# 3SAT

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### 3SAT

Given a boolean function in conjunctive normal form such that every clause contains exactly three literals, decide whether the formula is satisfiable.

[This a special case of SAT]

## Proving NP-Completeness

How do you prove that a decision problem L is NP-complete? (1) Show that L is in NP. (2.a) Choose an appropriate known NP-complete language L'. (2.b) Show L'  $\leq_p$  L

# Proof Strategy

(1) 3SAT is in NP, since we can check in polynomial time whether a given truth assignment evaluates to true. (2.a) Choose SAT as a known NP-complete problem. (2.b) Describe a reduction from SAT inputs to 3SAT inputs computable in polynomial time 

SAT input is satisfiable iff constructed 3SAT input is satisfiable 

### General Idea of the Reduction

We're given an arbitrary CNF formula  $C = c_1 \wedge c_2 \wedge \dots \wedge c_m$  over set of variables, where each  $c_i$  is a clause (a disjunction of literals).

We will replace each clause  $c_i$  with a conjunction of clauses  $c_i'$ , and may use some extra variables. Each clause in  $c_i'$  will have exactly 3 literals. The transformed input will be conjunction of all the clauses in all the  $c_i'$ .

Let  $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$ 

Case 1: k = 1. Use extra variables  $y_i^1$  and  $y_i^2$ . Replace  $c_i$  with 4 clauses:  $(z_1 \vee y_i^1 \vee y_i^2) \wedge (z_1 \vee y_i^1 \vee y_i^2) \wedge (z_1 \vee y_i^1 \vee y_i^2) \wedge (z_1 \vee y_i^2) \wedge (z_1 \vee y_i^2)$ 

Let  $c_i = z_1 \vee z_2 \vee \dots \vee z_k$ 

Case 2: k = 2. Use extra variable  $y_i^1$ . Replace  $c_i$  with 2 clauses:  $(z_1 \vee z_2 \vee \neg y_i^1) \wedge (z_1 \vee z_2 \vee y_i^1).$ 



Let  $c_i = z_1 \vee z_2 \vee \dots \vee z_k$ 

Case 3: k = 3. No extra variables are needed. Keep  $c_i: (z_1 \vee z_2 \vee z_3)$ 



Let  $C_i = Z_1 \vee Z_2 \vee ... \vee Z_k$ 

Case 4: k > 3. Use extra variables  $y_i^1$ , ...,  $y_i^{k-3}$ . Replace  $c_i$  with k-2 clauses:  $(\mathbf{Z}_1 \vee \mathbf{Z}_2 \vee \mathbf{y}_i^1)$ Text  $\wedge (\neg y_i^1 \vee Z_3 \vee y_i^2) \wedge (\neg y_i^2 \vee Z_4 \vee y_i^3) \wedge \dots$  $\wedge (\neg y_i^{k-5} \vee z_{k-3} \vee y_i^{k-4}) \wedge (\neg y_i^{k-4} \vee z_{k-2} \vee y_i^{k-3})$  $\wedge (\neg y_i^{k-3} \vee z_{k-1} \vee z_k)$ 



# Polynomial Time Reduction

Each new formula is at most a constant times larger than the original formula, and the translation is straightforward. Therefore, the reduction is polynomial time.

### Correctness of the Reduction

Show that CNF formula C is satisfiable iff the 3-CNF formula C' constructed is satisfiable.

=>: Suppose that C is satisfiable. We need to construct a satisfying truth assignment for C'.

For variables in C' that are already in C, we use same truth assignments as for C.

How should we assign T/F to the new variables?

### Truth Assignment for New Variables

Let  $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$ 

Case 1: k = 1. Use extra variables  $y_i^1$  and  $y_i^2$ . Replace  $c_i$  with 4 clauses:  $(z_1 \vee y_i^1 \vee y_i^2) \wedge (z_1 \vee y_i^1 \vee y_i^2) \wedge (z_1 \vee y_i^1 \vee y_i^2) \wedge (z_1 \vee y_i^2) \wedge (z_1 \vee y_i^2)$ 

Assign  $y_i$ 's with arbitrary values, as  $z_1$  is true

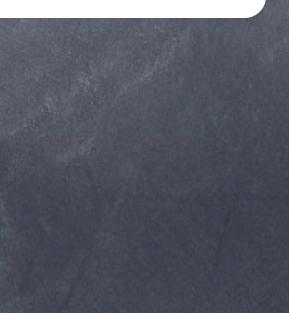
Let  $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$ 

