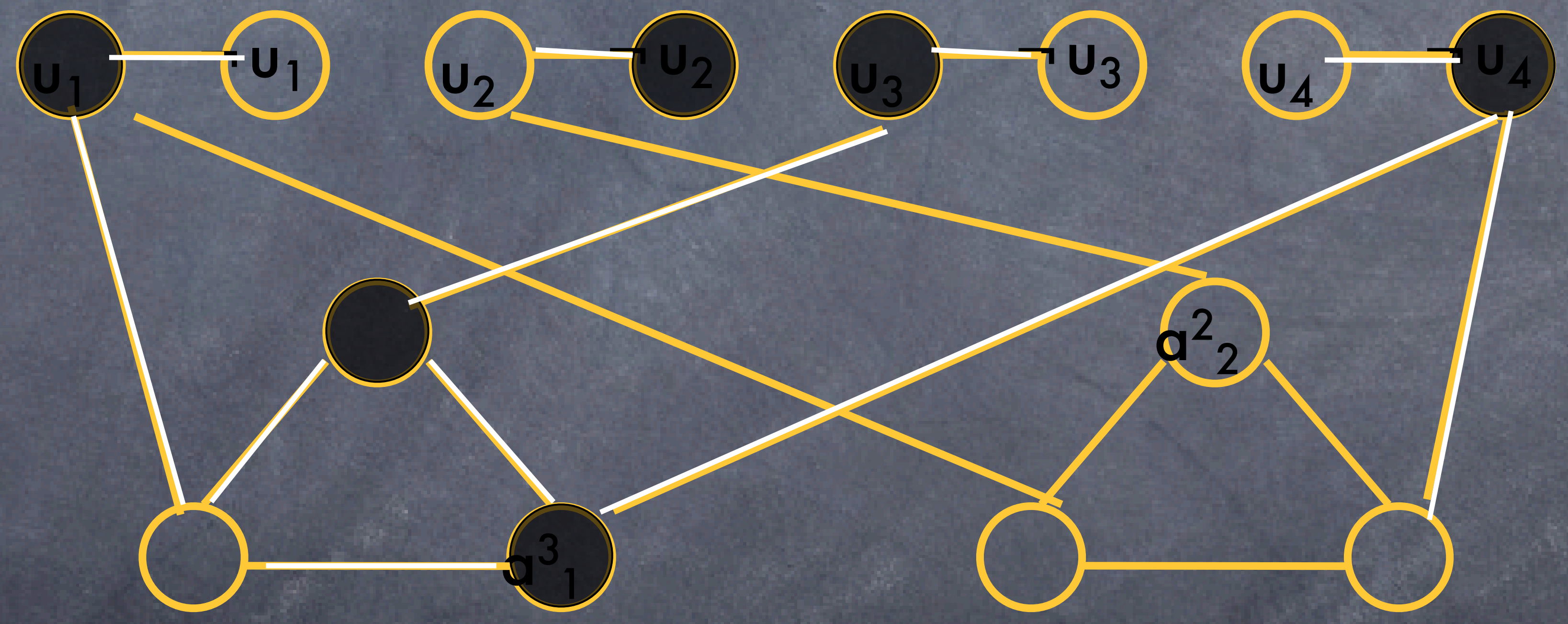
$(u_1 \vee \neg u_3 \vee \neg u_4) \wedge (\neg u_1 \vee u_2 \vee \neg u_4)$

$u_1 = T$

$u_2 = F$

$u_3 = T$

$u_4 = F$



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$(u_1 \vee \neg u_3 \vee \neg u_4) \wedge (\neg u_1 \vee u_2 \vee \neg u_4)$

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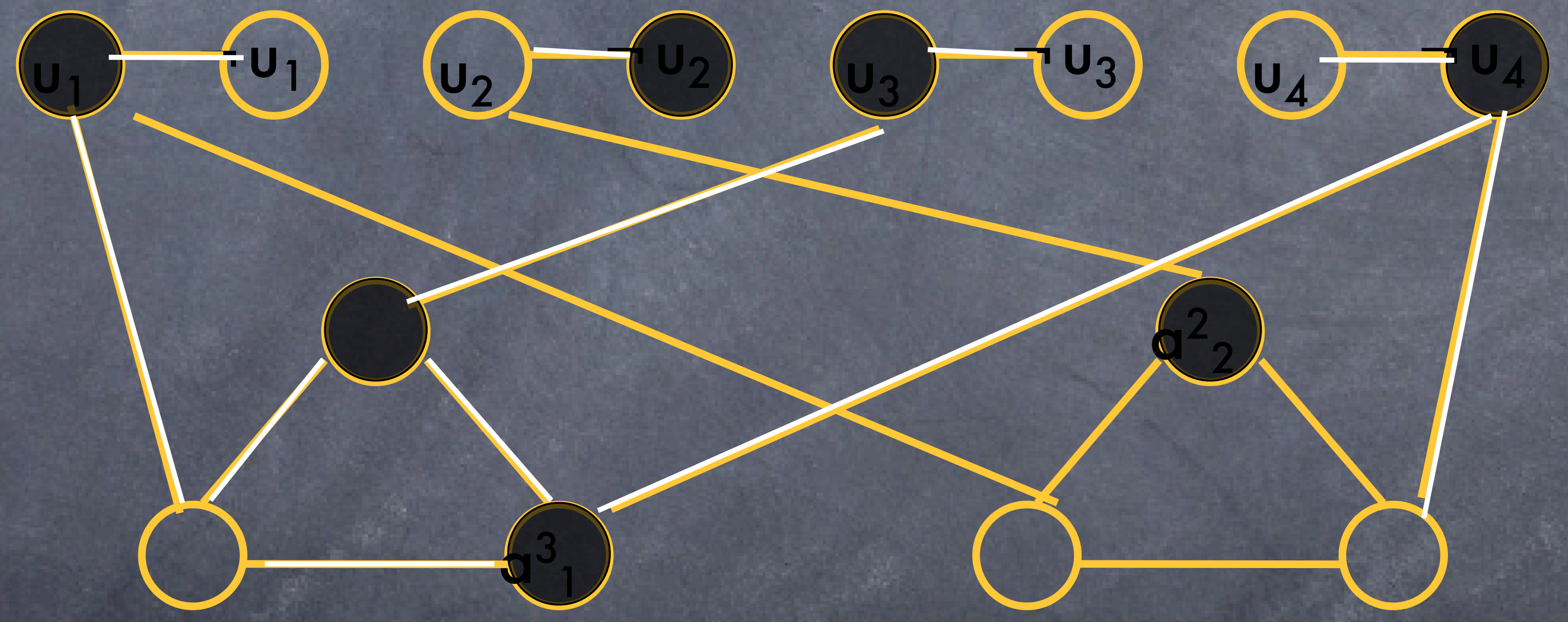

