Sorting Lower Bound

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based on slides by Prof. Welch

Insertion Sort Review

- How it works:
  - incrementally build up longer and longer prefix of the array of keys that is in sorted order
  - take the current key, find correct place in sorted prefix, and shift to make room to insert it
- Finding the correct place relies on comparing current key to keys in sorted prefix
- Worst-case running time is $\Theta(n^2)$

Insertion Sort Demo

- http://sorting-algorithms.com

Heapsort Review

- How it works:
  - put the keys in a heap data structure
  - repeatedly remove the min from the heap
- Manipulating the heap involves comparing keys to each other
- Worst-case running time is $\Theta(n \log n)$

Heapsort Demo

- http://www.sorting-algorithms.com

Mergesort Review

- How it works:
  - split the array of keys in half
  - recursively sort the two halves
  - merge the two sorted halves
- Merging the two sorted halves involves comparing keys to each other
- Worst-case running time is $\Theta(n \log n)$

Mergesort Demo

- http://www.sorting-algorithms.com

Quicksort Review

- How it works:
  - choose one key to be the pivot
  - partition the array of keys into those keys < the pivot and those ≥ the pivot
  - recursively sort the two partitions
- Partitioning the array involves comparing keys to the pivot
- Worst-case running time is $\Theta(n^2)$

Quicksort Demo

- http://www.sorting-algorithms.com

9

Comparison-Based Sorting

- All these algorithms are comparison-based
  - the behavior depends on relative values of keys, not exact values
  - behavior on [1,3,2,4] is same as on [9,25,23,99]
- Fastest of these algorithms was O(n log n).
- We will show that's the best you can get with comparison-based sorting.

Decision Tree

- Consider any comparison based sorting algorithm
- Represent its behavior on all inputs of a fixed size with a decision tree
- Each tree node corresponds to the execution of a comparison
- Each tree node has two children, depending on whether the parent comparison was true or false
- Each leaf represents correct sorted order for that path

Decision Tree Diagram

first comparison:
check if $a_i \leq a_j$

Decision Tree Diagram

first comparison:
check if $a_i \leq a_j$

YES

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first comparison:
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YES

