# The Birthday Problem

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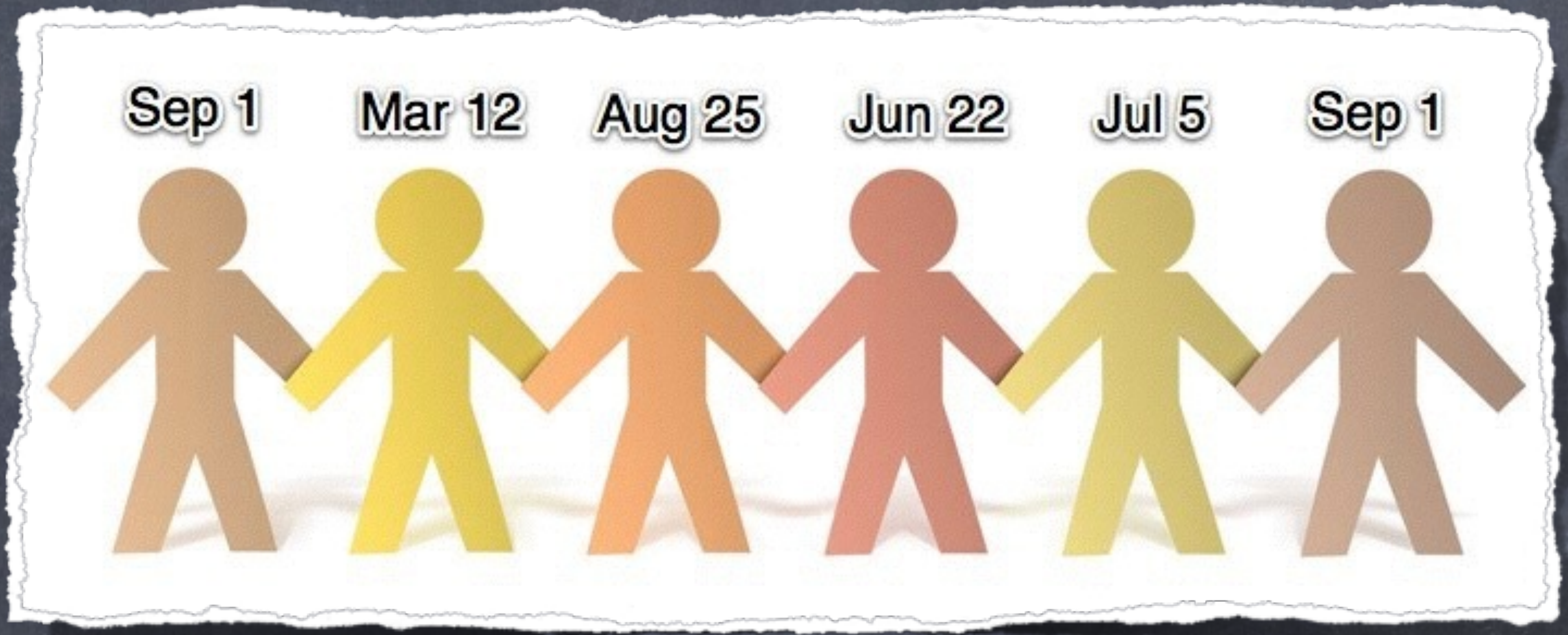
# The Birthday Problem




What is the probability $p_{uni}$ that among a group of m people, at least two share the same birthday?

# Solution

Let's solve the problem for arbitrary planets. Let's assume that the m people live on a planet that has n days per year. Then

$$\frac{n(n-1)\cdots(n-m+1)}{n^m}$$

is the probability that no two share a birthday, so

$$p_{uni} = 1 - \frac{n(n-1)\cdots(n-m+1)}{n^m} = 1 - \prod_{i=1}^{m-1}\left(1 - \frac{i}{n}\right),$$

assuming that m <= n and the birthdays are independent and uniformly distributed.

# Lower Bound

Since 1-x <= exp(-x) holds for all real numbers x, we have

$$p_{uni} = 1 - \prod_{i=1}^{m-1} \left(1 - \frac{i}{n}\right)$$

$$\geq 1 - \exp\left(-\sum_{i=1}^{m-1} \frac{i}{n}\right) = 1 - \exp\left(-\frac{(m-1)m}{2n}\right).$$

# Consequence

Therefore, if we consider $m \geq \frac{1}{2}\left(1+\sqrt{1-8n\ln\delta}\right)$ people, where $\delta$ is a real number in the range $0 < \delta \leq 1$, then the probability $p_{uni}$ that at least two of them have a common birthday satisfies $p_{uni} \geq 1-\delta$. For example, when $n = 365$, we have

| $m$ | 23 | 42 | 59 | 72 |
|---|---|---|---|---|
| $p_{uni}$ | 0.5 | 0.9 | 0.99 | 0.999 |

# The Flaw

There are fewer births on weekends than during the week.

There are fewer births on July 4 than on other days in July.

There are significant seasonal variations.

=> Birthdays are not uniformly distributed.

# Nonuniform Birthday Problem

Let $p_k$ denote the probability that a person is born on the $k$-th day of the year, where $1 \leq k \leq n$. Then the probability $p_{nu}$ that among $m$ people at least two have the same birthday using the distribution $(p_1, p_2, \ldots, p_n)$ of birthdays is given by

$$p_{nu} = 1 - e_m(p_1, p_2, \ldots, p_n),$$

where $e_m$ denotes the $m$-th elementary symmetric function,

$$e_m(x_1, \ldots, x_n) = \sum_{1 \leq j_1 < j_2 < \cdots < j_m \leq n} x_{j_1} x_{j_2} \cdots x_{j_m}.$$

# Relation

Any probability distribution majorizes the uniform distribution,

$$(1/n, 1/n, \ldots, 1/n) \prec (p_1, p_2, \ldots, p_n),$$

which means that the sum of the k largest probabilities in $\{p_1, \ldots, p_n\}$ is at least $k/n$ for all $k$ in the range $1 \leq k \leq n$. Since the elementary symmetric functions are Schur-concave (meaning that they are monotonically decreasing with respect to the relation $\prec$), it follows that $e_m(1/n, 1/n, \ldots, 1/n) \geq e_m(p_1, p_2, \ldots, p_n)$.

# Relation

Therefore, we can conclude that

$$\begin{aligned} p_{uni} &= 1 - \frac{n(n-1)\cdots(n-m+1)}{n^m} \\ &= 1 - e_m(1/n, 1/n, \dots, 1/n) \\ &\leq 1 - e_m(p_1, p_2, \dots, p_n) = p_{nu}. \end{aligned}$$

# Relation

One can show the following relation between uniform and nonuniform distribution case:

$$\begin{aligned} p_{uni} &= 1 - \frac{n(n-1)\cdots(n-m+1)}{n^m} \\ &= 1 - e_m(1/n, 1/n, \ldots, 1/n) \\ &\leq 1 - e_m(p_1, p_2, \ldots, p_n) = p_{nu}, \end{aligned}$$

as $e_m$ is a so-called Schur-concave function.

# References


- J. Buchmann, Introduction to Cryptography, Springer, 2004
- J. Michael Steele, The Cauchy Schwarz Master Class, Cambridge University Press, 2004