Example of Reduction

$(u_1 \vee \neg u_3 \vee \neg u_4) \wedge (\neg u_1 \vee u_2 \vee \neg u_4)$

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$u_2 = F$
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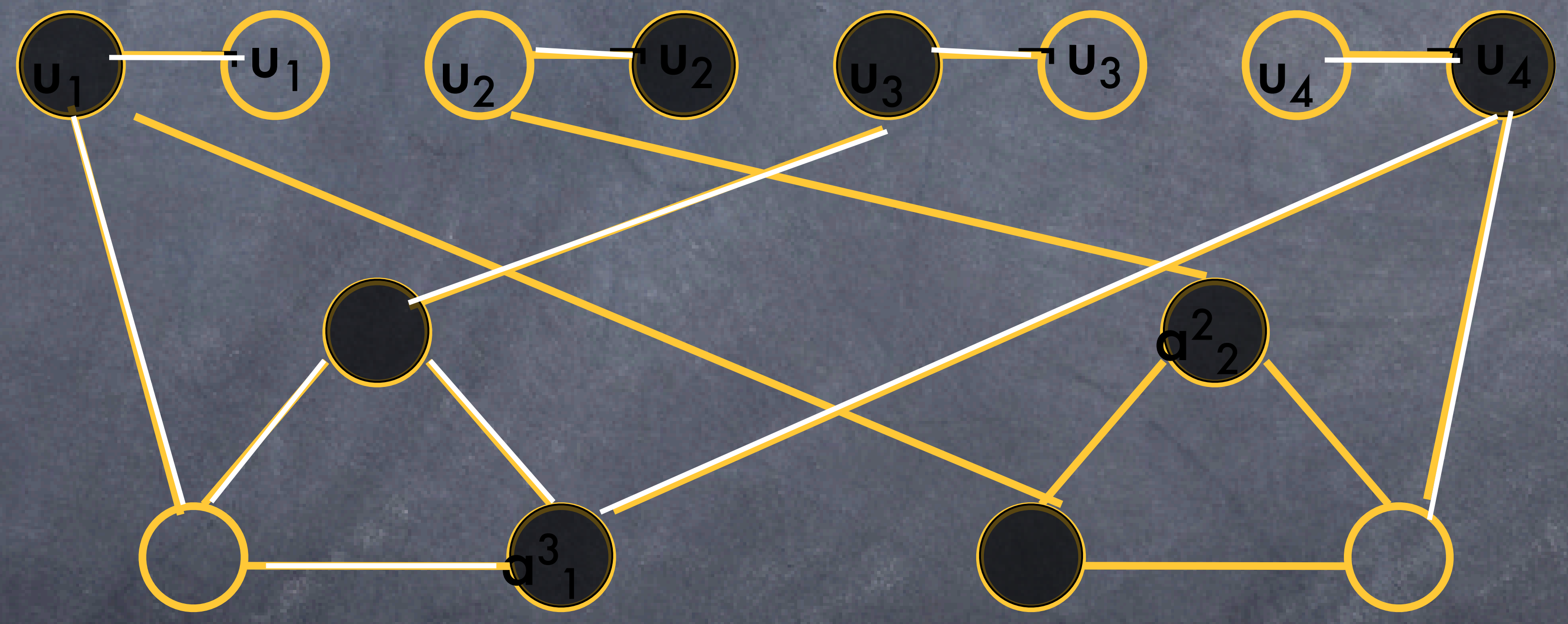