second comparison
if $a_i \leq a_j$: check if
$a_k \leq a_l$

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Decision Tree Diagram

first comparison:
check if $a_i \leq a_j$

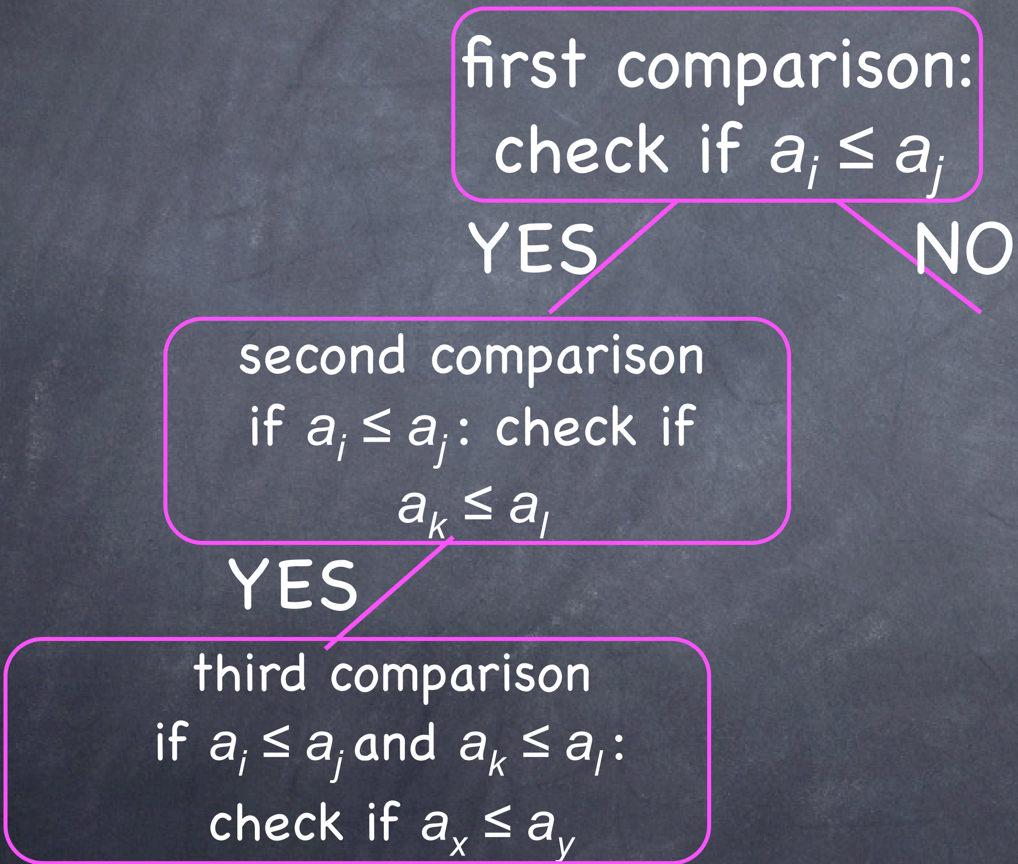
YES

second comparison
if $a_i \leq a_j$: check if
$a_k \leq a_l$

YES

third comparison
if $a_i \leq a_j$ and $a_k \leq a_l$:
check if $a_x \leq a_y$

Decision Tree Diagram



Decision Tree Diagram

first comparison:
check if $a_i \leq a_j$

YES

second comparison
if $a_i \leq a_j$: check if
$a_k \leq a_l$

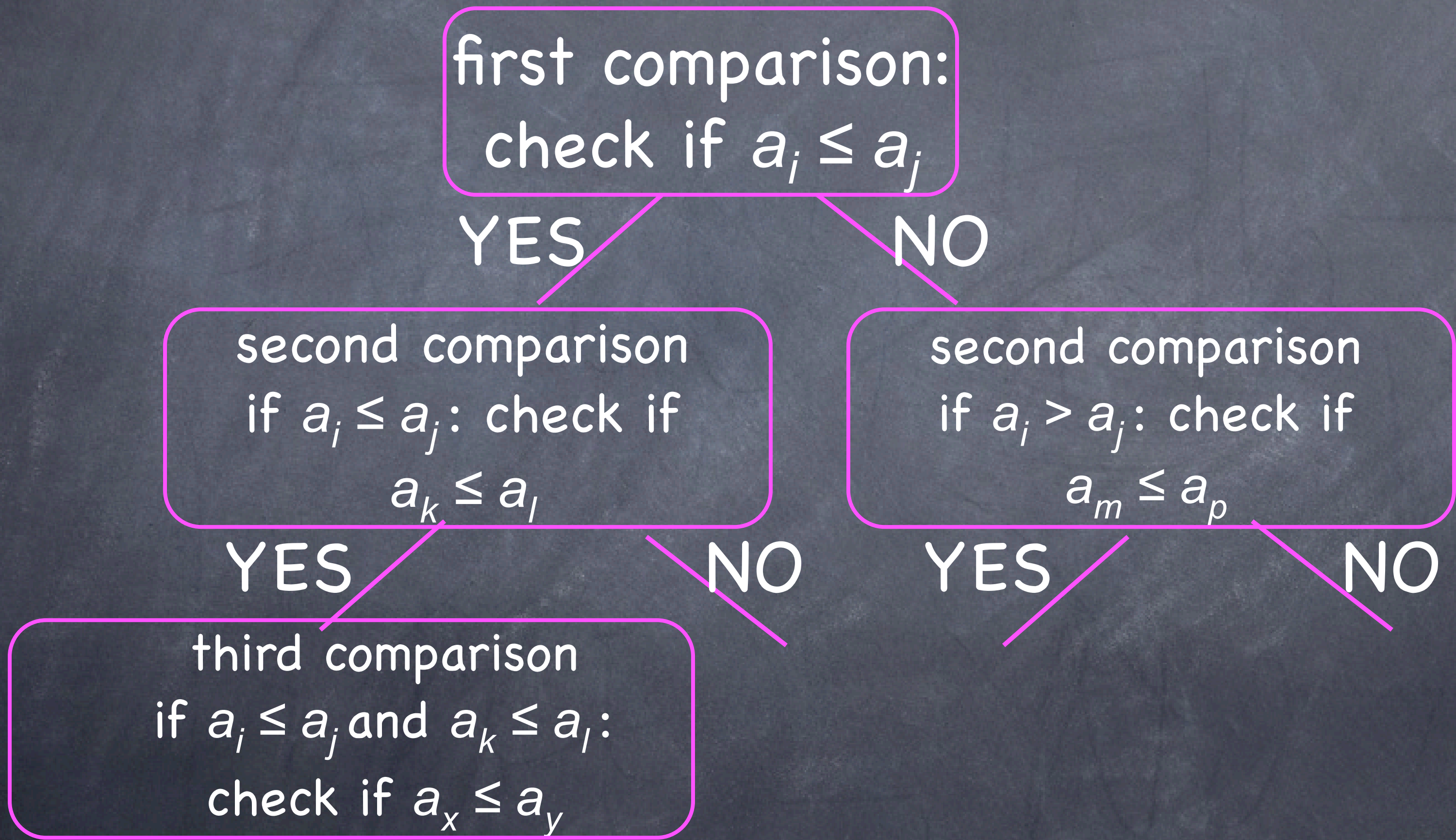
YES

third comparison
if $a_i \leq a_j$ and $a_k \leq a_l$:
check if $a_x \leq a_y$

NO

second comparison
if $a_i > a_j$: check if
$a_m \leq a_p$

Decision Tree Diagram



Insertion Sort

```
for j := 2 to n to
   key := a[j]
   i := j-1
   while i > 0 and a[i] > key do
      a[i+1] := a[i]
      i := i -1
   endwhile
   a[i+1] := key
endfor
```

comparison

Insertion Sort for n = 3

$a_1 \leq a_2$?

14

Insertion Sort for n = 3

$a_1 \leq a_2$?

YES

Insertion Sort for n = 3

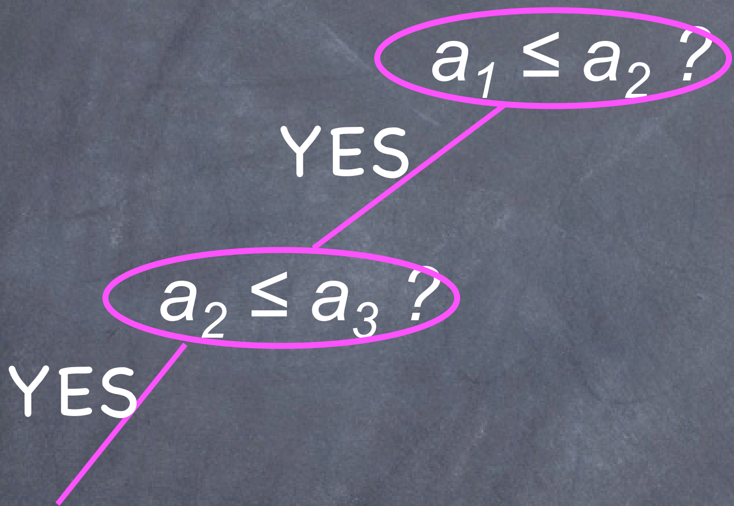
$a_1 \leq a_2$?

YES

$a_2 \leq a_3$?

14

Insertion Sort for n = 3



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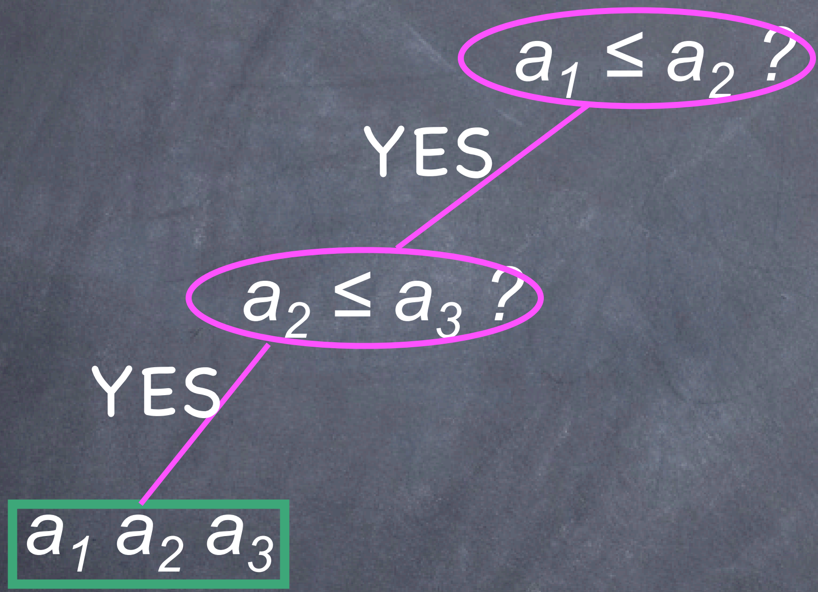

