Shor's Algorithm
Part 2

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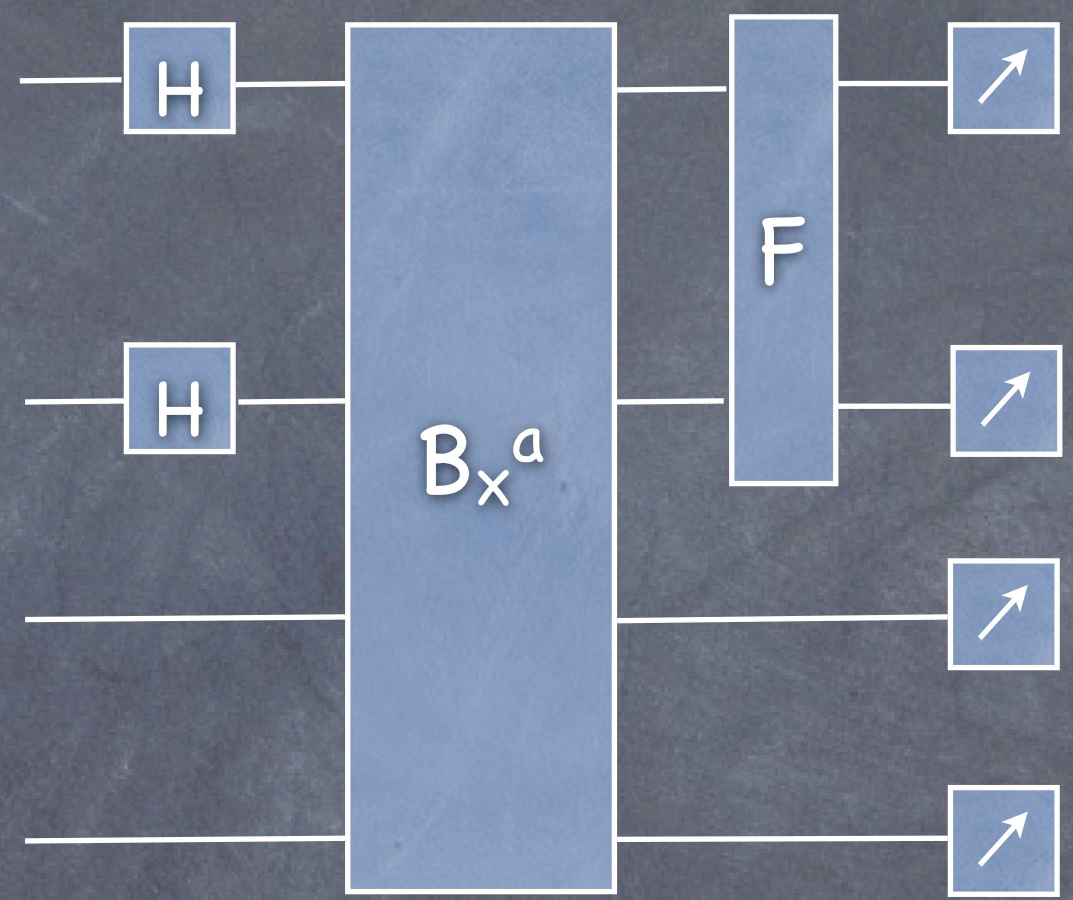
Given: integer n and an integer c coprime to n

Let q be a power of 2 such that $n^2 <= q=2^l < 2n^2$.

We use two registers, each with $l=\log_2 q$ bits.

The state space is $\mathbf{C}^q \otimes \mathbf{C}^q$

Quantum Algorithm



$$B_{x^a} = \begin{cases} \mathbf{C}^q \otimes \mathbf{C}^q & \rightarrow \quad \mathbf{C}^q \otimes \mathbf{C}^q \\ |a\rangle \otimes |y\rangle & \mapsto \quad |a\rangle \otimes |y \oplus x^a \mod n\rangle \end{cases}$$

Analysis

The initial state is $|0\rangle \otimes |0\rangle$.

After applying Hadamard gates, we get

$$\frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} |a\rangle \otimes |0\rangle.$$

Applying the black box function yields

$$\frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} |a\rangle \otimes |x^a \pmod{n}\rangle.$$

The Boolean function depends on n and x. It can be constructed in poly time.