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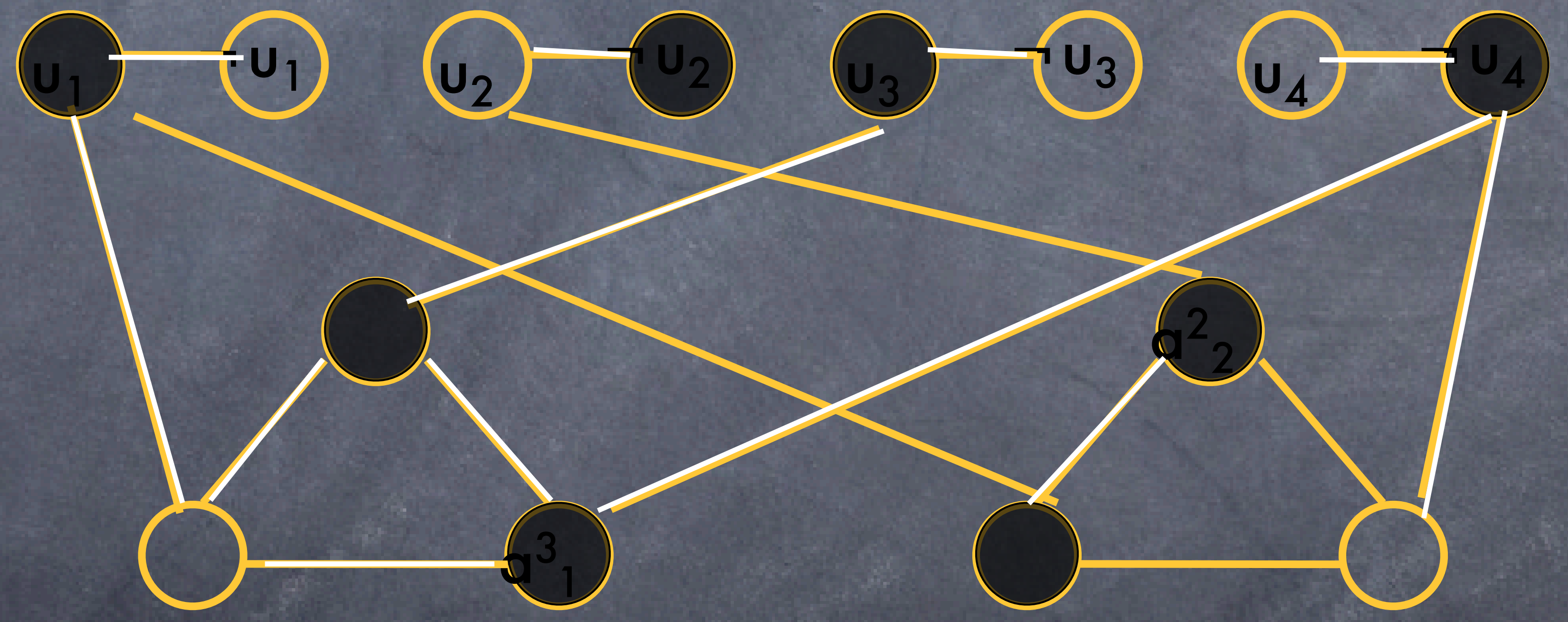

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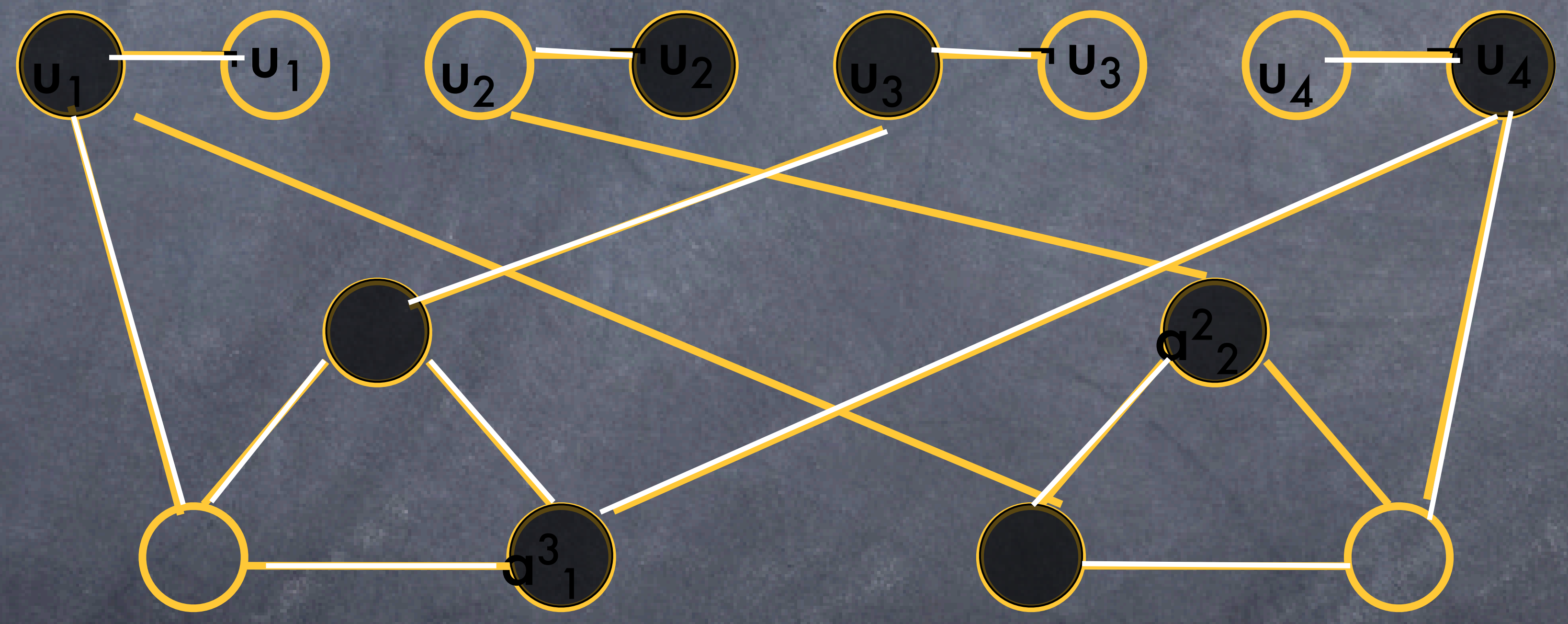