Case 2: k = 2. Use extra variable  $y_i^1$ . Replace  $c_i$  with 2 clauses:

 $(z_1 \vee z_2 \vee \neg y_i^1) \wedge (z_1 \vee z_2 \vee y_i^1).$ 

Assign  $y_i$ 's with arbitrary values, as  $z_1 \vee z_2$  is true





Let  $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$ 

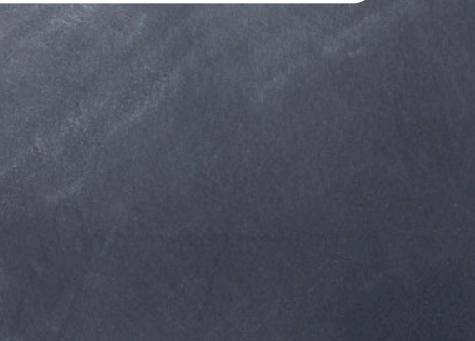
Case 3: k = 3. No extra variables are needed. Keep  $c_i$ :  $(z_1 \vee z_2 \vee z_3)$ 



Let  $C_i = Z_1 \vee Z_2 \vee ... \vee Z_k$ 

Case 4: k > 3. Use extra variables  $y_i^1$ , ...,  $y_i^{k-3}$ . Replace  $c_i$  with k-2 clauses:  $(\mathbf{z}_1 \vee \mathbf{z}_2 \vee \mathbf{y}^1)$  $\wedge (\neg y_i^1 \vee Z_3 \vee y_i^2) \wedge (\neg y_i^2 \vee Z_4 \vee y_i^3) \wedge \dots$  $\wedge (\neg y_i^{k-5} \vee z_{k-3} \vee y_i^{k-4}) \wedge (\neg y_i^{k-4} \vee z_{k-2} \vee y_i^{k-3})$  $\wedge (\neg y_i^{k-3} \vee z_{k-1} \vee z_k)$ 

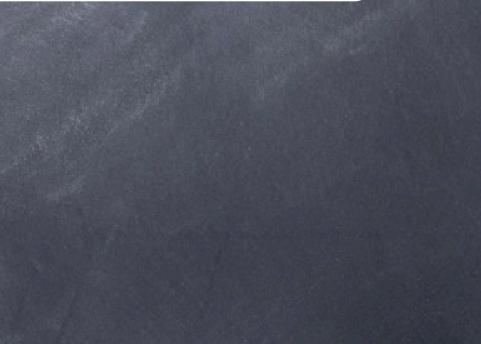
If  $z_1$  or  $z_2$  is true, set all  $y_i$ 's to false, so all later clauses have a true literal.



Let  $C_i = Z_1 \vee Z_2 \vee ... \vee Z_k$ 

Case 4: k > 3. Use extra variables  $y_i^1$ , ...,  $y_i^{k-3}$ . Replace  $c_i$  with k-2 clauses:  $(\mathbf{Z}_1 \vee \mathbf{Z}_2 \vee \mathbf{y}^1)$  $\wedge (\neg y_i^1 \vee Z_3 \vee y_i^2) \wedge (\neg y_i^2 \vee Z_4 \vee y_i^3) \wedge \dots$  $\wedge (\neg y_i^{k-5} \vee z_{k-3} \vee y_i^{k-4}) \wedge (\neg y_i^{k-4} \vee z_{k-2} \vee y_i^{k-3})$  $\wedge (\neg y_i^{k-3} \vee z_{k-1} \vee z_k)$ 

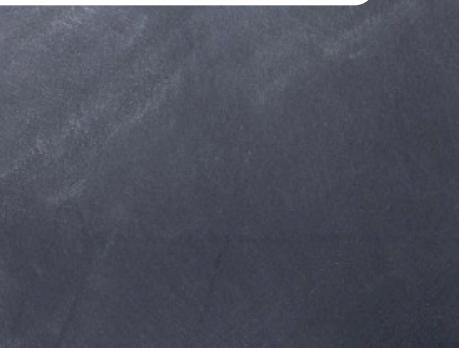
If  $z_{k-1}$  or  $z_k$  is the first true literal of  $c_i$ , set all  $y_i$ 's to true, so all earlier clauses have a true literal.