Analysis

$$\frac{1}{\sqrt{q}}\sum_{a=0}^{q-1}|a\rangle \otimes |x^a \pmod n\rangle.$$

Applying the Fourier transform yields

$$\frac{1}{q}\sum_{a=0}^{q-1}\sum_{c=0}^{q-1}\exp(2\pi i a c/q)|c\rangle \otimes |x^a \pmod n\rangle.$$

Analysis

$$\frac{1}{q}\sum_{a=0}^{q-1}\sum_{c=0}^{q-1}\exp(2\pi iac/q)|c\rangle\otimes|x^a \pmod n\rangle.$$

We now measure.

Let's assume that x has order r mod n. Then

$$\begin{aligned}\Pr[\text{observe } (c, x^k \bmod n)] &= \left|\frac{1}{q}\sum_{a:x^a\equiv x^k}\exp(2\pi iac/q)\right|^2\\ &= \left|\frac{1}{q}\sum_{b=0}^{\lfloor (q-k-1)/r\rfloor}\exp(2\pi i(br+k)c/q)\right|^2\end{aligned}$$

$0 \leq k < r$

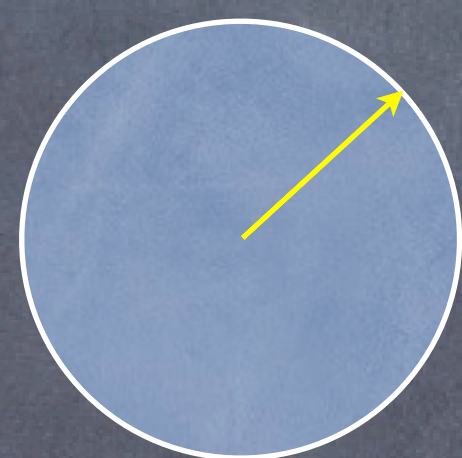
Analysis

$$\Pr[\text{observe } (c, x^k \bmod n)] = \left| \frac{1}{q} \sum_{b=0}^{\lfloor (q-k-1)/r \rfloor} \exp(2\pi i(br+k)c/q) \right|^2$$

$$= \frac{1}{q^2} \left| \sum_{b=0}^{\lfloor (q-k-1)/r \rfloor} \exp(2\pi i\, bcr/q) \right|^2$$

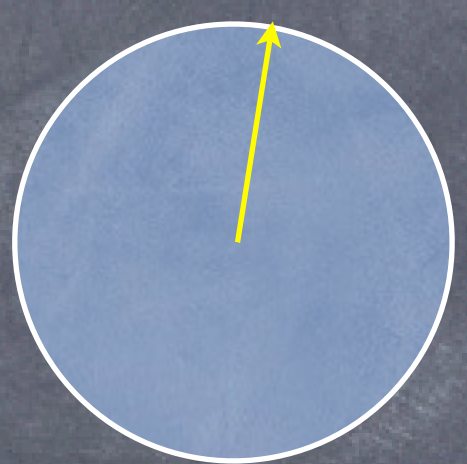
The powers of exp(2πi cr/q) nearly cancel out unless cr/q is close to an integer. This is called destructive interference.

Destructive Interference



exp(2πi bcr/q) for various b

when cr/q is not close to an integer



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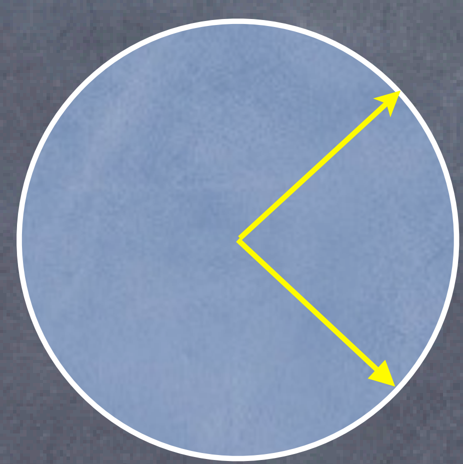

