On the Smoothness of Real-Valued Functions Generated by Subdivision Schemes Using Nonlinear Binary Averaging

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Abstract

Our main result is that two point interpolatory subdivision schemes using C^k nonlinear averaging rules on pairs of real numbers generate real-valued functions that are also C^k. The significance of this result is the following consequence: Suppose that S is a subdivision algorithm operating on sequences of real numbers using linear binary averaging that generates C^m real-valued functions and $\overline{S}$ is the same subdivision procedure where linear binary averaging is replaced everywhere in the algorithm by a C^n nonlinear binary averaging rule on pairs of real numbers; then the functions generated by the nonlinear subdivision scheme $\overline{S}$ are C^k , where $k = \min(m, n)$.

Classification: CCScat{I.3.5}{Computer Graphics}{Curve, surface, and solid representations}

1 Introduction

Nonlinear subdivision algorithms can be generated from linear subdivision algorithms by replacing linear averages by nonlinear averages [3]. For example, the de Casteljau subdivision algorithm for Bezier curves and the Lane-Riesenfeld algorithm for uniform B-splines generate polynomials and piecewise polynomials by successively averaging adjacent coefficients. If we start with positive real numbers and we replace the arithmetic mean $A(x, y) = (x + y)/2$ by the geometric mean $G(x, y) = \sqrt{xy}$, then instead of generating polynomials and piecewise polynomials, these algorithms generate exponential and piecewise exponential functions (See Figure 1). The goal of this paper is to investigate the smoothness of the functions generated by subdivision algorithms when linear averages on pairs of real numbers are replaced by nonlinear averages on pairs of real numbers.

We begin in Section 2 by introducing the general notion of an averaging rule for pairs of real numbers. We then explain the connection between averaging rules, monotone functions satisfying functional equations, and nonlinear subdivision algorithms. In Section 3 we prove our main result: that C^k averaging rules generate via two point interpolatory subdivision C^k monotone functions. The most difficult cases are $k = 0, 1, 2$, which we establish in separate subsections. The general result for arbitrary k then follows easily by induction on k. We conclude in Section 4 with a brief summary of our main results.

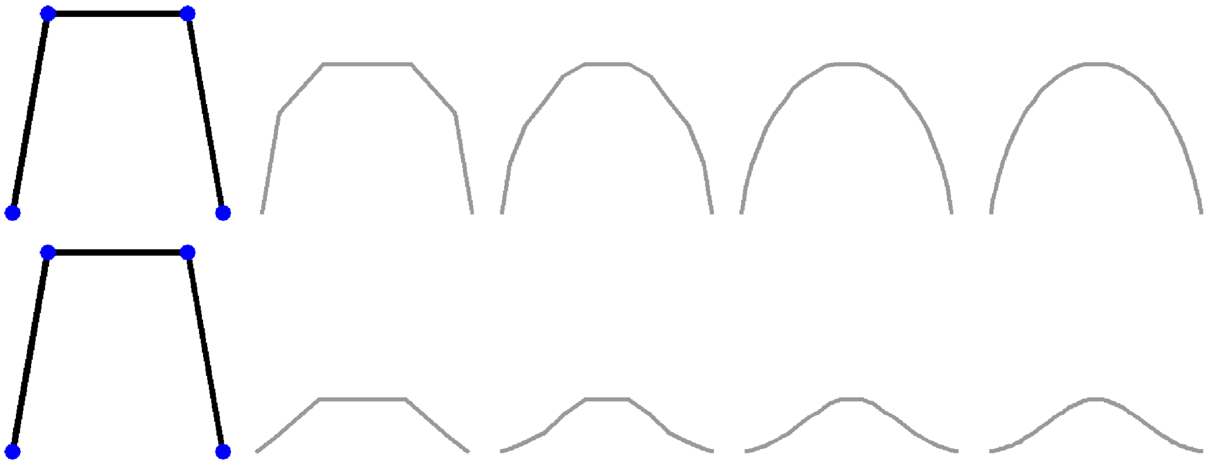

Figure 1: A function generated by the de Casteljau subdivision algorithm using the arithmetic mean (Top), and a function generated by the de Casteljau subdivision algorithm starting with the same data but using the geometric mean (Bottom). The limit function on top is a polynomial (a parabola); the limit function below is an exponential (a Gaussian).