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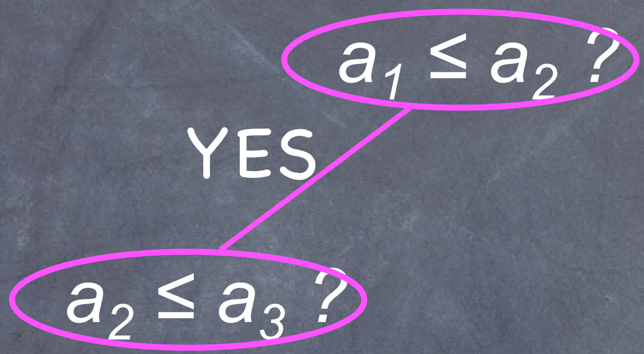

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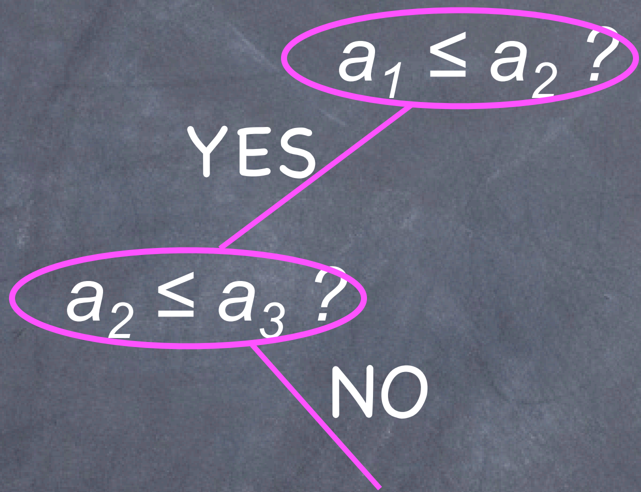

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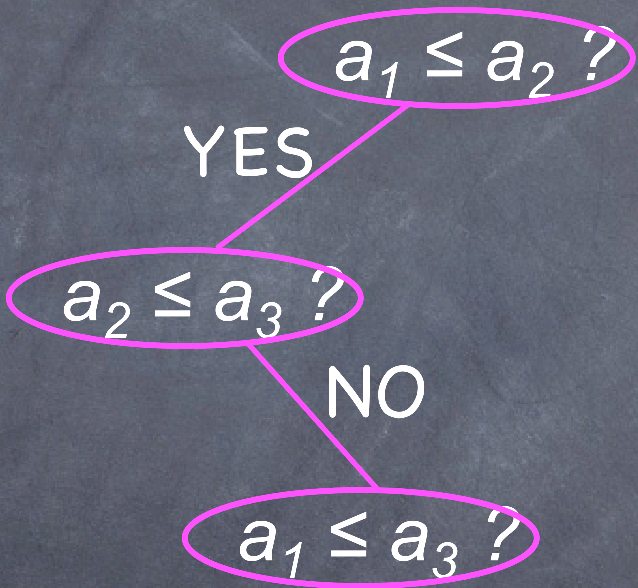

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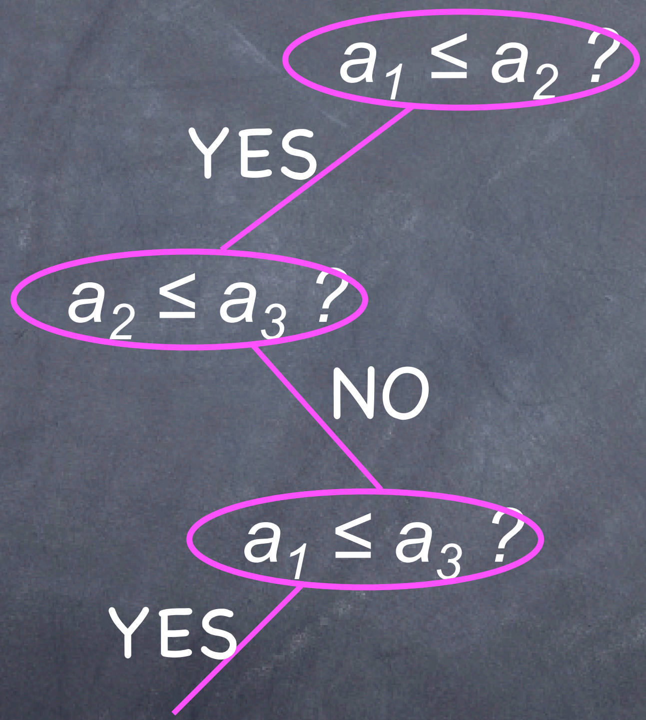

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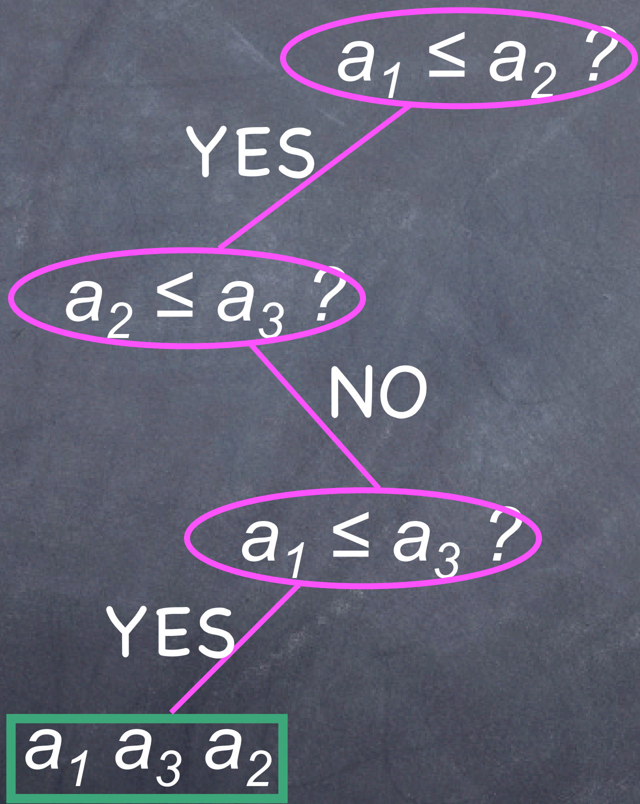