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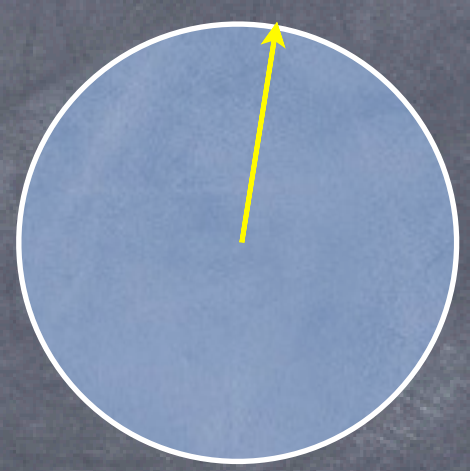

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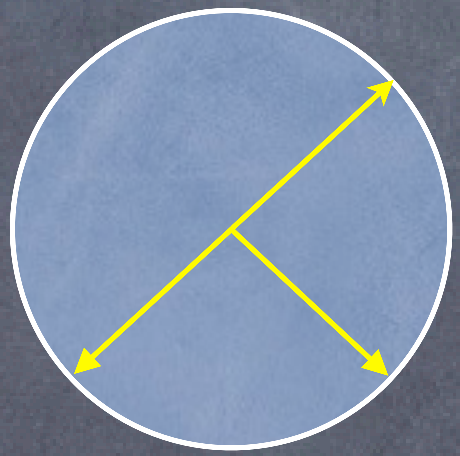

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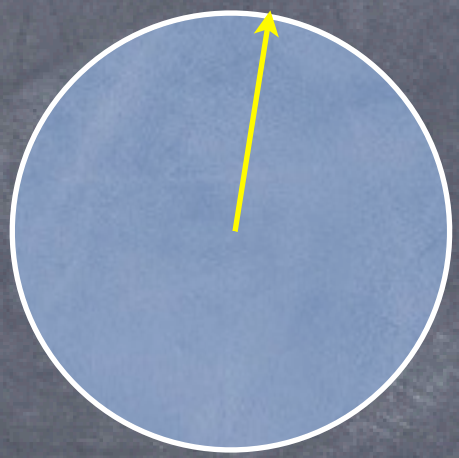

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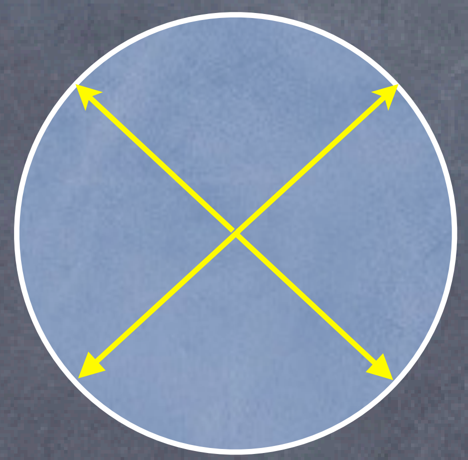

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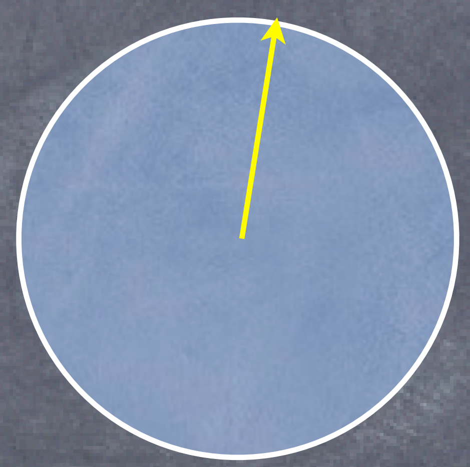

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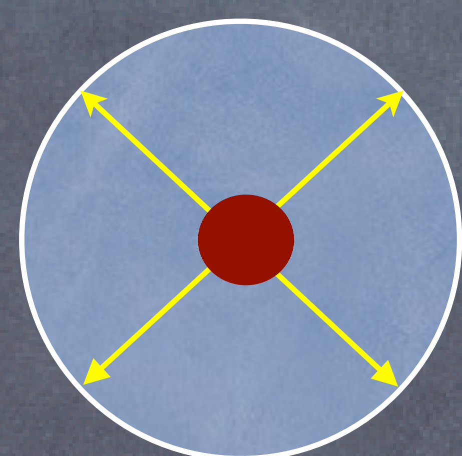