2 Nonlinear Averaging Rules and Monotonic Functions

A function $Av : I \times I \to I$, where I is an interval (open or closed) in R, is called an *averaging rule* if

i. $Av(a,a) = a$

ii. $\min(a,b) < Av(a,b) < \max(a,b)$

iii. $Av(a,b) = Av(b,a)$

iv. $Av(Av(a,b), Av(c,d)) = Av(Av(a,c), Av(b,d))$

The first three properties are self-explanatory; the fourth property simply states that if we take the average of four numbers in pairs, then the result is independent of the way we group the pairs. This property certainly holds for standard averaging rules such as the arithmetic and geometric means. One immediate consequence of property iv is that

v. $Av(a, Av(b,c)) = Av(Av(a,b), Av(a,c))$

because

$$Av(a, Av(b,c)) = Av(Av(a,a), Av(b,c)) = Av(Av(a,b), Av(a,c)).$$

The properties of averaging rules may be easier to understand if we think of an averaging rule as a binary operation $\oplus : I \times I \to I$, where I is an interval (open or closed) in R. With this notation properties i-v become

i. $a \oplus a = a$

ii. $\min(a,b) < a \oplus b < \max(a,b)$

iii. $a \oplus b = b \oplus a$

iv. $(a \oplus b) \oplus (c \oplus d) = (a \oplus c) \oplus (b \oplus d)$

v. $a \oplus (b \oplus c) = (a \oplus b) \oplus (a \oplus c)$

Thus $\oplus$ is idempotent, commutative, but not associative; rather $\oplus$ distributes through itself.

A function F generated by starting with two arbitrary values $F(a)$ and $F(b)$ and iterating the subdivision rule

$$F\left(\frac{x+y}{2}\right) = Av(F(x), F(y)) \tag{2.1}$$

is called a *function generated by the averaging rule Av*. We shall show in Section 3.1 that if the averaging rule Av is continuous, then the subdivision procedure defined by Equation 2.1 converges to a continuous function F that satisfies this functional equation. Notice that if $F(a) \neq F(b)$, then F is strictly monotone on the interval $[a, b]$.

A continuous averaging rule Av together with two initial values $F(a)$ and $F(b)$ generates a monotonic function F on the interval $[a, b]$ by the subdivision rule

$$F\left(\frac{x+y}{2}\right) = Av(F(x), F(y)).$$

Similarly, a monotonic function F on the interval $[a, b]$ induces an averaging rule Av^* on the domain $[a, b] \times [a, b]$ by the formula

$$Av^*(x, y) = F\left(\frac{F^{-1}(x) + F^{-1}(y)}{2}\right). \tag{2.2}$$

Properties i,ii,and iii are easy to verify; property iv holds because

$$Av^*(Av^*(a, b), Av^*(c, d)) = F\left(\frac{F^{-1}(a) + F^{-1}(b) + F^{-1}(c) + F^{-1}(d)}{4}\right) = Av^*(Av^*(a, c), Av^*(b, d)).$$

The rule Av^* in Equation 2.2 is called *the averaging rule induced by the function F*. Averaging rules of this form are studied in detail by Hardy et al [2].

Hardy et al [2] also introduce a generic collection of such averaging rules. Let $F_p : (0, \infty) \to (0, \infty)$ be defined by

$$F_p(x) = \begin{cases} x^{\frac{1}{p}} & p \neq 0 \\ e^x & p = 0 \end{cases}$$

Then the corresponding averaging rules $Av_p^* : (0, \infty) \times (0, \infty) \to (0, \infty)$ induced by the functions F_p are

$$Av_p^*(x, y) = \begin{cases} \left(\frac{x^p + y^p}{2}\right)^{\frac{1}{p}} & p \neq 0 \\ \sqrt{xy} & p = 0. \end{cases}$$

In particular,

$$Av_{-1}^*(x, y) = \frac{2xy}{x+y} \quad \text{(Harmonic Mean)}$$
$$Av_0^*(x, y) = \sqrt{xy} \quad \text{(Geometric Mean)}$$
$$Av_1^*(x, y) = \frac{x+y}{2} \quad \text{(Arithmetic Mean)}$$

Notice that

$$\begin{aligned} \lim_{p \to -\infty} Av_p^*(x, y) &= \min(x, y) \\ \lim_{p \to \infty} Av_p^*(x, y) &= \max(x, y) \\ \lim_{p \to 0} Av_p^*(x, y) &= Av_0^*(x, y) \end{aligned}$$

All three of these limits can be derived by considering $\log(Av_p^*(x, y))$ and applying L'Hopital's Rule.

Thus we see that averaging rules and monotonic functions are closely linked. We begin with a proposition summarizing the interrelationship between monotonic functions and averaging rules.

