The asymptotic equality is often a bit too strict. Sometimes it is desirable to relax the constraints and consider
(a) the growth up to a constant factor and
(b) without the need for the existence of a limit.
Let $f$ and $g$ denote functions from the natural numbers to the real numbers. We say that $f$ and $g$ have the **same order of growth** and write $f \asymp g$ or $f \in \Theta(g)$ if and only if there exist positive real constants $c$ and $C$ and a natural number $n_0$ such that

$$c|g(n)| \leq |f(n)| \leq C|g(n)|$$

holds for all $n \geq n_0$. 
The notation $f \asymp g$ goes back to Hardy and is popular in mathematics. Computer scientists like to express this in the form $f \in \Theta(g)$, where

$$
\Theta(g) = \{ f : \mathbb{N} \rightarrow \mathbb{R} \mid f \asymp g \}
$$

is the set of functions that have the same order of growth as $g$. If $f \in \Theta(g)$ or $f \asymp g$, then we also say that $g$ is an **asymptotically tight bound** for $f$. 


Proposition

Let $f$ and $g$ be functions from the set of natural numbers to the set of real numbers. If $g$ is a positive function and the limit

$$d = \lim_{n \to \infty} \frac{|f(n)|}{|g(n)|}$$

exists and is a nonzero real number $d$, then $f \in \Theta(g)$. 
Proof

It follows from the definition of the limit that for each $\epsilon > 0$ there exists a natural number $n_\epsilon$ such that

$$d - \epsilon \leq \frac{|f(n)|}{|g(n)|} \leq d + \epsilon$$

for all $n \geq n_\epsilon$. In other words, for the constants $c = d - \epsilon$ and $C = d + \epsilon$ there exists an $n_\epsilon$ such that $c|g(n)| \leq |f(n)| \leq C|g(n)|$ holds for all $n \geq n_\epsilon$, which proves $f \in \Theta(g)$. 
Corollary

If two functions $f$ and $g$ are asymptotically equal, $f \sim g$, then they have the same order of growth, that is, $f \asymp g$. 
Example

Let $f(n) = (2 + (-1)^n)n^2$ and $g(n) = n^2$. Then the limit

$$\lim_{n \to \infty} \frac{|f(n)|}{|g(n)|}$$

does not exist, as the quotient fluctuates between 3 and 1, but

$$|g(n)| \leq |f(n)| \leq 3|g(n)|$$

holds for all $n \geq 1$; hence, $f \in \Theta(g)$. 
We can characterize \( f \in \Theta(g) \) using limit superior and limit inferior from calculus. We recall the relevant terminology.

Let \( f \) be a function from the set of natural numbers to the set of real numbers. The real number \( u \) is an **upper accumulation point** of \( f \) if and only if the following two conditions are met:

**U1.** For each \( \epsilon > 0 \) there exist infinitely many natural numbers \( n \) such that \( f(n) > u - \epsilon \),

**U2.** For each \( \epsilon > 0 \) there exist at most finitely many natural numbers such that \( f(n) > u + \epsilon \).

If an upper accumulation point of \( f \) exists, then it is unique.
The function $f$ is called **bounded above** if and only if there exists a real number $u$ such that $f(n) \leq u$ holds for all natural numbers $n$. If $f$ has an upper accumulation point, then it is bounded above. The **limit superior** of $f$ is defined as

$$\limsup_{n \to \infty} f(n) = \begin{cases} +\infty & \text{if } f \text{ is not bounded above}, \\ u & \text{if the upper accumulation point } u \text{ of } f \text{ exists}, \\ -\infty & \text{otherwise}. \end{cases}$$

Unlike the limit of $f$, the limit superior of $f$ always exists.
Example

Let \( f(n) \) denote the function given by

\[
f(n) = \begin{cases} 
2 + 1/n & \text{if } n \text{ is even} \\
1 - 1/n & \text{if } n \text{ is odd}
\end{cases}
\]

We can conclude that \( \limsup_{n \to \infty} f(n) = 2 \) and \( \liminf_{n \to \infty} f(n) = 1 \).
Proposition

Let $f$ be a function from the natural numbers to the real numbers, and $g$ an eventually nonzero function from the natural numbers to the real numbers. Then $f \in \Theta(g)$ if and only if

$$\liminf_{n \to \infty} \frac{|f(n)|}{|g(n)|} > 0 \quad \text{and} \quad \limsup_{n \to \infty} \frac{|f(n)|}{|g(n)|} < \infty.$$
Proof

If \( f \in \Theta(g) \), then there exists a positive constants \( c \) and \( C \) and a natural number \( n_0 \) such that \( c \leq \frac{|f(n)|}{|g(n)|} \leq C \) holds for all \( n \geq n_0 \). This implies that

\[
\liminf_{n \to \infty} \frac{|f(n)|}{|g(n)|} \geq c > 0 \quad \text{and} \quad \limsup_{n \to \infty} \frac{|f(n)|}{|g(n)|} \leq C < \infty
\]

hold.
If \( f \in \Theta(g) \), then there exists a positive constants \( c \) and \( C \) and a natural number \( n_0 \) such that \( c \leq \frac{|f(n)|}{|g(n)|} \leq C \) holds for all \( n \geq n_0 \). This implies that
\[
\liminf_{n \to \infty} \frac{|f(n)|}{|g(n)|} \geq c > 0 \quad \text{and} \quad \limsup_{n \to \infty} \frac{|f(n)|}{|g(n)|} \leq C < \infty
\]
hold.

Conversely, suppose that \( f \) and \( g \) are functions satisfying
\[
c := \liminf_{n \to \infty} \frac{|f(n)|}{|g(n)|} > 0 \quad \text{and} \quad C := \limsup_{n \to \infty} \frac{|f(n)|}{|g(n)|} < \infty.
\]
By definition of the limit superior and inferior, for any \( \epsilon \) in the range \( 0 < \epsilon < c \) there exists a natural number \( n_0 \) such that
\[
0 < c - \epsilon \leq \frac{|f(n)|}{|g(n)|} \leq (C + \epsilon)
\]
holds for all \( n \geq n_0 \). Multiplying these inequalities by \( |g(n)| \) shows that \( f \in \Theta(g) \) holds.