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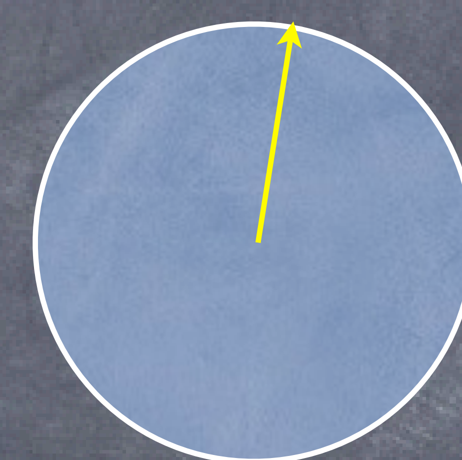

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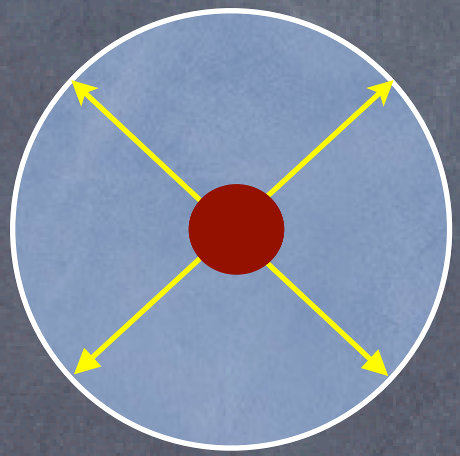

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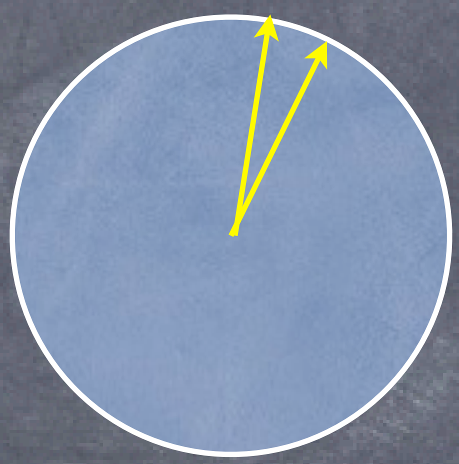

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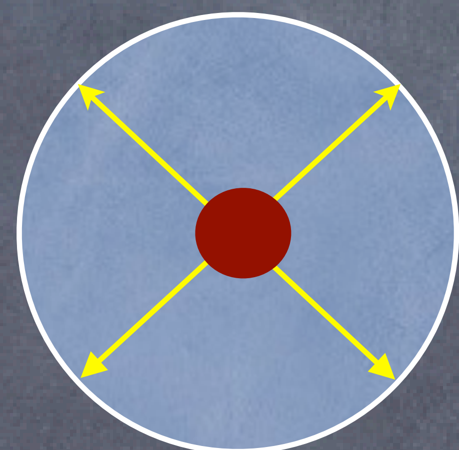

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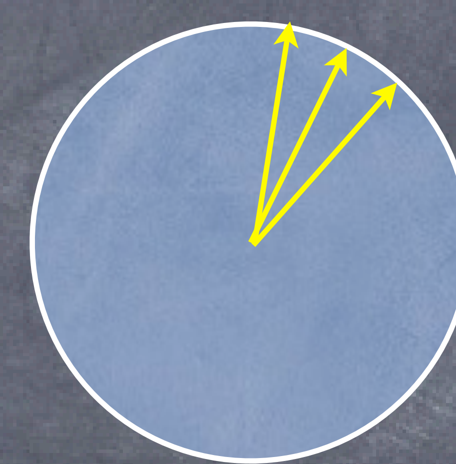