Proposition 2.1. *The following properties hold:*

i. A continuous averaging rule Av and two initial values $F(a)$ and $F(b)$ generate a continuous monotonic function F on the interval $[a, b]$ that satisfies the functional Equation 2.1.

ii. A monotonic function F on $[a, b]$ induces an averaging rule Av on $[a, b] \times [a, b]$.

iii. If F is generated by a continuous averaging rule Av, then the averaging rule Av^* induced by F is the same as the original averaging rule Av that generates F.

iv. If F and F^{-1} are continuous, then the monotonic function F is generated by the averaging rule induced by F.

v. Every continuous monotonic function with continuous inverse is generated by some continuous averaging rule.

vi. Every continuous averaging rule is induced by some continuous monotonic function.

vii. Two monotonic functions F, F_* induce the same averaging rule if and only if there is a linear function L such that $F_* = F \circ L$.

viii. Two monotonic function F, F_* are generated by the same averaging rule (with different initial values) if and only if there is a linear function L such that $F_* = F \circ L$.

Proof. Property i will be proved in Section 3.1, and Property ii follows immediately from Equation 2.2.

To prove Property iii, suppose that a continuous averaging rule Av generates the monotonic function F and that F induces the averaging rule Av^*. Then since by Property i F satisfies the functional Equation 2.1,

$$Av^*(x,y) = F\left(\frac{F^{-1}(x)+F^{-1}(y)}{2}\right) = Av(F(F^{-1}(x)), F(F^{-1}(y))) = Av(x,y).$$

Property iv follows because if Av^* is the averaging rule induced by F, then

$$Av^*(x,y) = F\left(\frac{F^{-1}(x)+F^{-1}(y)}{2}\right),$$

so

$$Av^*(F(x), F(y)) = F\left(\frac{x+y}{2}\right).$$

Thus since Av^* is continuous, F is generated by Av^*.

Property v is an immediate consequence of Property iv, and Property vi is an immediate consequence of Property iii.

To prove Property vii, let F be a monotonic function and let L be a linear function. Suppose that $F_* = F \circ L$, and set

$$Av(x,y) = F\left(\frac{F^{-1}(x)+F^{-1}(y)}{2}\right)$$

and

$$Av_*(x,y) = F_*\left(\frac{F_*^{-1}(x)+F_*^{-1}(y)}{2}\right).$$

Then since L is linear,

$$\begin{aligned} Av_*(x,y) &= F_*\left(\frac{F_*^{-1}(x)+F_*^{-1}(y)}{2}\right) = F \circ L\left(\frac{L^{-1}\circ F^{-1}(x)+L^{-1}\circ F^{-1}(y)}{2}\right) \\ &= F\left(\frac{F^{-1}(x)+F^{-1}(y)}{2}\right) = Av(x,y). \end{aligned}$$

Thus F and $F_* = F \circ L$ induce the same averaging rule. Conversely if F and F_* induce the same averaging rule, then

$$F_*\left(\frac{F_*^{-1}(x)+F_*^{-1}(y)}{2}\right) = F\left(\frac{F^{-1}(x)+F^{-1}(y)}{2}\right).$$

Setting $a = F^{-1}(x)$, $b = F^{-1}(y)$, and composing F^{-1} with both sides yields

$$(F^{-1}\circ F_*)\left(\frac{F_*^{-1}(F(a))+F_*^{-1}(F(b))}{2}\right) = \frac{a+b}{2}.$$

Now let $L = F^{-1} \circ F_*$. Then

$$L\left(\frac{L^{-1}(a)+L^{-1}(b)}{2}\right) = \frac{a+b}{2}$$

or equivalently

$$\frac{L^{-1}(a)+L^{-1}(b)}{2}=L^{-1}\left(\frac{a+b}{2}\right).$$

Thus L^{-1} is linear. Hence L is linear and $F_*=F\circ L$. This result is also proved in [2]; we include the proof here for completeness.

Finally, Property viii follows immediately from Properties vii and iii. □

We are interested in functions generated by averaging rules because these functions are the simplest examples of functions built by nonlinear subdivision algorithms. Moreover, the smoothness of these functions is linked to the smoothness of arbitrary functions built by nonlinear subdivision from nonlinear averaging rules. Indeed the following theorem is the main result proved in [3] concerning the smoothness of the functions built by subdivision algorithms when linear averages are replaced by nonlinear averages.