Example of Reduction

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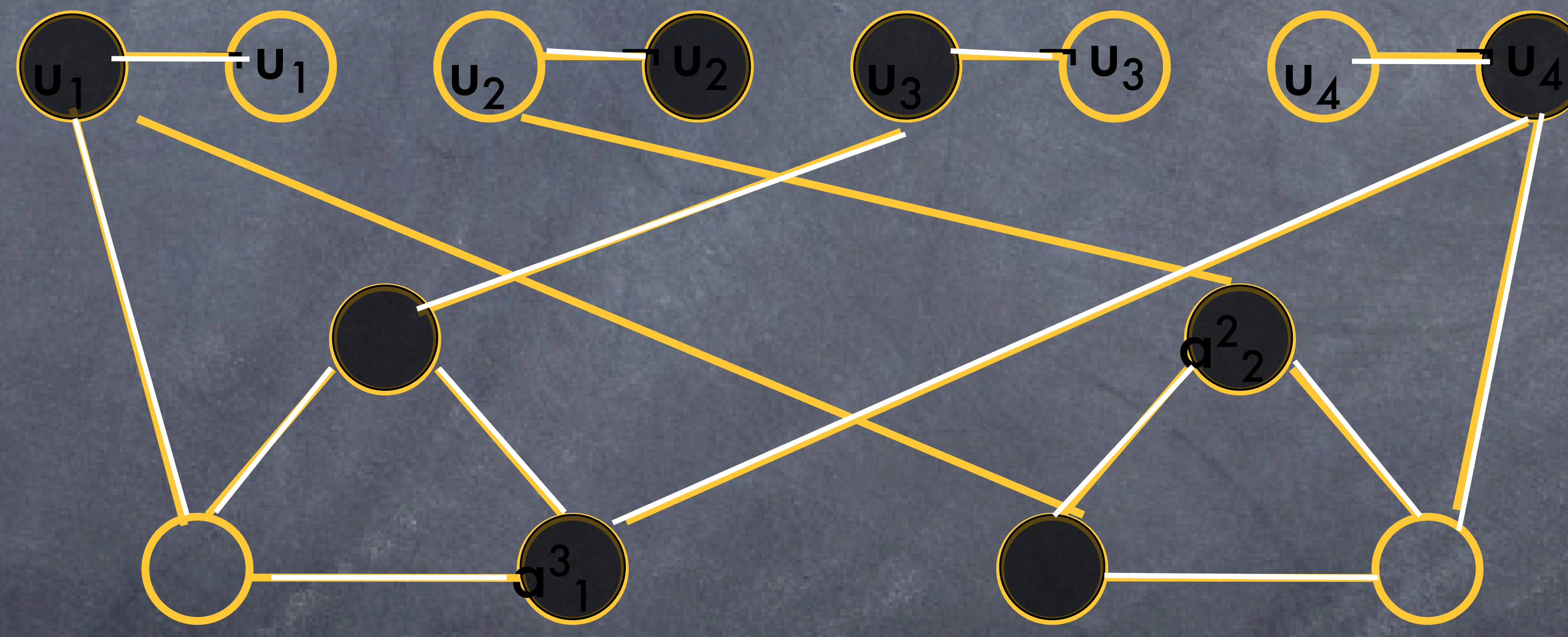