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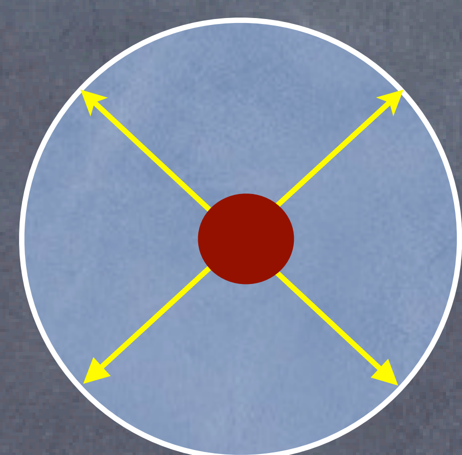

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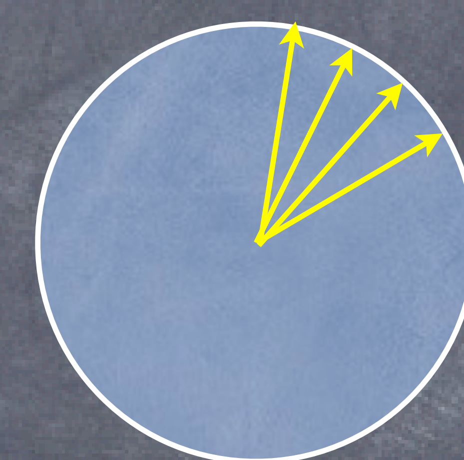

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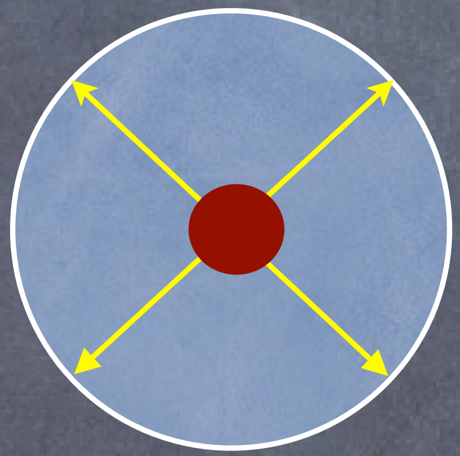

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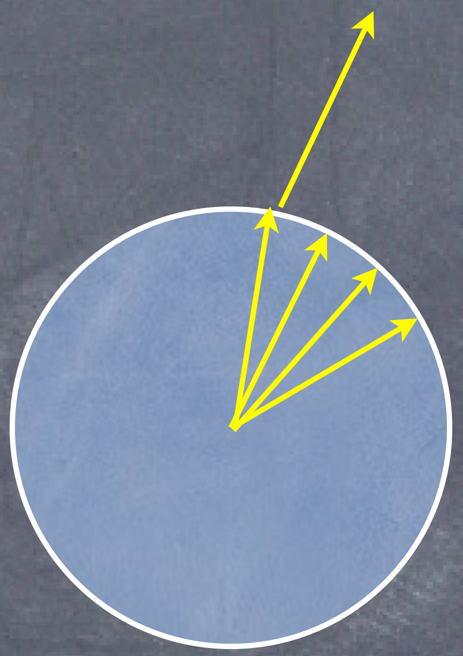

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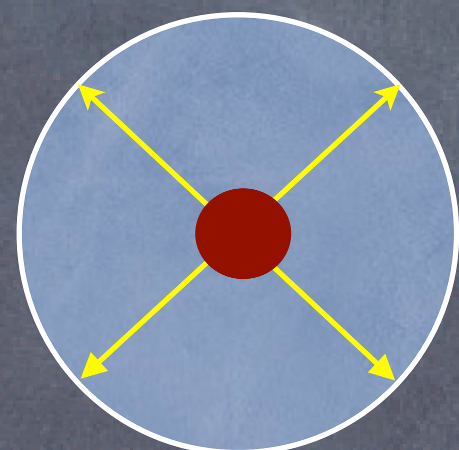