Theorem 2.2. *Let S be a subdivision algorithm based on linear averaging and let $\overline{S}$ be the same subdivision procedure where the linear averaging rule $A(a,b)=(a+b)/2$ is replaced everywhere in the algorithm by a nonlinear averaging rule $Av(a,b)$. If the functions generated by S are C^n and the functions generated by Av are C^m, then the functions generated by $\overline{S}$ are C^k, where $k=\min(m,n)$.*

Thus to determine the smoothness of the functions built by nonlinear subdivision algorithms where linear averages are replaced by nonlinear averages, we need only determine the smoothness of the functions generated by the nonlinear averages. The purpose of this paper is to show that if the averaging rule Av is C^m, then the functions F generated by Av are also C^m. As a consequence of Theorem 2.2 we will then have the following result:

Corollary 2.3. *Let S be a subdivision algorithm based on linear averaging and let $\overline{S}$ be the same subdivision procedure where the linear averaging rule $A(a,b)=(a+b)/2$ is replaced everywhere in the algorithm by a nonlinear averaging rule $Av(a,b)$. If the functions generated by S are C^n and the averaging rule Av is C^m, then the functions generated by $\overline{S}$ are C^k, where $k=\min(m,n)$.*

In general, if we start with a monotone function F that is known to be C^m, and F' is never equal to zero, then the function F^{-1} is also C^m. Hence the averaging rule

$$Av^*(x,y)=F\left(\frac{F^{-1}(x),F^{-1}(y)}{2}\right)$$

induced by F is also C^m. Thus it follows from Proposition 2.1 and Theorem 2.2 that if we replace the arithmetic average $A(a,b)=(a+b)/2$ by the averaging rule $Av^*(a,b)$ in a subdivision algorithm that generates functions that are C^n, we will generate functions that are at least C^k, where $k=\min(m,n)$. This observation allows us to build many smooth nonlinear subdivision algorithms. For example, if we let $F(x)=e^x$, then $F^{-1}(x)=\log x$. Hence in this case

$$Av^*(x,y)=e^{\frac{\log x+\log y}{2}}=\sqrt{xy}.$$

Therefore it follows from Proposition 2.1 and Theorem 2.2 that if we replace the arithmetic mean $A(a,b)=(a+b)/2$ with the geometric mean $G(a,b)=\sqrt{ab}$ in a subdivision algorithm that generates functions that are C^n, the algorithm will still generate functions that are C^n.

A problem arises, however, when we know the averaging rule Av, but we do not have an explicit formula for the functions F generated by Av. For example, suppose that Av_1 and Av_2 are two averaging rules and for some fixed value of t we set

$$Av(x,y)=(1-t)Av_1(x,y)+tAv_2(x,y).$$

Then Av surely satisfies Properties i-iii of an averaging rule. If Av also satisfies property iv, then Av is an averaging rule. Moreover if Av_1 and Av_2 are C^m, then Av is also C^m. But is it true that if the functions generated by Av_1,Av_2 are C^m, then the functions generated by Av are also C^m? This result is not at all obvious. Indeed, if F is a function generated by Av, then it is *not* necessarily true that $F(x)=(1-t)F_1(x)+tF_2(x)$, where F_1 and F_2 are functions generated by Av_1 and Av_2. For example, in our generic example for the averages Av_p^*:

$$Av^*_{\frac{1}{2}}(a,b)=\frac{1}{2}\left(\sqrt{ab}+\frac{a+b}{2}\right)=\frac{1}{2}Av^*_0(a,b)+\frac{1}{2}Av^*_1(a,b).$$

Therefore the function generated by $Av^*_{\frac{1}{2}}$ is

$$F_{\frac{1}{2}}(x) = x^2 \neq \frac{1}{2}F_0(x) + \frac{1}{2}F_1(x) = \frac{1}{2}e^x + \frac{1}{2}x.$$

Nevertheless, even if we do not have any explicit formula for the functions F generated by the averaging rule Av, we would still like to know that if Av is C^m, then the functions F generated by Av are also C^m. The purpose of Section 3 is to prove exactly this result.

3 Smoothness of Functions Generated from Smooth Averaging Rules

We are now going to show that if an averaging rule Av is C^m, then the functions F generated by Av are also C^m. The most difficult cases turn out to be $m = 0, 1, 2$, so we will treat each of these cases in a separate subsection.