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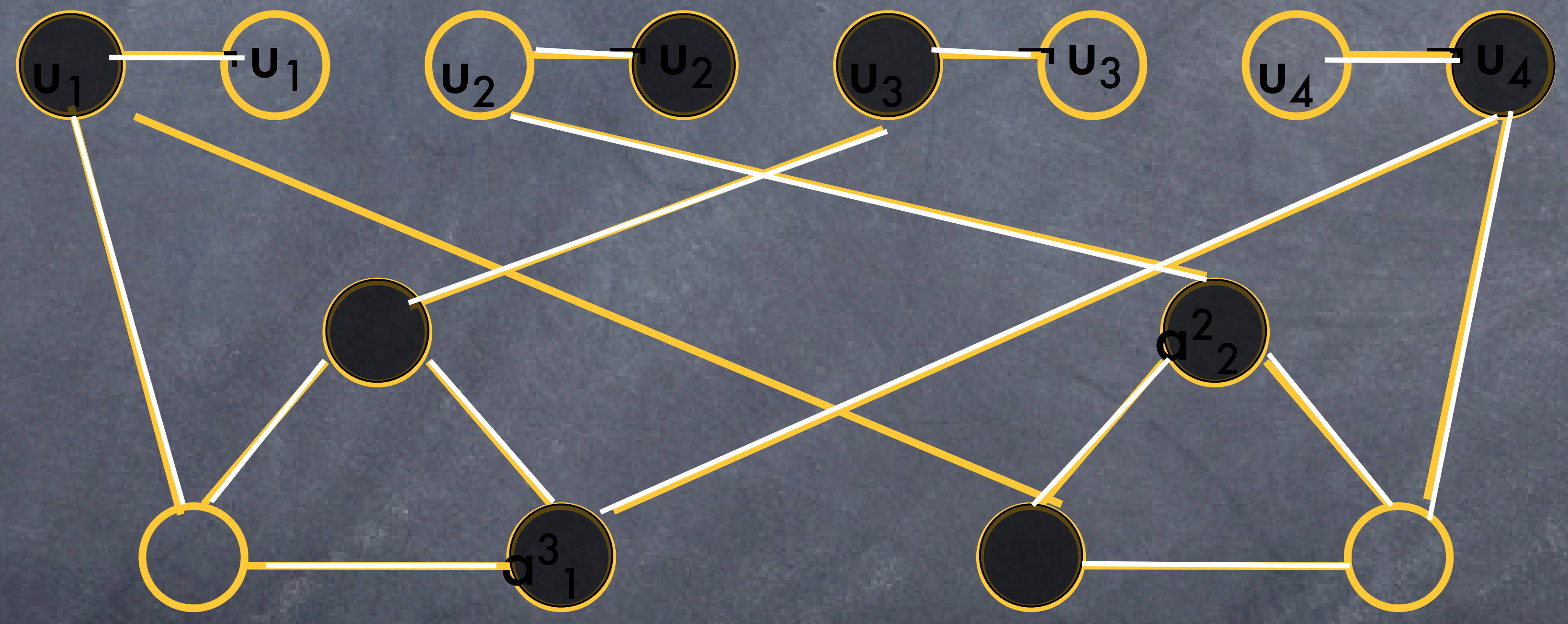

Example of Reduction

$(u_1 \vee \neg u_3 \vee \neg u_4) \wedge (\neg u_1 \vee u_2 \vee \neg u_4)$

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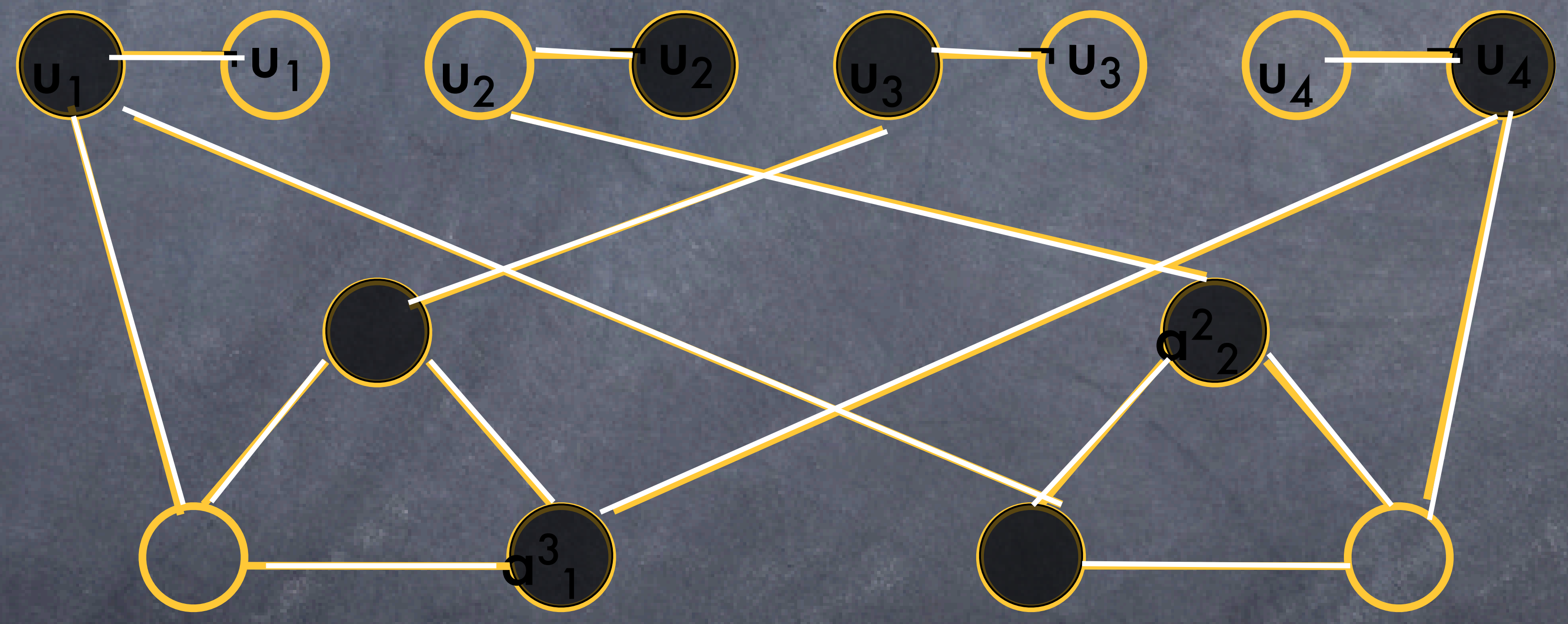


Correctness of Reduction

Since one from each pair is chosen, the edges in the pairs are covered.

