§1 Analysis of a Minimum Cut Algorithm

We introduced in Chapter 1 a randomized algorihm to compute the size of a minimum cut of a loopfree multigraph. You might recall that this algorithm repeatedly selects and contracts edges as follows:

Contract(G)
Input: A connected loopfree multigraph $G = (V, E)$ with at least 2 vertices.

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while |V| > 2 do
    Select e ∈ E uniformly at random;
    G := G/e;
od;
return |E|.
```

Output: An upper bound on the minimum cut of G.

We now want to analyze this algorithm. Our goal is to determine a lower bound on the probability that the algorithm correctly determines the minimum cut. We will see that this algorithm produces the correct answer with probability $\Omega(1/n^2)$. We need remarkably few tools from probability theory in this proof: All we need is the innocuous formula

$$\boxed{\Pr[E \cap F] = \Pr[E|F]\Pr[F]}$$

Exercise 3.1 *Prove the following straightforward consequence of the previous formula*

$$\Pr[\cap_{\ell=1}^{n} E_\ell] = \left(\prod_{m=2}^{n} \Pr[E_m | \cap_{\ell=1}^{m-1} E_\ell]\right) \Pr[E_1].$$

If you expand the formula then you will immediately see the pattern.

Let me motivate the approach taken in the analysis by emphasizing a special case. Suppose that the multigraph has a uniquely determined minimum cut. If the algorithm selects in this case *any* edge crossing this cut, then the algorithm will fail to produce the correct result. The analysis is largely guided by this observation.

Exercise 3.2 *Give an example of a connected, loopfree multigraph with at least four vertices that has a uniquely determined minimum cut.*

Let $G = (V, E)$ be a loopfree connected multigraph with $n = |V|$ vertices. Note that each contraction reduces the number of vertices by one, so the algorithm terminates after $n - 2$ steps.

Suppose that C is a particular minimum cut of G. Let E_i denote the event that the algorithm selects in the ith step an edge that does not cross the cut C. Therefore, the probability that no edge crossing the cut C is ever picked during an execution of the algorithm is $\Pr[\cap_{j=1}^{n-2} E_j]$. By Exercise 3.1, this probability can be calculated by

$$\Pr[\cap_{m=1}^{n-2} E_m] = \left(\prod_{m=2}^{n-2} \Pr[E_m | \cap_{\ell=1}^{m-1} E_\ell] \right) \Pr[E_1]. \tag{3.1}$$

Suppose that the size of the minimum cut is k. This means that the degree of each vertex is at least k, hence there exist at least $kn/2$ edges. The probability to select an edge crossing the cut C in the first step is at most $k/(kn/2) = 2/n$. Consequently, $\Pr[E_1] \geq 1 - 2/n = (n-2)/n$.

Similarly, at the beginning of the mth step, with $m \geq 2$, there are $n-m+1$ remaining vertices. The minimum cut is still at least k, hence the multigraph has at this stage at least $k(n-m+1)/2$ edges. Assuming that none of the edges crossing C was selected in an earlier step, the probability to select an edge crossing the cut C is $2/(n-m+1)$. It follows that

$$\Pr\left[E_m | \bigcap_{j=1}^{m-1} E_j\right] \geq 1 - \frac{2}{n-m+1} = \frac{n-m-1}{n-m+1}.$$

Applying these lower bounds to the terms in equation (3.1) yields the result:

$$\Pr\left[\bigcap_{j=1}^{n-2} E_j\right] \geq \prod_{m=1}^{n-2} \left(\frac{n-m-1}{n-m+1} \right) = \frac{2}{n(n-1)}.$$

The last equality is obtained by canceling terms in the telescoping product.

In conclusion, we have shown that the contraction algorithm yields the correct answer with probability at least $\Omega(1/n^2)$.

Repetitions. We can repeatedly execute the randomized algorithm Contract and take the minimum of all results. Recall from calculus that

$$\left(1 + \frac{x}{n}\right)^n \leq e^x,$$

and, in fact, $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$. The probability that the algorithm fails to produce the correct result in one execution is $\Pr[\text{failure}] = (1 - 2/n^2)$. Recall that for independent event E and F, the probability is given by $\Pr[E \cap F] = \Pr[E]\Pr[F]$. Therefore, if we execute the algorithm $n^2/2$ times, then the probability that the repeated executions will never reveal the correct size of the minimum cut is given by $(1 - 2/n^2)^{n^2/2} \leq e^{-1}$. We can conclude that repeating the contraction algorithm $O(n^2 \log n)$ times yields the correct size of the minimum cut with high probability.