It remains to show that V is a unitary matrix. Recall that the unitary matrix U can be diagonalized by a base change with some unitary matrix P, say $\mathrm{diag}(\lambda_1, \lambda_2) = PUP^\dagger$. Then P diagonalizes V as well, so $PVP^\dagger = \mathrm{diag}(a, b)$. Since

$$\mathrm{diag}(\lambda_1, \lambda_2) = PUP^\dagger = (PVP^\dagger)(PVP^\dagger) = \mathrm{diag}(a^2, b^2),$$

it follows that $a = \sqrt{\lambda_1}$ and $b = \sqrt{\lambda_2}$ are complex numbers of absolute value 1. Therefore, $\mathrm{diag}(a, b)$ is a unitary matrix and we can conclude that $V = P^\dagger \mathrm{diag}(a, b) P$ is a unitary matrix as well. ■