Let \( f \) and \( g \) be functions from the natural numbers to the real numbers. We say that \( g \) is an **asymptotic upper bound** for \( f \) and write \( f \in O(g) \) if and only if there exists a positive real constant \( C \) and a natural number \( n_0 \) such that

\[
|f(n)| \leq C|g(n)|
\]

holds for all \( n \geq n_0 \).
Proposition

Let $f$ be a function from the natural numbers to the real numbers, and $g$ an eventually nonzero function from the natural numbers to the real numbers. Then $f(n) \in O(g(n))$ if and only if

$$\limsup_{n \to \infty} \frac{|f(n)|}{|g(n)|} < \infty.$$
Corollary

Let $f$ and $g$ be functions from the set of natural numbers to the set of real numbers. If the limit

$$\lim_{n \to \infty} \frac{|f(n)|}{|g(n)|}$$

exists and is finite, then $f \in O(g)$.
We say that $g$ is a strict asymptotic upper bound for $f$ and write $f \in o(g)$ if and only if for every $\epsilon > 0$ there exists a natural number $n_\epsilon$ such that

$$|f(n)| \leq \epsilon |g(n)|$$

holds for all $n \geq n_\epsilon$. By definition, $f \in o(g)$ implies that $f \in O(g)$. 
Proposition

Let $f$ and $g$ be functions from the set of natural numbers to the set of real numbers such that $g$ is eventually nonzero. Then $f \in o(g)$ if and only if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

holds.
Proof

Suppose that (1) holds. By definition of the limit, this means that for any \( \epsilon > 0 \) there exists a natural number \( n_\epsilon \) such that

\[
\left| \frac{f(n)}{g(n)} \right| < \epsilon
\]

holds for all \( n \geq n_\epsilon \). This is equivalent to the condition that for each \( \epsilon > 0 \) there exists an \( n_\epsilon \) such that

\[
|f(n)| \leq \epsilon |g(n)|
\]

holds for all \( n \geq n_\epsilon \). In other words, (1) is equivalent to \( f \in o(g) \).
Corollary

Let $f$ and $g$ be functions from the set of real natural numbers to the set of real numbers. Suppose that $f = o(g)$. Then

$$g + f = O(g).$$

Example

Since $n^{1000} + n^2 + 1 \in o(\exp(n))$, we have

$$\exp(n) + n^{1000} + n^2 + 1 \in O(\exp(n)).$$
Recall that the Harmonic number satisfies

\[ H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - E(n), \]

where \( \gamma \) is Euler’s constant \( \gamma \approx 0.5772156649 \), and the value of the error term \( E(n) \) is in the range \( 0 < E(n) < 1/(252n^6) \). It follows that

\[ H_n = \log n + \gamma + O \left( \frac{1}{n} \right). \]
**Rules**

**Constants**
If $c$ is a nonzero constant, then

\[
c O(f(n)) = O(f(n)),
\]
(2)
\[
O(cf(n)) = O(f(n)).
\]
(3)

**Idempotency**
The Big Oh operator is idempotent, meaning that

\[
O(O(f(n)) = O(f(n)).
\]
(4)
Multiplications

The multiplication of Big Oh expressions follows the rules

\[ O(f(n))O(g(n)) = O(f(n)g(n)), \]
\[ O(f(n)g(n)) = f(n)O(g(n)). \]

Absorbtion.

We can simplify Big Oh expressions using the rule

\[ O(f(n)) + O(g(n)) = O(g(n)) \text{ provided that } f(n) = O(g(n)). \]
Rules

Powers
For all positive integers $k$, we have

$$(f(n) + g(n))^k = O((f(n))^k) + O((g(n))^k). \quad (8)$$

Linear Combinations
If $f(n) = O(h(n))$ and $g(n) = O(h(n))$, then

$$af(n) + bg(n) = O(h(n)) \quad \text{for all } a, b \in \mathbb{C}. \quad (9)$$
Swap

The next rule allows you to swap Big Oh terms.

If $f(n) = g(n) + O(h(n))$ then $g(n) = f(n) + O(h(n))$.  \hspace{1cm} (10)