Since two from each triangle are chosen, the edges in the triangles are covered.

For edges between triangles and pairs:

- edge to a true literal is covered by pair choice
- edges to false literals are covered by triangle choices

Correctness of Reduction

Conversely, suppose that G has a vertex cover C of size at most K.

To cover the edges in the pairs, C must contain at least one node in each pair

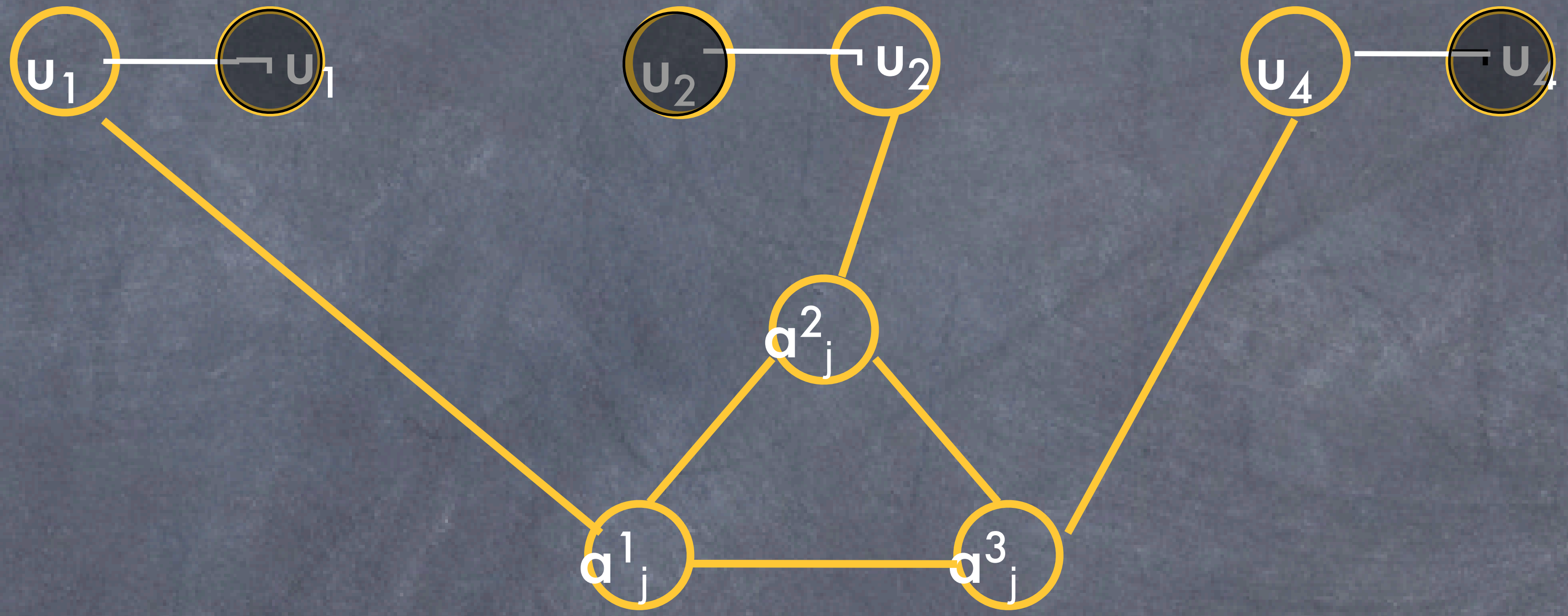
To cover the edges in the triangles, C must contain at least two nodes in each triangle

Since there are n pairs and m triangles, and since $K = n + 2m$, C contains exactly one from each pair and two from each triangle.

Correctness of Reduction

- Use choice of nodes in pairs to define a truth assignment:
  - if node u_i is chosen, then set variable u_i to T
  - if node $\neg u_i$ is chosen, then set variable u_i to F
- Why is this a satisfying truth assignment?
- Seeking a contradiction, suppose that some clause has no true literal … [complete the proof!] …

Correctness of Reduction



In order to cover the triangle-to-literal edges, all three nodes in this triangle must be chosen, contradicting fact that only two can be chosen (since size is n + 2m).

Running Time of the Reduction

- Show graph constructed is not too much bigger than the input 3SAT formula:
  - number of nodes is $2n + 3m$
  - number of edges is $n + 3m + 3m$
- Size of VC input is polynomial in size of 3SAT input, and rules for constructing the VC input are quick to calculate, so running time is polynomial.

Clique

Clique

Let $G=(V,E)$ be a graph. A subset Q of the set of vertices is called a clique if and only if there is an edge between any two vertices in Q.

Clique Problem: Given a graph G and a positive integer K, does there exist an independent set of size K in G?

CLIQUE vs. VC

- The complement of graph $G = (V,E)$ is the graph $G_c = (V,E_c)$, where E_c consists of all the edges that are missing in G.