Let  $C_i = Z_1 \vee Z_2 \vee ... \vee Z_k$ 

Case 4: k > 3. Use extra variables  $y_i^1$ , ...,  $y_i^{k-3}$ . Replace  $c_i$  with k-2 clauses:  $(\mathbf{Z}_1 \vee \mathbf{Z}_2 \vee \mathbf{y}^1)$ later  $y_i$ 's to false.  $\wedge (\neg y_i^1 \vee Z_3 \vee y_i^2) \wedge (\neg y_i^2 \vee Z_4 \vee y_i^3) \wedge \dots$  $\wedge (\neg y_i^{k-5} \vee z_{k-3} \vee y_i^{k-4}) \wedge (\neg y_i^{k-4} \vee z_{k-2} \vee y_i^{k-3})$  $\wedge (\neg y_i^{k-3} \vee z_{k-1} \vee z_k)$ 

If first true literal is in between, set all earlier  $y_i$ 's to true and all



# Correctness of Reduction

<=: Suppose the newly constructed 3SAT formula C' is satisfiable. We must show that the original SAT formula C is also satisfiable.

Use the same satisfying truth assignment for C as for C' (ignoring new variables).

Show each original clause has at least one true literal in it.

### Original Clause is True

Let  $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$ 

Case 1: k = 1. Use extra variables  $y_i^1$  and  $y_i^2$ . Replace  $c_i$  with 4 clauses:  $c_i = (z_1 \vee y_i^1 \vee y_i^2) \wedge (z_1 \vee y_i^1 \vee y_i^2) \wedge (z_1 \vee y_i^1 \vee y_i^2) \wedge (z_1 \vee y_i^2) \wedge (z_1 \vee y_i^1 \vee y_i^2).$ 

> If  $c_i$  is true, then  $c_i = z_1$  must be true, since one pair of literals in  $y_i^1$  and  $y_i^2$  must be true

Let  $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$ 

Case 2: k = 2. Use extra variable  $y_i^1$ . Replace  $c_i$  with 2 clauses:  $C_i = (Z_1 \vee Z_2 \vee \neg Y_i^1) \wedge (Z_1 \vee Z_2 \vee Y_i^1).$ 

If  $c_i'$  is true, then  $c_i = z_1 \vee z_2$  must be true





Let  $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$ 

Case 3: k = 3. No extra variables are needed. Keep  $c_i$ :  $(z_1 \vee z_2 \vee z_3)$ 



Let  $C_i = Z_1 \vee Z_2 \vee ... \vee Z_k$ 

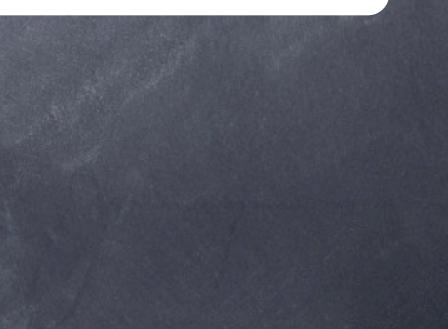
Case 4: k > 3. Use extra variables  $y_i^1$ , ...,  $y_i^{k-3}$ . Replace  $c_i$  with k-2 clauses:

 $(\mathbf{Z}_1 \vee \mathbf{Z}_2 \vee \mathbf{y}_1^1)$  $\wedge (\neg y_i^1 \vee z_3 \vee y_i^2) \wedge (\neg y_i^2 \vee z_4 \vee y_i^3) \wedge \dots$  last clause in  $c_i$  must be false,

contradiction.

 $\wedge (\neg y_i^{k-5} \vee z_{k-3} \vee y_i^{k-4}) \wedge (\neg y_i^{k-4} \vee z_{k-2} \vee y_i^{k-3})$  $\wedge (\neg y_i^{k-3} \vee z_{k-1} \vee z_k)$ 

Suppose that there is a valuation such that  $c_i$  is true and  $c_i$  is false. Then  $y_i^k$  must be true for all k, so the



### Conclusions

We have shown that SAT is in NP There exists a polynomial time reduction from SAT to 3SAT. Therefore, 3SAT is NP-complete.