Problem Set 4

CPSC 440/640 Quantum Algorithms

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The assignment is due on Wednesday, November 8, before class.

Problem 1. Let G_n denote the Pauli group $G_n = \{\pm X(a)Z(b) \mid a, b \in \mathbf{F}_2^n\}$. Show that two matrices A, B in G_n either commute or anticommute, that is, $AB = \pm BA$.

Problem 2. Recall that the **trace** $\operatorname{tr} P$ of a matrix P is given by the sum of its diagonal elements, that is, $\operatorname{tr} P = \sum P_{ii}$. It is easy to show that $\operatorname{tr}(APA^{-1}) = \operatorname{tr}(P)$ holds for all invertible matrices A.

Let Q be a quantum code of length n. Let $P\colon \mathbf{C}^{2^n} \to \mathbf{C}^{2^n}$ denote the orthogonal projector onto Q, that is, P is the unique linear map that satisfies $P^2 = P$, $P^\dagger = P$, and $\operatorname{image}(P) = Q$. Prove that $\dim Q = \operatorname{tr} P$.

Problem 3. Recall that a **group** G is a set with a binary operation $\circ\colon G \times G \to G$ such that (i) $a \circ (b \circ c) = (a \circ b) \circ c$ holds for all $a, b, c \in G$; (ii) there exists an element 1 in G, called the identity, such that $a \circ 1 = 1 \circ a = a$ for all a in G; (iii) for each a in G there exists an element a^{-1} in G such that $a \circ a^{-1} = a^{-1} \circ a = 1$. The group G is called **abelian** if $a \circ b = b \circ a$ holds for all a, b in G.

Let S denote an abelian subgroup of the Pauli group G_n.

(a) Let A be an element of S. Prove that the map $m_A\colon S \to S$ given by $m_A(x) = Ax$ is a bijective map (=one-to-one and onto).

(b) Prove that $P = |S|^{-1} \sum_{B \in S} B$ is an orthogonal projector.

(c) Prove that if the group S has 2^{n-k} elements, then the quantum code given by the image of P has dimension 2^k.

Problem 4. Let Q denote the quantum code $Q = \{a|001\rangle + b|110\rangle \mid a, b \in \mathbf{C}\}$.

(a) Determine all matrices A in S such that $A|x\rangle = |x\rangle$ for all $|x\rangle$ in Q. Prove that this set forms an abelian group T.

(b) Let P denote the orthogonal projector corresponding to the abelian group T, as define in the previous problem. Show that the image of P is Q.