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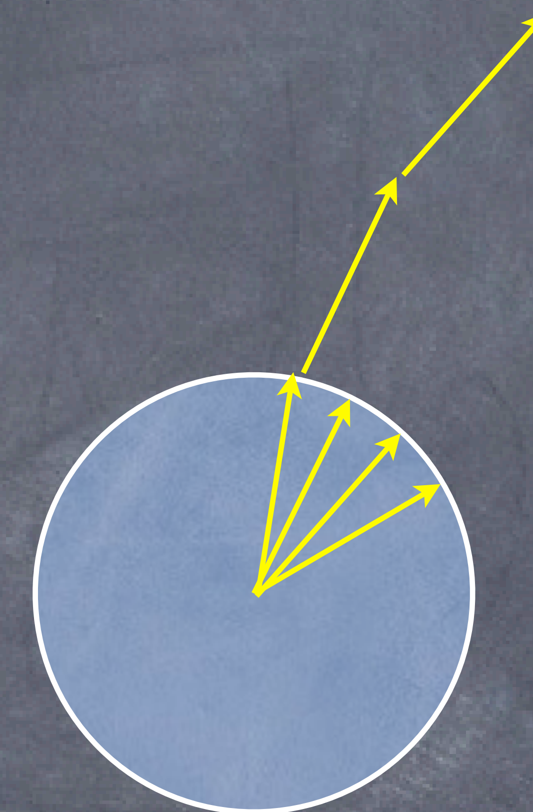

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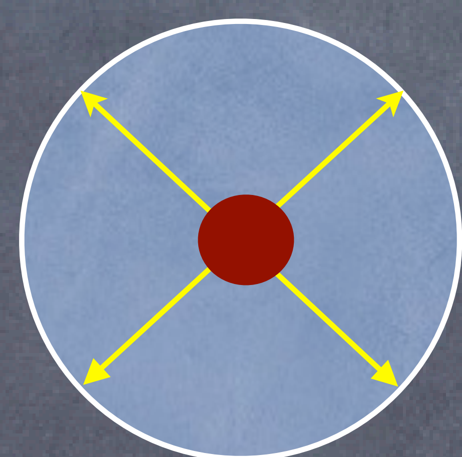

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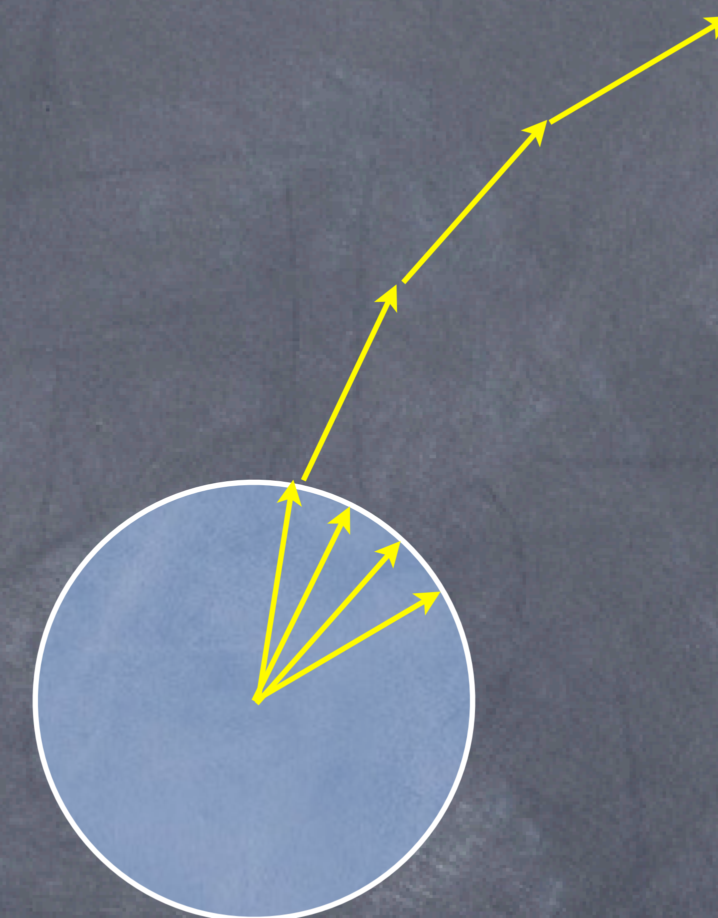

