3SAT

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[partially based on slides by Jennifer Welch]
3SAT

Given a boolean function in conjunctive normal form such that every clause contains exactly three literals, decide whether the formula is satisfiable.

[This a special case of SAT]
Proving NP-Completeness

How do you prove that a decision problem L is NP-complete?

(1) Show that L is in NP.

(2.a) Choose an appropriate known NP-complete language L'.

(2.b) Show $L' \leq_p L$
Proof Strategy

(1) 3SAT is in NP, since we can check in polynomial time whether a given truth assignment evaluates to true.

(2.a) Choose SAT as a known NP-complete problem.

(2.b) Describe a reduction from SAT inputs to 3SAT inputs

- computable in polynomial time
- SAT input is satisfiable iff constructed 3SAT input is satisfiable
General Idea of the Reduction

We're given an arbitrary CNF formula $C = c_1 \land c_2 \land \ldots \land c_m$ over set of variables, where each $c_i$ is a clause (a disjunction of literals).

We will replace each clause $c_i$ with a conjunction of clauses $c_i'$, and may use some extra variables. Each clause in $c_i'$ will have exactly 3 literals. The transformed input will be conjunction of all the clauses in all the $c_i'$. 
Reduction from SAT to 3SAT

Let $c_i = z_1 \lor z_2 \lor \ldots \lor z_k$

Case 1: $k = 1$. Use extra variables $y_i^1$ and $y_i^2$. Replace $c_i$ with 4 clauses:

$$(z_1 \lor y_i^1 \lor y_i^2) \land (z_1 \lor \neg y_i^1 \lor y_i^2) \land (z_1 \lor y_i^1 \lor \neg y_i^2) \land (z_1 \lor \neg y_i^1 \lor \neg y_i^2).$$
Reduction from SAT to 3SAT

Let $c_i = z_1 \lor z_2 \lor \ldots \lor z_k$

Case 2: $k = 2$. Use extra variable $y_i^1$. Replace $c_i$ with 2 clauses:

$$(z_1 \lor z_2 \lor \neg y_i^1) \land (z_1 \lor z_2 \lor y_i^1).$$
Reduction from SAT to 3SAT

Let $c_i = z_1 \lor z_2 \lor \ldots \lor z_k$

Case 3: $k = 3$. No extra variables are needed.

Keep $c_i: (z_1 \lor z_2 \lor z_3)$
Reduction from SAT to 3SAT

Let $c_i = z_1 \lor z_2 \lor \ldots \lor z_k$

Case 4: $k > 3$. Use extra variables $y_i^1, \ldots, y_i^{k-3}$. Replace $c_i$ with $k-2$ clauses:

$$(z_1 \lor z_2 \lor y_i^1) \land (\neg y_i^1 \lor z_3 \lor y_i^2) \land (\neg y_i^2 \lor z_4 \lor y_i^3) \land \ldots$$

$$\land (\neg y_i^{k-5} \lor z_{k-3} \lor y_i^{k-4}) \land (\neg y_i^{k-4} \lor z_{k-2} \lor y_i^{k-3})$$

$$\land (\neg y_i^{k-3} \lor z_{k-1} \lor z_k)$$
Each new formula is at most a constant times larger than the original formula, and the translation is straightforward. Therefore, the reduction is polynomial time.
Correctness of the Reduction

Show that CNF formula $C$ is satisfiable iff the 3-CNF formula $C'$ constructed is satisfiable.

$\implies$: Suppose that $C$ is satisfiable. We need to construct a satisfying truth assignment for $C'$.

For variables in $C'$ that are already in $C$, we use same truth assignments as for $C$.

How should we assign T/F to the new variables?
Truth Assignment for New Variables

Let $c_i = z_1 \lor z_2 \lor \ldots \lor z_k$.

Case 1: $k = 1$. Use extra variables $y_i^1$ and $y_i^2$. Replace $c_i$ with 4 clauses:

$$(z_1 \lor y_i^1 \lor y_i^2) \land (z_1 \lor \neg y_i^1 \lor y_i^2) \land (z_1 \lor y_i^1 \lor \neg y_i^2) \land (z_1 \lor \neg y_i^1 \lor \neg y_i^2).$$

Assign $y_i$'s with arbitrary values, as $z_1$ is true.
Reduction from SAT to 3SAT

Let \( c_i = z_1 \lor z_2 \lor \ldots \lor z_k \)

Case 2: \( k = 2 \). Use extra variable \( y_{i1} \). Replace \( c_i \) with 2 clauses:

\[
(z_1 \lor z_2 \lor \neg y_{i1}) \land (z_1 \lor z_2 \lor y_{i1}).
\]

Assign \( y_i \)'s with arbitrary values, as \( z_1 \lor z_2 \) is true.
Reduction from SAT to 3SAT

Let $c_i = z_1 \lor z_2 \lor \ldots \lor z_k$

Case 3: $k = 3$. No extra variables are needed.