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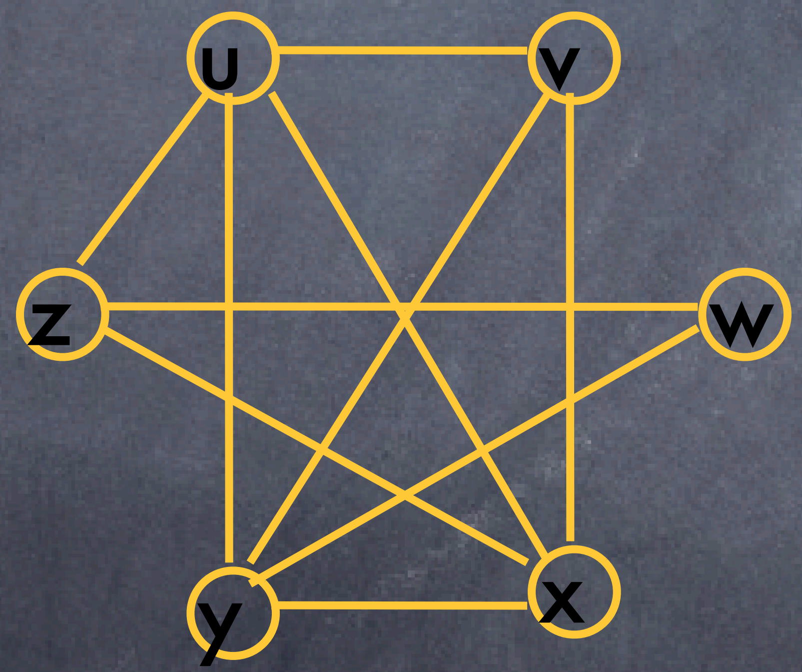

CLIQUE vs. VC

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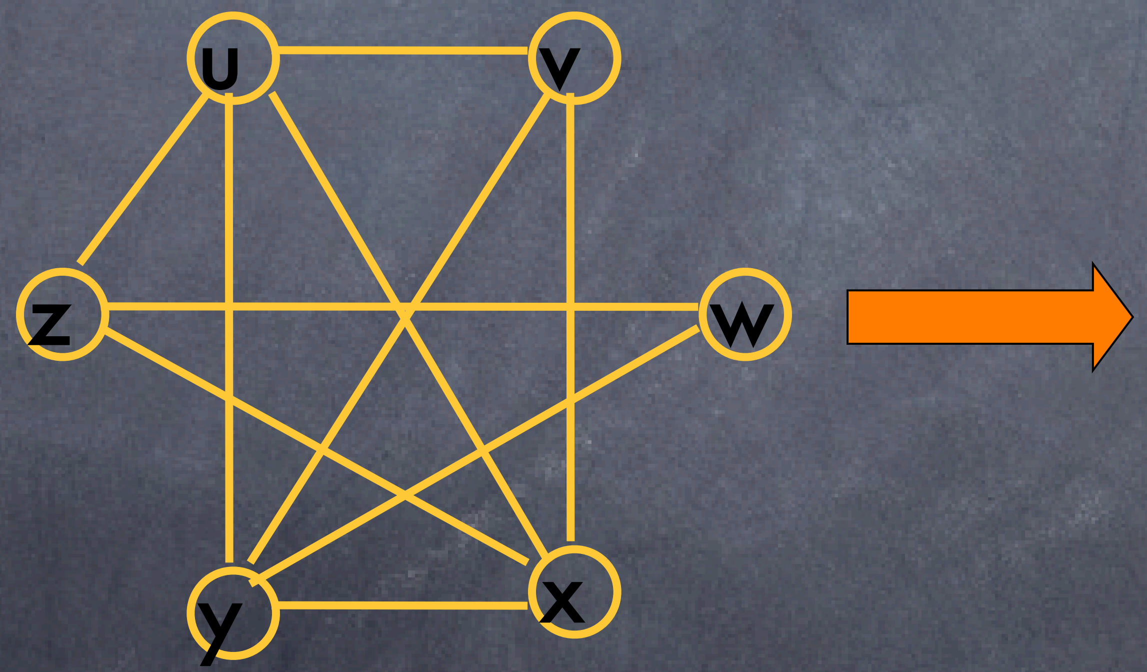

