Proof. Let us first show that V is a well-defined matrix. Seeking a contradiction, we assume that $\operatorname{tr} U \pm 2\sqrt{\det U} = 0$. Let λ_1, λ_2 be the eigenvalues of U. We have $\det U = \lambda_1 \lambda_2$ and $\operatorname{tr} U = \lambda_1 + \lambda_2$. It follows that

$$\lambda_1 + \lambda_2 = \operatorname{tr} U = \mp 2\sqrt{\det U} = 2\sqrt{\lambda_1 \lambda_2}.$$

Since U is unitary, $|\lambda_1| = |\lambda_2| = 1$. Therefore, $|\lambda_1 + \lambda_2| = 2|\sqrt{\lambda_1 \lambda_2}| = 2$. This means that the triangle inequality $|\lambda_1 + \lambda_2| \leq 2 = |\lambda_1| + |\lambda_2|$ holds with equality, which implies that $\lambda_1 = r\lambda_2$ for some positive real number r. Since $|\lambda_1| = |\lambda_2| = 1$, we have $|r| = r = 1$, which means that the eigenvalues λ_1 and λ_2 must be the same. This would imply that U is a multiple of the identity, contradicting our hypothesis. Therefore, $\operatorname{tr} U \pm 2\sqrt{\det U}$ is nonzero and the matrix V is well-defined.