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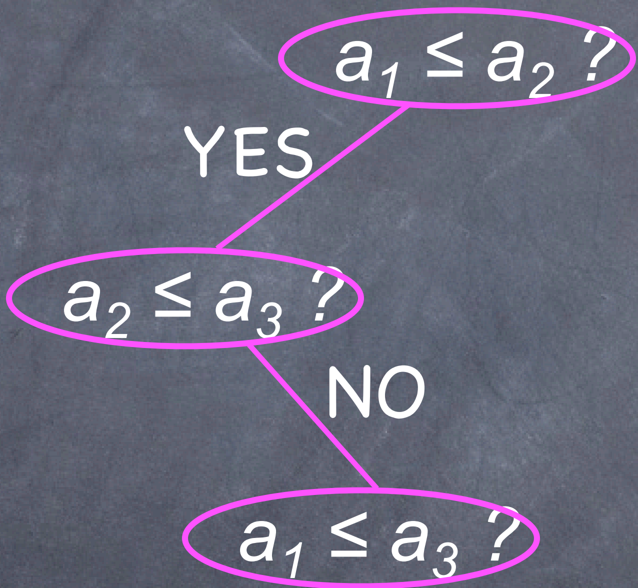

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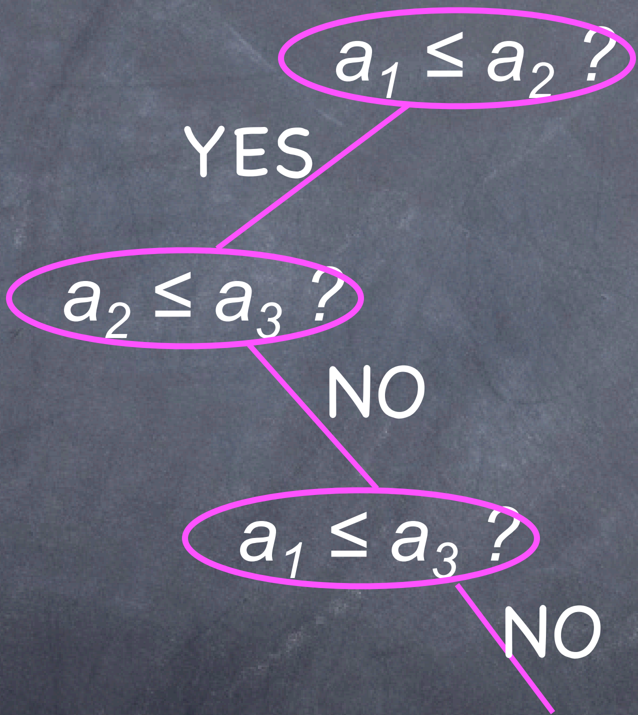

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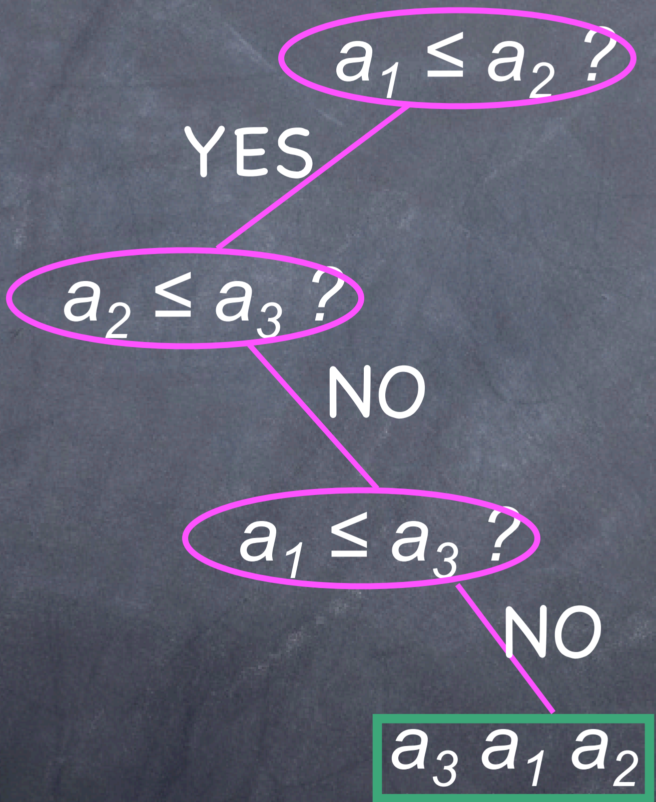

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$a_1 \leq a_2$?

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$a_1 \leq a_2$?

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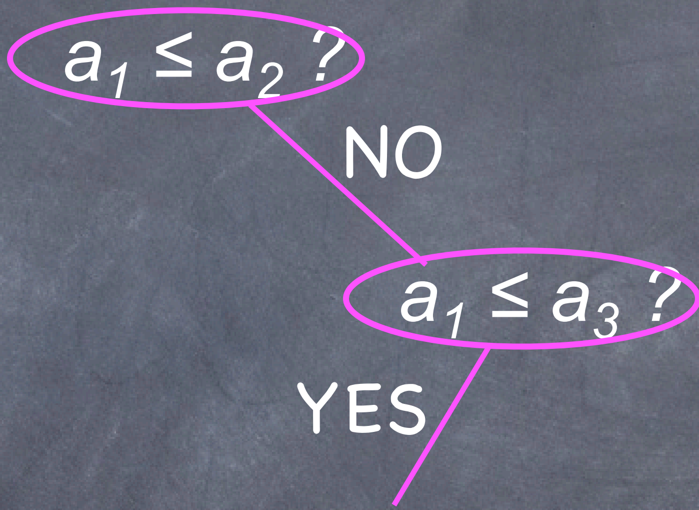