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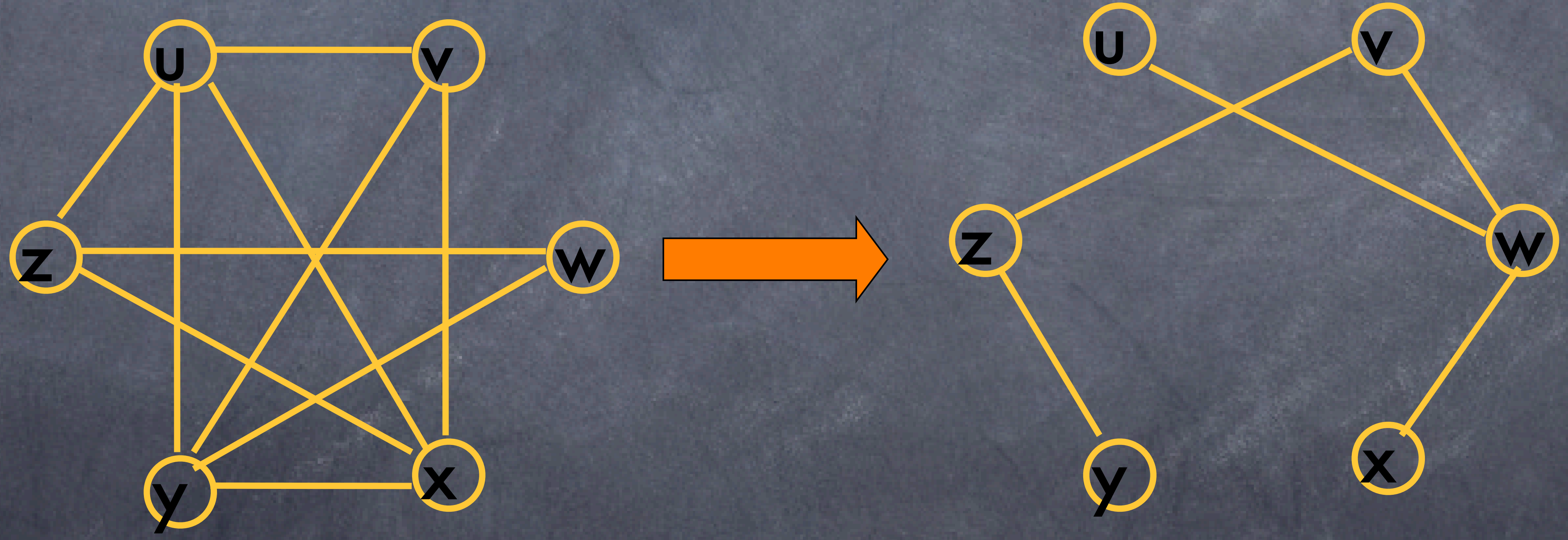
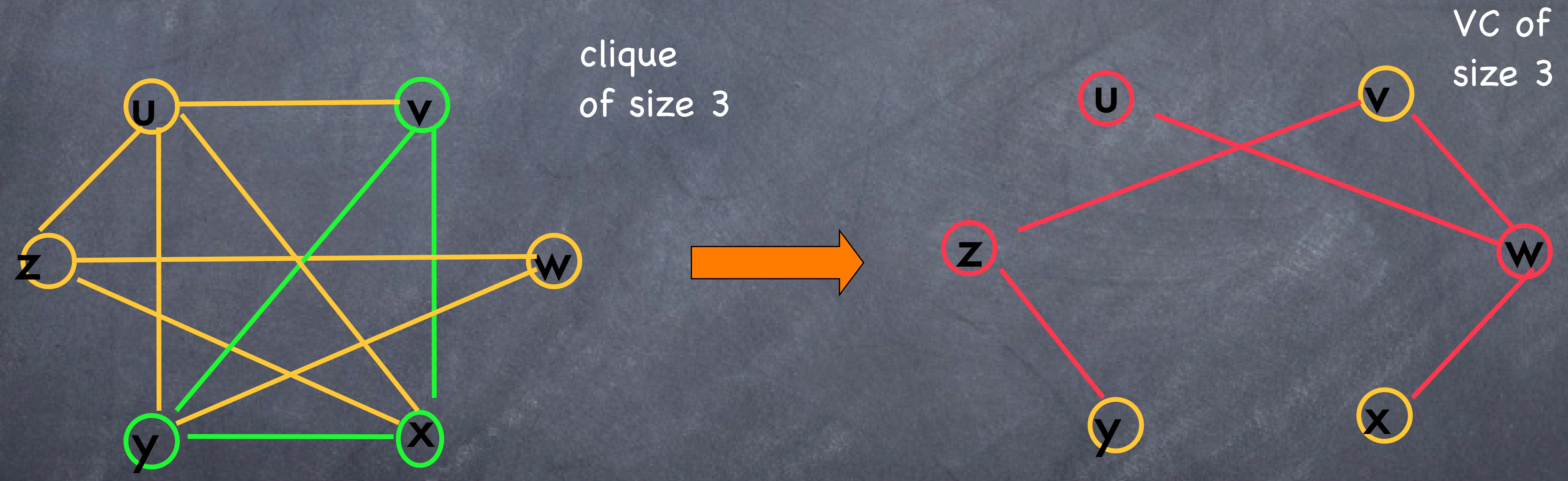


CLIQUE vs. VC

Theorem: Q is a clique of G if and only if C = V – Q is a vertex cover of the complementary graph G_c.



The nodes in Q would only "cover" missing edges and thus are not needed in G_c

VC and CLIQUE

Corollary: VC $\leq_p$ CLIQUE and CLIQUE $\leq_p$ VC.

NP-Completeness of Clique

Since CLIQUE is obviously in NP, and the NP-complete problem Vertex Cover satisfies Vertex Cover $\leq_p$ CLIQUE, it follows that CLIQUE is NP-complete.

Independent Set

Independent Set

Let $G=(V,E)$ be a graph. A subset I of the set of vertices is called an independent set if and only if there are no edges between the elements of I, that is, $(I \times I) \cap E = \emptyset$.

Independent Set Problem: Given a graph G and a positive integer K, does there exist an independent set of size K in G?

[This is the same as the clique problem in the complementary graph. So CLIQUE $\leq_p$ Independent set. Therefore, Independent Set is NP-complete, as it is obviously in NP.]

Useful Reference

- Additional source: Computers and Intractability, A Guide to the Theory of Intractability, M. Garey and D. Johnson, W. H. Freeman and Co., 1979