To fix our notation once and for all, let F_k denote the piecewise linear function generated after k levels of subdivision starting from a straight line F_0 joining two initial values $F_0(a), F_0(b)$ and inserting new vertices at the dyadic points

$$d_{j,k} = a + \frac{j}{2^k}(b-a) \qquad j = 0, \ldots, 2^k$$

by applying the subdivision rule

$$F_{k+1}\left(\frac{x+y}{2}\right) = Av(F_k(x), F_k(y)),$$

–that is, by setting

$$F_{k+1}(d_{2j,k+1}) = Av\left(F_k(d_{j,k}), F_k(d_{j,k})\right) = F_k(d_{j,k}) \tag{3.1}$$

$$F_{k+1}(d_{2j+1,k+1}) = Av\left(F_k(d_{j,k}), F_k(d_{j+1,k})\right). \tag{3.2}$$

Notice that this subdivision scheme is interpolatory; old vertices are retained since $d_{2j,k+1} = d_{j,k}$ and $F_{k+1}(d_{2j,k+1}) = F_k(d_{j,k})$. Moreover, since

$$\min(F(d_{j,k}), F(d_{j+1,k})) < Av(F(d_{j,k}), F(d_{j+1,k})) < \max(F(d_{j,k}), F(d_{j+1,k})),$$

the functions F_k are monotone functions on the interval $[a, b]$. When the functions F_k converge, we shall use F to denote the limit of the functions F_k on the interval $[a, b]$.

3.1 C^0

We are now going to prove that if Av is a continuous averaging rule, then the piecewise linear functions $F_k(x)$ converge to a continuous function $F(x)$. We begin with a somewhat technical lemma.

Lemma 3.1. $Av(F_k(d_{p,k}), F_k(d_{q,k})) = F_{k+1}(d_{p+q,k+1})$.

Proof. We proceed by induction on k. The result is certainly true for $k = 0$. We shall now assume that the result is valid for some $k \geq 0$ and prove that the result is true as well for $k+1$. There are three cases to consider:

- p and q are both even.

  By Equation 3.1,

  $$Av(F_{k+1}(d_{p,k+1}), F_{k+1}(d_{q,k+1})) = Av(F_k(d_{p/2,k}), F_k(d_{q/2,k})) \tag{3.3}$$

  But by the inductive hypothesis and Equation 3.1

  $$Av(F_k(d_{p/2,k}), F_k(d_{q/2,k})) = F_{k+1}(d_{(p+q)/2,k+1}) = F_{k+2}(d_{p+q,k+2}),$$

  so by Equation 3.3

  $$Av(F_{k+1}(d_{p,k+1}), F_{k+1}(d_{q,k+1})) = F_{k+2}(d_{p+q,k+2}).$$

- p and q are both odd.

  By Equation 3.2 and Property iv of averaging rules,

$$\begin{aligned} & Av(F_{k+1}(d_{p,k+1}), F_{k+1}(d_{q,k+1})) \\ = \; & Av(Av(F_k(d_{(p-1)/2,k}), F_k(d_{(p+1)/2,k})), Av(F_k(d_{(q-1)/2,k}), F_k(d_{(q+1)/2,k}))) \\ = \; & Av(Av(F_k(d_{(p-1)/2,k}), F_k(d_{(q+1)/2,k})), Av(F_k(d_{(p+1)/2,k}), F_k(d_{(q-1)/2,k}))). \end{aligned} \tag{3.4}$$

  But by the inductive hypothesis

$$\begin{aligned} Av(F_k(d_{(p-1)/2,k}), F_k(d_{(q+1)/2,k})) &= F_{k+1}(d_{(p+q)/2,k+1}) \\ Av(F_k(d_{(p+1)/2,k}), F_k(d_{(q-1)/2,k})) &= F_{k+1}(d_{(p+q)/2,k+1}). \end{aligned}$$

  Therefore by Equation 3.4

$$\begin{aligned} Av(F_{k+1}(d_{p,k+1}), F_{k+1}(d_{q,k+1})) &= Av(F_{k+1}(d_{(p+q)/2,k+1}), F_{k+1}(d_{(p+q)/2,k+1})) \\ &= F_{k+1}(d_{(p+q)/2,k+1}) = F_{k+2}(d_{p+q,k+2}). \end{aligned}$$

- p is even and q is odd.

  Since q is odd, it follows from Equation 3.2 that

$$Av(F_k(d_{(q-1)/2,k}), F_k(d_{(q+1)/2,k})) = F_{k+1}(d_{q,k+1}).$$

  Moreover, since p is even, we know from Equation 3.1 that

$$F_{k+1}(d_{p,k+1}) = F_k(d_{p/2,k}).$$

  Therefore by Property v of averaging rules,

$$\begin{aligned} & Av(F_{k+1}(d_{p,k+1}), F_{k+1}(d_{q,k+1})) \\ = \; & Av(F_k(d_{p/2,k}), Av(F_k(d_{(q-1)/2,k}), F_k(d_{(q+1)/2,l}))) \\ = \; & Av(Av(F_k(d_{p/2,k}), F_k(d_{(q-1)/2,k})), Av(F_k(d_{p/2,k}), F_k(d_{(q+1)/2,k}))). \end{aligned} \tag{3.5}$$