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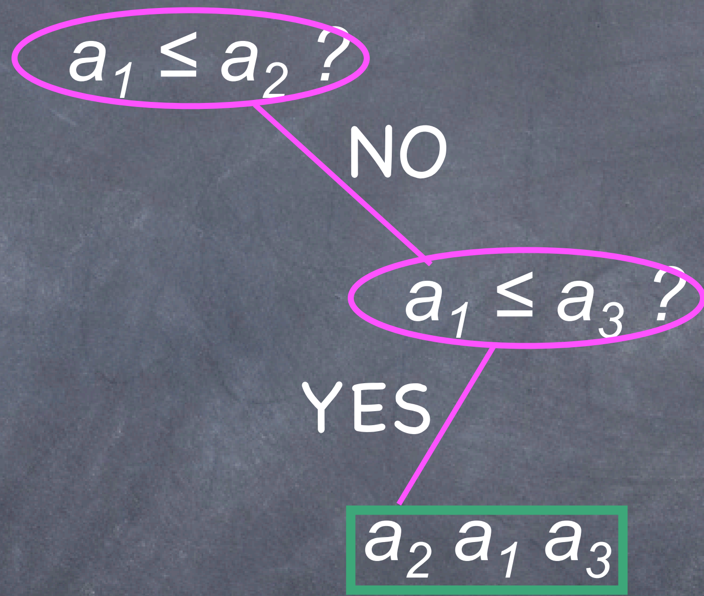

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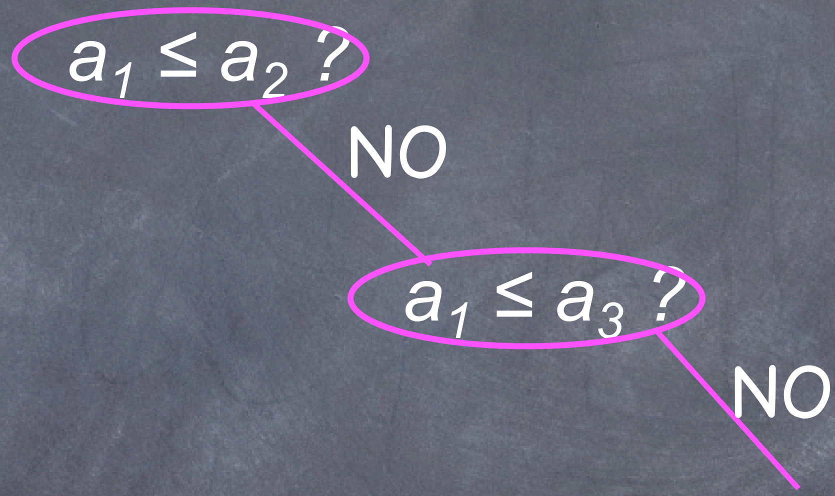

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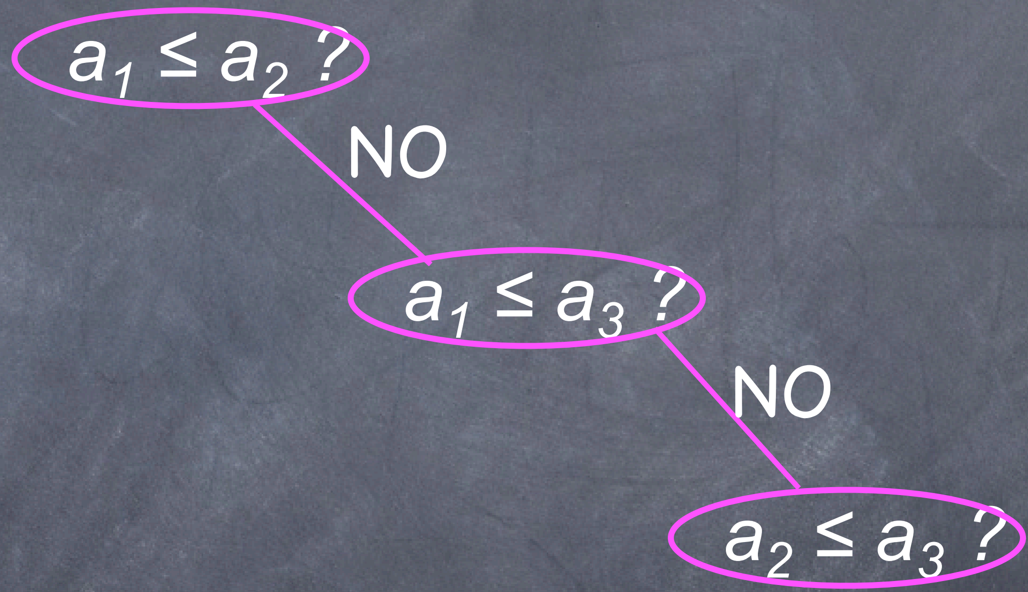