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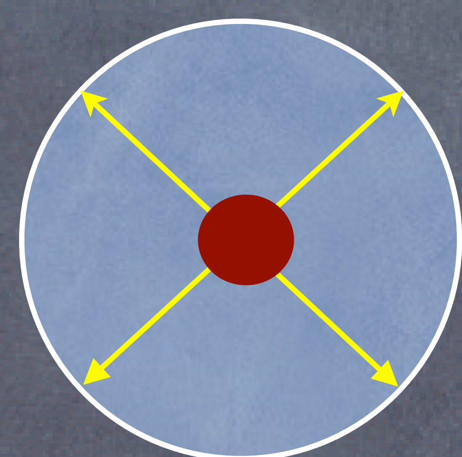

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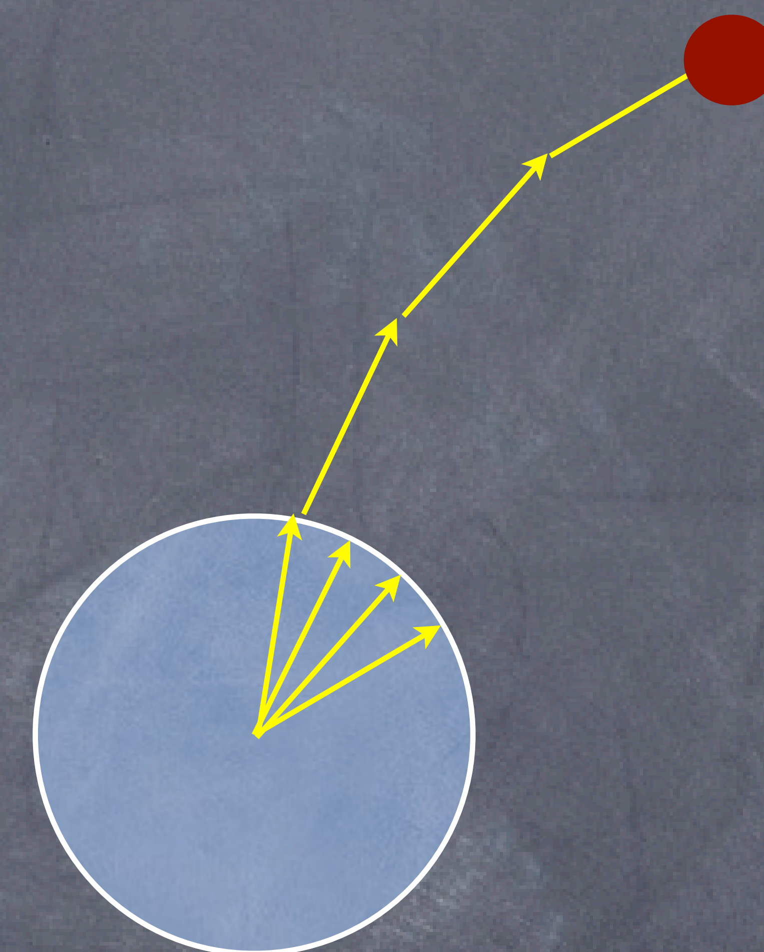

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Learning from Observations

In general, $rc \bmod q$ lies in the large interval $[-q/2, q/2]$.

If $rc \bmod q$ in $[-r/2, r/2]$ then $\Pr[\text{observe } (c, x^k \bmod n)] \geq 1/3r^2$

$rc \bmod q$ in $[-r/2, r/2]$ means that there exists an integer d s.t.

$$-r/2 <= rc - dq <= r/2$$

Dividing by rq yields

$$|c/q - d/r| <= 1/2q \qquad (*)$$

Since $q > n^2$ there is at most one fraction d/r with $r<n$ satisfying (*)

Continued Fractions

Since we know c and q, we obtain the fraction d/r in lowest terms by rounding c/q to the nearest fraction having a denominator smaller than n.

We can find this fraction by using the continued fraction expansion of c/q. This can be done by a variation of the Euclidean algorithm.

Loose Ends

We remains to show that

- the Fourier transform can be computed in $O((\log q)^2)$ time.
- the modular exponentiation can be computed in poly time
- the continued fraction can yield the result

Assuming these results, we obtained a polynomial time algorithm to factor a given composite integer n.