  But by the inductive hypothesis

$$\begin{aligned} Av(F_k(d_{p/2,k}), F_k(d_{(q-1)/2,k})) &= F_{k+1}(d_{(p+q-1)/2,k+1}) \\ Av(F_k(d_{p/2,k}), F_k(d_{(q+1)/2,k})) &= F_{k+1}(d_{(p+q+1)/2,k+1}). \end{aligned}$$

  Therefore by Equation 3.5 and Equation 3.2

$$\begin{aligned} Av(F_{k+1}(d_{p,k+1}), F_{k+1}(d_{q,k+1})) &= Av(F_{k+1}(d_{(p+q-1)/2,k+1}), F_{k+1}(d_{(p+q+1)/2,k+1})) \\ &= F_{k+2}(d_{p+q,k+2}). \end{aligned}$$

□

Proposition 3.2. *Let Av be a continuous averaging rule. Then the piecewise linear functions $F_k(x)$ converge pointwise for each value of x.*

Proof. If $F_0(a) = F_0(b)$, then by construction the functions F_k converge to a constant function F. Therefore, without loss of generality, we shall assume that $F_0(a) < F_0(b)$. (The case $F_0(a) > F_0(b)$ can be treated symmetrically.) At each dyadic value $d_{j,k}$, we know that the piecewise linear functions $F_n(d_{j,k})$ converge to a fixed value because, since our subdivision scheme is interpolatory, the values of the functions $F_n(d_{j,k})$ are all equal for $n > k$. Therefore we only need to consider non-dyadic values. Let c be a non-dyadic value. Since the dyadic values are dense in the reals, there is a sequence of dyadic values $d_{j_1,1}, d_{j_2,2}, \ldots$ approaching c from below, and another sequence of dyadic values $d_{j_1^*,1}, d_{j_2^*,2}, \ldots$ approaching c from above. Moreover, since the functions F_k are monotone increasing,

$$F_k(d_{j_k,k}) < F_k(c) < F_k(d_{j_k^*,k}).$$

To show that $\lim_{k\to\infty} F_k(c)$ exists, we shall show that $\lim_{k\to\infty} F_k(d_{j_k,k})$ and $\lim_{k\to\infty} F_k(d_{j_k^*,k})$ both exist and are equal. Since the functions F_k are monotone increasing, the sequence $F_1(d_{j_1,1}), F_2(d_{j_2,2}), \ldots$ is a monotone increasing sequence bounded above. Therefore this sequence has a limit which we shall denote by d. Similarly, the sequence $F_1(d_{j_1^*,1}), F_2(d_{j_2^*,2}), \ldots$ is a monotone decreasing sequence bounded below, and so also has a limit, which we shall denote by d^*. We need to show $d = d^*$. Suppose that $d \neq d^*$. Then since Av is continuous,

$$\lim_{k\to\infty} Av(F_k(d_{j_k,k})F_k(d_{j_k^*,k})) = Av(\lim_{k\to\infty} F_k(d_{j_k,k}), \lim_{k\to\infty} F_k(d_{j_k^*,k})) = Av(d, d^*).$$

Therefore since $d < d^*$,

$$d < \lim_{k\to\infty} Av(F_k(d_{j_k,k}), F_k(d_{j_k^*,k})) < d^*.$$

But, and here is the key point, since $d_{j_k,k}$ and $d_{j_k^*,k}$ are dyadic values,

$$d_{j_k+j_k^*,k+1} = \frac{d_{j_k,k} + d_{j_k^*,k}}{2}$$

is also a dyadic value. Moreover, since c is not dyadic, $d_{j_k+j_k^*,k+1} \neq c$. Thus either $d_{j_k+j_k^*,k+1} < c$ or $d_{j_k+j_k^*,k+1} > c$. Now without loss of generality we can assume that for infinitely many k, $d_{j_k+j_k^*,k+1} < c$. Since $d_{j_k,k} \to c$ and $d_{j_k^*,k} \to c$, it follows that $d_{j_k+j_k^*,k+1} \to c$. Moreover, since by assumption $d_{j_k+j_k^*,k+1} \to c$ from below,

$$\lim_{k\to\infty} F_{k+1}(d_{j_k+j_k^*,k+1}) = d.$$

But by Lemma 3.1

$$Av(F_k(d_{j_k,k}), F_k(d_{j_k^*,k})) = F_{k+1}(d_{j_k+j_k^*,k+1}).$$

Therefore

$$\lim_{k\to\infty} Av(F_k(d_{j_k,k}), F_k(d_{j_k^*,k})) = \lim_{k\to\infty} F_{k+1}(d_{j_k+j_k^*,k+1}) = d,$$