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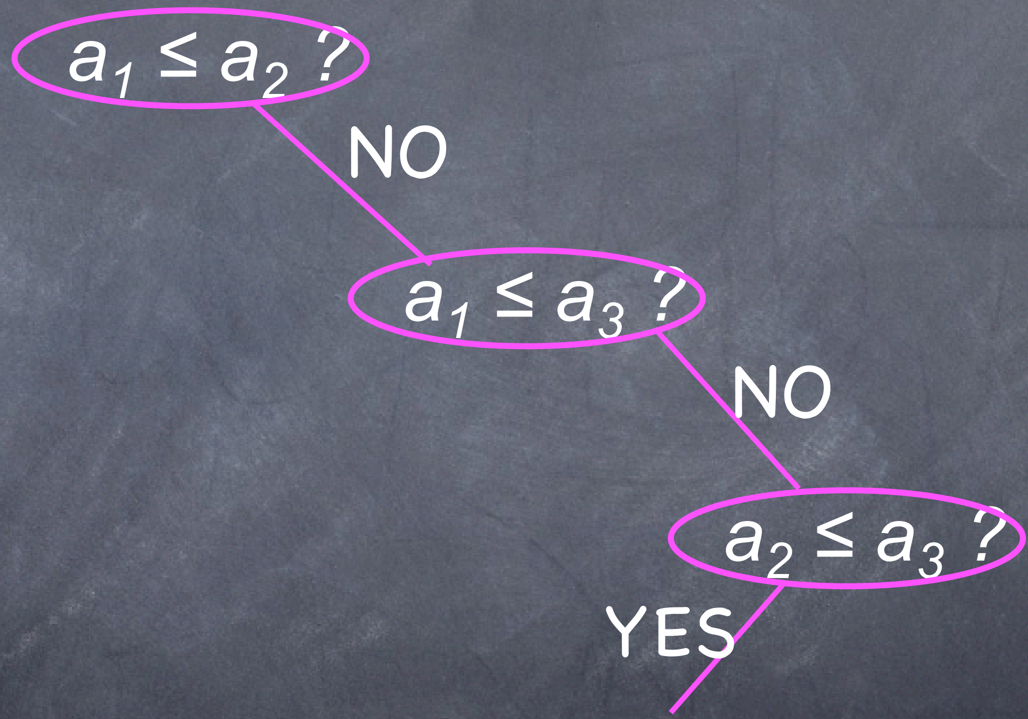

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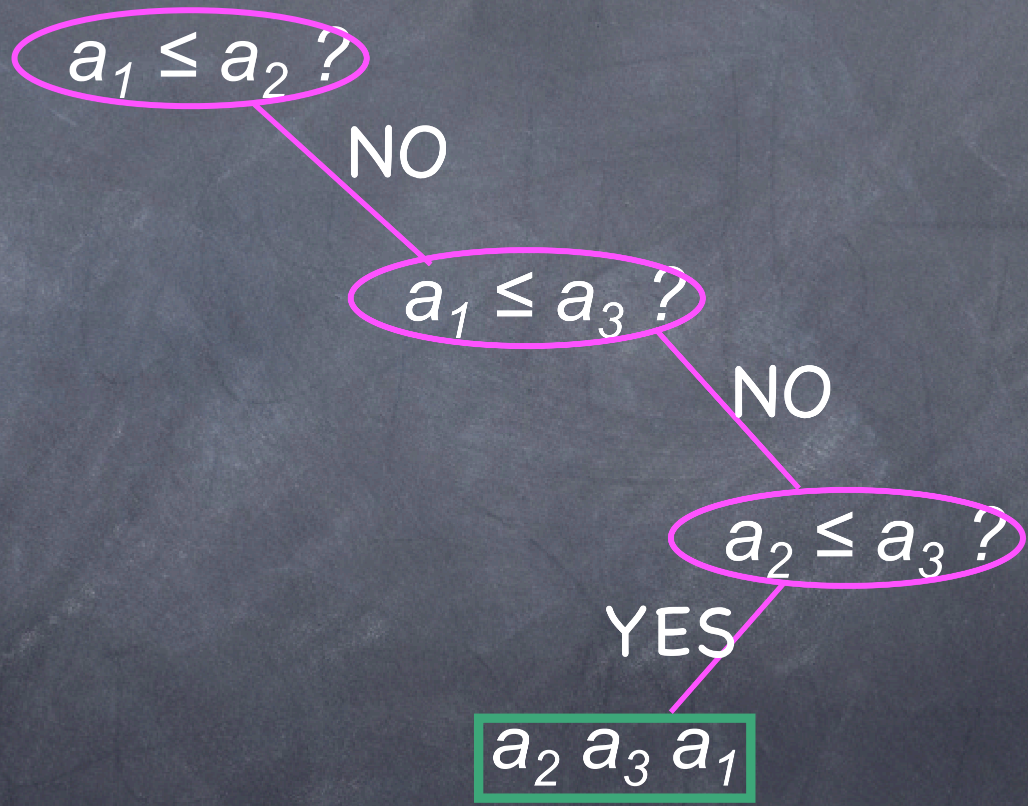

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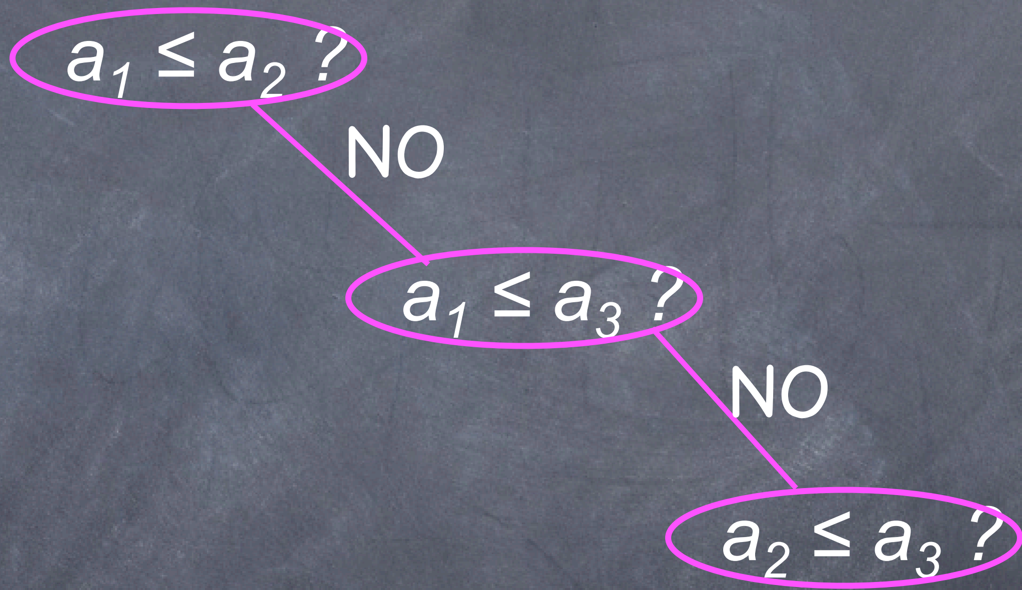