Keep $c_i$: $(z_1 \lor z_2 \lor z_3)$
Reduction from SAT to 3SAT

Let \( c_i = z_1 \lor z_2 \lor ... \lor z_k \)

Case 4: \( k > 3 \). Use extra variables \( y_i^1, ..., y_i^{k-3} \). Replace \( c_i \) with \( k-2 \) clauses:

\[
(z_1 \lor z_2 \lor y_i^1) \\
\land (\neg y_i^1 \lor z_3 \lor y_i^2) \land (\neg y_i^2 \lor z_4 \lor y_i^3) \land ...
\land (\neg y_i^{k-5} \lor z_{k-3} \lor y_i^{k-4}) \land (\neg y_i^{k-4} \lor z_{k-2} \lor y_i^{k-3})
\land (\neg y_i^{k-3} \lor z_{k-1} \lor z_k)
\]

If \( z_1 \) or \( z_2 \) is true, set all \( y_i \)'s to false, so all later clauses have a true literal.
Reduction from SAT to 3SAT

Let $c_i = z_1 \lor z_2 \lor \ldots \lor z_k$

Case 4: $k > 3$. Use extra variables $y_i^1, \ldots, y_i^{k-3}$. Replace $c_i$ with $k-2$ clauses:

$$(z_1 \lor z_2 \lor y_i^1)$$

$$\land (\neg y_i^1 \lor z_3 \lor y_i^2) \land (\neg y_i^2 \lor z_4 \lor y_i^3) \land \ldots$$

$$\land (\neg y_i^{k-5} \lor z_{k-3} \lor y_i^{k-4}) \land (\neg y_i^{k-4} \lor z_{k-2} \lor y_i^{k-3})$$

$$\land (\neg y_i^{k-3} \lor z_{k-1} \lor z_k)$$

If $z_{k-1}$ or $z_k$ is the first true literal of $c_i$, set all $y_i$'s to true, so all earlier clauses have a true literal.
Reduction from SAT to 3SAT

Let \( c_i = z_1 \lor z_2 \lor ... \lor z_k \)

Case 4: \( k > 3 \). Use extra variables \( y_i^1, ..., y_i^{k-3} \). Replace \( c_i \) with \( k-2 \) clauses:

\[
(z_1 \lor z_2 \lor y_i^1) \\
\land (-y_i^1 \lor z_3 \lor y_i^2) \land (-y_i^2 \lor z_4 \lor y_i^3) \land ... \\
\land (-y_i^{k-5} \lor z_{k-3} \lor y_i^{k-4}) \land (-y_i^{k-4} \lor z_{k-2} \lor y_i^{k-3}) \\
\land (-y_i^{k-3} \lor z_{k-1} \lor z_k)
\]

If first true literal is in between, set all earlier \( y_i \)'s to true and all later \( y_i \)'s to false.
Correctness of Reduction

\[ \leq : \text{Suppose the newly constructed 3SAT formula } C' \text{ is satisfiable. We must show that the original SAT formula } C \text{ is also satisfiable.} \]

Use the same satisfying truth assignment for } C \text{ as for } C' \text{ (ignoring new variables).}

Show each original clause has at least one true literal in it.
Original Clause is True

Let $c_i = z_1 \lor z_2 \lor \ldots \lor z_k$

Case 1: $k = 1$. Use extra variables $y_{i1}$ and $y_{i2}$. Replace $c_i$ with 4 clauses:

$$c'_i = (z_1 \lor y_{i1} \lor y_{i2}) \land (z_1 \lor \neg y_{i1} \lor y_{i2}) \land (z_1 \lor y_{i1} \lor \neg y_{i2}) \land (z_1 \lor \neg y_{i1} \lor \neg y_{i2}).$$

If $c'_i$ is true, then $c_i = z_1$ must be true, since one pair of literals in $y_{i1}$ and $y_{i2}$ must be true.
Reduction from SAT to 3SAT

Let \( c_i = z_1 \lor z_2 \lor ... \lor z_k \)

Case 2: \( k = 2 \). Use extra variable \( y_{i1} \). Replace \( c_i \) with 2 clauses:

\[
\begin{align*}
  c_i' &= (z_1 \lor z_2 \lor \neg y_{i1}) \land (z_1 \lor z_2 \lor y_{i1}).
\end{align*}
\]

If \( c_i' \) is true, then \( c_i = z_1 \lor z_2 \) must be true.
Reduction from SAT to 3SAT

Let $c_i = z_1 \lor z_2 \lor \ldots \lor z_k$

Case 3: $k = 3$. No extra variables are needed.

Keep $c_i$: $(z_1 \lor z_2 \lor z_3)$
Reduction from SAT to 3SAT

Let \( c_i = z_1 \lor z_2 \lor ... \lor z_k \)

Case 4: \( k > 3 \). Use extra variables \( y_{i1}, ..., y_{ik-3} \). Replace \( c_i \) with \( k-2 \) clauses:

\[
(z_1 \lor z_2 \lor y_{i1}) \land (\neg y_{i1} \lor z_3 \lor y_{i2}) \land (\neg y_{i2} \lor z_4 \lor y_{i3}) \land ...
\land (\neg y_{i(k-5)} \lor z_{k-3} \lor y_{i(k-4)}) \land (\neg y_{i(k-4)} \lor z_{k-2} \lor y_{i(k-3)})
\land (\neg y_{i(k-3)} \lor z_{k-1} \lor z_k)
\]

Suppose that there is a valuation such that \( c_i \) is true and \( c_i \) is false. Then \( y_{ik} \) must be true for all \( k \), so the last clause in \( c_i \) must be false, contradiction.
Conclusions

We have shown that

- 3SAT is in NP
- there exists a polynomial time reduction from SAT to 3SAT.

Therefore, 3SAT is NP-complete.