which contradicts the result proved earlier that

$$d < \lim_{k\to\infty} Av(F_k(d_{j_k,k}), F_k(d_{j_k^*,k})) < d^*.$$

Hence the assumption that $d \neq d^*$ must be false, so $d = d^*$. Thus

$$d = \lim_{k\to\infty} F_k(d_{j_k,k}) \leq \lim_{k\to\infty} F_k(c) \leq \lim_{k\to\infty} F_k(d_{j_k^*,k}) = d,$$

so $\lim_{k\to\infty} F_k(c)$ exists. Therefore the functions $F_k(x)$ converge pointwise for all values of x. □

Lemma 3.3. *Let Av be a continuous averaging rule, and let F be the pointwise limit of the functions F_k. Then for any two dyadic values d_1,d_2,*

$$F\left(\frac{d_1 + d_2}{2}\right) = Av(F(d_1), F_k(d_2)).$$

Proof. By Lemma 3.1

$$F_{k+1}(d_{p+q,k+1}) = Av(F_k(d_{p,k}), F_k(d_{q,k})).$$

Since the subdivision scheme that generates the function F_k is interpolatory, it follows that for any integer $n > k$,

$$F_n(d_{p+q,k+1}) = Av(F_n(d_{p,k}), F_n(d_{q,k})).$$

Moreover, since by assumption Av is continuous,

$$\begin{aligned}\lim_{n\to\infty} F_n(d_{p+q,k+1}) &= \lim_{n\to\infty} Av(F_n(d_{p,k}), F_n(d_{q,k})) \\ &= Av(\lim_{n\to\infty} F_n(d_{p,k}), \lim_{n\to\infty} F_n(d_{q,k})),\end{aligned}$$

so

$$F(d_{p+q,k+1}) = Av(F(d_{p,k}), F(d_{q,k})). \qquad (3.6)$$

Now let d_1, d_2 be any two dyadic values. Then for k sufficiently large, $d_1 = d_{p,k}$ and $d_2 = d_{q,k}$, so

$$\frac{d_1 + d_2}{2} = d_{p+q,k+1}.$$

Therefore by Equation 3.6

$$F\left(\frac{d_1+d_2}{2}\right) = Av(F(d_1), F(d_2)).$$

□

Proposition 3.4. *Let Av be a continuous averaging rule. Then the piecewise linear functions $F_k(x)$ converge pointwise to a continuous monotone function $F(x)$.*

Proof. If $F_0(a) = F_0(b)$, then by construction the functions F_k converge to a constant function F. Hence, without loss of generality, we shall assume $F_0(a) < F_0(b)$. From Proposition 3.2 we know that the functions $F_k(x)$ converge pointwise for each value of x. Therefore the limit function $F(x)$ exists. Moreover, since the functions $F_k(x)$ are monotonic, the function $F(x)$ is also monotonic. It remains only to show that the limit function $F(x)$ is a continuous function.

Consider first the dyadic values $d_{j,k}$. Let

$$\begin{aligned} d_0 &= d_{j-1,k} \\ d_{n+1} &= \frac{d_{j,k}+d_n}{2}. \end{aligned}$$

Then $\{d_n\}$ is a monotone increasing sequence converging to $d_{j,k}$. Since F is a monotone increasing function, $\{F(d_n)\}$ is a monotone increasing sequence bounded above by $F(d_{j,k})$. Therefore the sequence $\{F(d_n)\}$ converges to some limit value y. We claim that $y = F(d_{j,k})$. Indeed from Lemma 3.3 we know that

$$F(d_{n+1}) = F\left(\frac{d_{j,k}+d_n}{2}\right) = Av(F(d_{j,k}), F(d_n)).$$

Therefore by the continuity of Av

$$\lim_{n\to\infty} F(d_{n+1}) = \lim_{n\to\infty} Av(F(d_{j,k}), F(d_n)) = Av\left(F(d_{j,k}), \lim_{n\to\infty} F(d_n)\right),$$

so

$$y = Av(F(d_{j,k}), y).$$

But, by the definition of an averaging rule, we can have $Av(a,b) = b$ if and only if $a = b$. Hence $y = F(d_{j,k})$. Since F is monotonic, we conclude that F is continuous at $d_{j,k}$ from below. A similar argument shows that F is continuous at $d_{j,k}$ from above. Thus F is continuous at dyadic values.