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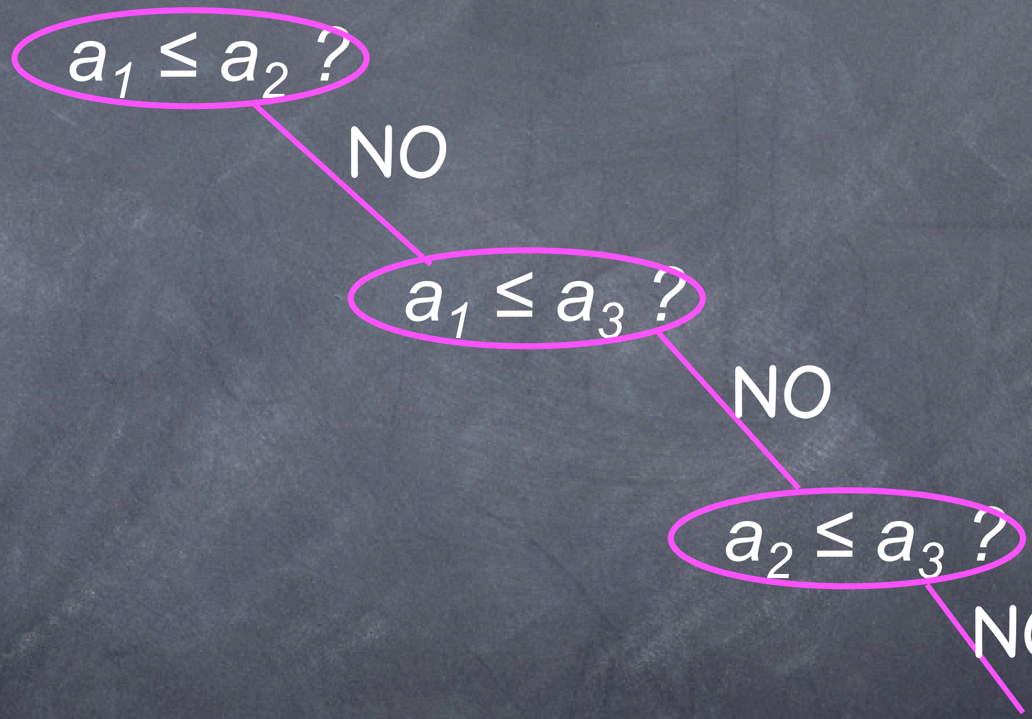

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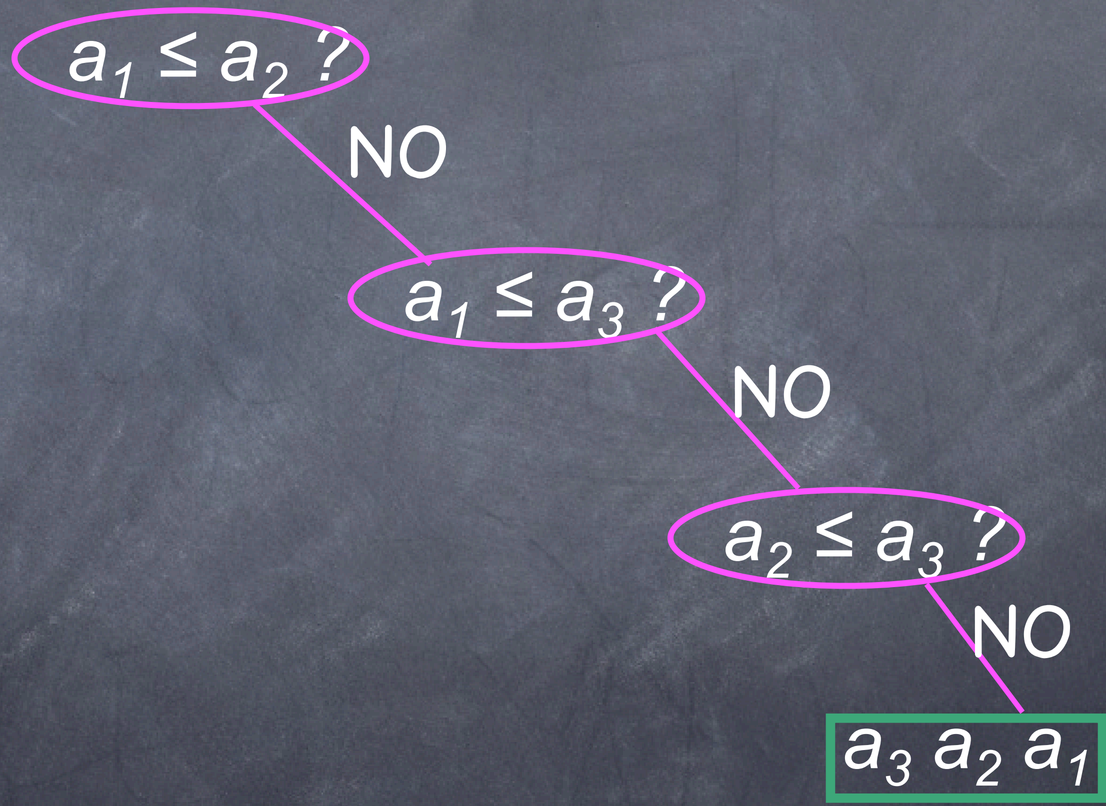

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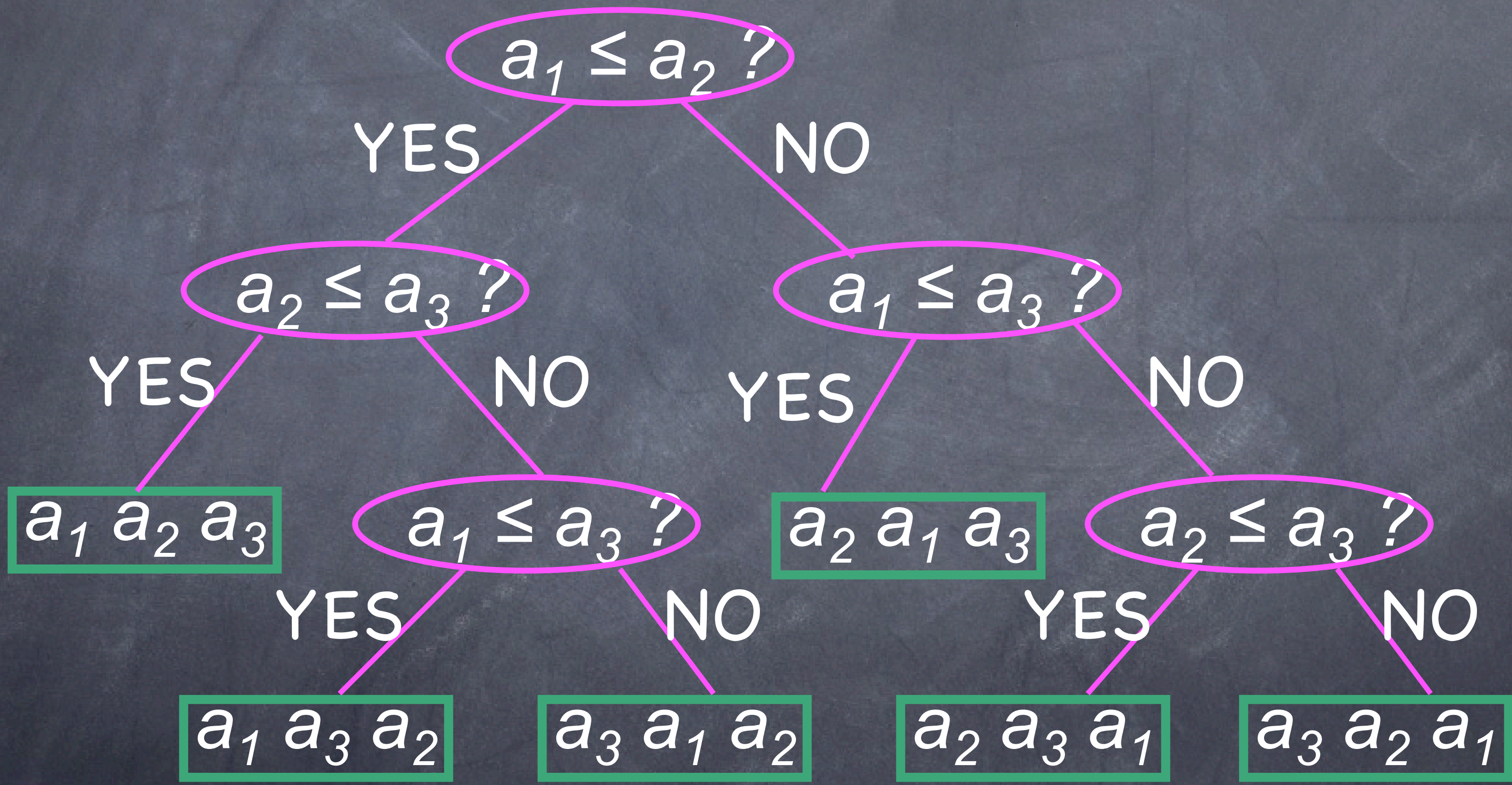

How Many Leaves?

- Must be at least one leaf for each permutation of the input
  - otherwise there would be a situation that was not correctly sorted
- Number of permutations of n keys is n!.
- Idea: since there must be a lot of leaves, but each decision tree node only has two children, tree cannot be too shallow
  - depth of tree is a lower bound on running time

Key Lemma

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Height of a binary tree with n! leaves is

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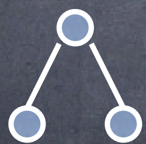

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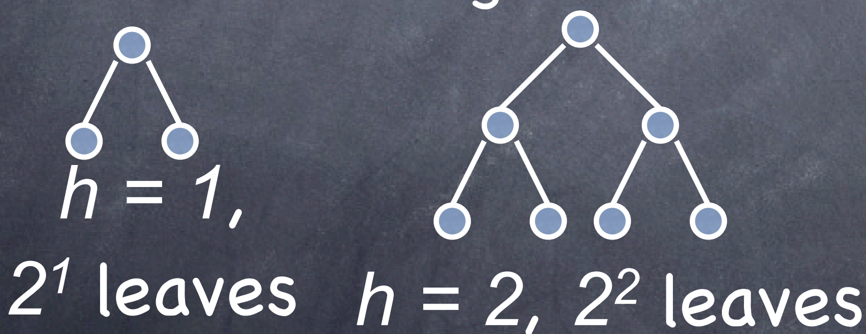

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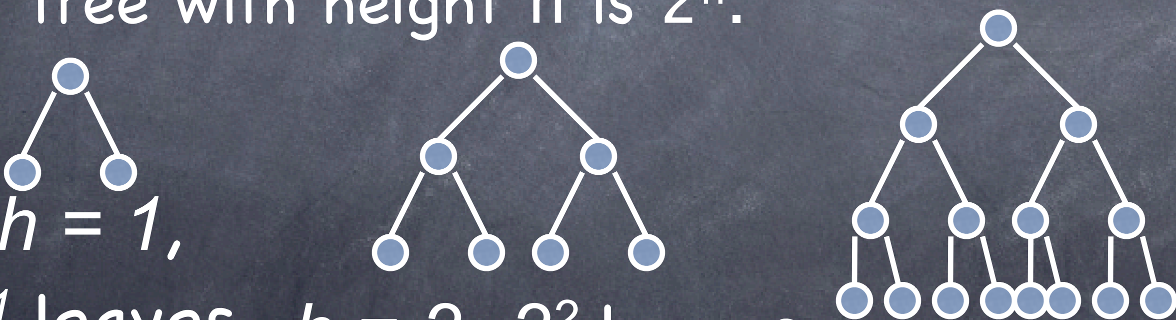

$h = 1$, 2^1 leaves $h = 2$, 2^2 leaves $h = 3$, 2^3 leaves

Proof of Lemma

- Let h be the height of decision tree, so it has at most 2^h leaves.
- The actual number of leaves is n!, hence

$$2^h \geq n!$$

$$h \geq \log(n!)$$

$$= \log(n(n-1)(n-1)\ldots(2)(1))$$

$$\geq (n/2)\log(n/2) \qquad \text{by algebra}$$

$$= \Omega(n \log n)$$

Finishing Up

- Any binary tree with n! leaves has height $\Omega(n \log n)$.
- Decision tree for any c-b sorting alg on n keys has height $\Omega(n \log n)$.
- Any c-b sorting alg has at least one execution with $\Omega(n \log n)$ comparisons
- Any c-b sorting alg has $\Omega(n \log n)$ worst-case running time.