Now let c be a non-dyadic value. In the proof of Proposition 3.2 we showed that there is a sequence of dyadic values $d_{j_1,1}, d_{j_2,2}, \ldots$ approaching c from below, and another sequence of dyadic values $d_{j_1^*,1}, d_{j_2^*,2}, \ldots$ approaching c from above such that

$$\lim_{k\to\infty} F_k(d_{j_k,k}) = F(c) = \lim_{k\to\infty} F_k(d_{j_k^*,k}).$$

Hence

$$\lim_{k\to\infty} F(d_{j_k,k}) = F(c) = \lim_{k\to\infty} F(d_{j_k^*,k}).$$

Therefore, since F is a monotonic function, F is continuous at c. □

Corollary 3.5. *Let Av be a continuous averaging rule. Then the piecewise linear functions $F_k(x)$ converge uniformly to a continuous monotone function $F(x)$.*

Proof. By Proposition 3.4, we know that the functions $F_k(x)$ converge pointwise to a continuous monotone function $F(x)$. Therefore we need only show that the convergence is uniform. Since F is continuous on a compact interval, F is uniformly continuous. Therefore for any $\epsilon > 0$ there exists a $\delta > 0$ such that for all x, y

$$|x-y| < \delta \Rightarrow |F(x) - F(y)| < \epsilon.$$

Now given any $\epsilon > 0$ we can choose N large enough that $\frac{1}{2}^N < \delta$. Then for $k > N$, adjacent dyadic points $d_{j,k}, d_{j+1,k}$ in the domain of F_k are within δ, so

$$|F_k(d_{j,k}) - F_k(d_{j+1,k})| = |F(d_{j,k}) - F(d_{j+1,k})| < \epsilon.$$

But by construction, for any value c, where $d_{j,k} < c < d_{j+1,k}$, and any $i > k$, we must have

$$\min(F_k(d_{j,k}), F_k(d_{j+1,k})) < F_i(c) < \max(F_k(d_{j,k}), F_k(d_{j+1,k})).$$

Moreover, since F_k is monotonic (in fact a straight line between $d_{j,k}$ and $d_{j+1,k}$),

$$\min(F_k(d_{j,k}), F_k(d_{j+1,k})) < F_k(c) < \max(F_k(d_{j,k}), F_k(d_{j+1,k})).$$

Hence

$$|F_i(c) - F_k(c)| < |F_k(d_{j,k}) - F_k(d_{j+1,k})| < \epsilon.$$

□

Corollary 3.6. *Let Av be a continuous averaging rule, and let F be the limit of the functions F_k. Then for all x, y in the domain of F,*

$$F\left(\frac{x+y}{2}\right) = Av(F(x), F(y)).$$

Proof. This result follows immediately from Lemma 3.3 because F is continuous and the dyadic points are dense in the reals. □

Theorem 3.7. *Let Av be a continuous averaging rule. Then the piecewise linear functions $F_k(x)$ converge uniformly to a continuous monotone function $F(x)$ that satisfies the functional equation*

$$F\left(\frac{x+y}{2}\right) = Av(F(x), F(y)).$$

Proof. This result follows immediately from Corollaries 3.5 and 3.6. □

3.2 C^1

If Av is C^1 and F is differentiable, then differentiating Equation 2.1 with respect to x and y by the chain rule yields

$$\begin{aligned}\frac{1}{2}F'\left(\frac{x+y}{2}\right) &= Av^{(1,0)}(F(x), F(y))F'(x)\\ \frac{1}{2}F'\left(\frac{x+y}{2}\right) &= Av^{(0,1)}(F(x), F(y))F'(y)\end{aligned}$$

Therefore,

$$F'(y) = \frac{Av^{(1,0)}(F(x), F(y))}{Av^{(0,1)}(F(x), F(y))}F'(x).$$

Thus we should expect that if F is differentiable anywhere, then F is differentiable everywhere.

Proposition 3.8. *Let F be a function generated by a C^1 averaging rule Av. Then F is either differentiable everywhere or differentiable nowhere. Moreover, if $F'(x)$ exists, then for all y,*

$$F'(y) = \frac{Av^{(1,0)}(F(x), F(y))}{Av^{(0,1)}(F(x), F(y))}F'(x). \tag{3.7}$$

Proof. Suppose that there is a point x where $F'(x)$ exists. We shall show that for any y, $F'(y)$ also exists. By definition,

$$\frac{Av^{(1,0)}(F(x), F(y))}{Av^{(0,1)}(F(x), F(y))} = \lim_{h\to 0} \frac{\frac{Av(F(x+h),F(y))-Av(F(x),F(y))}{F(x+h)-F(x)}}{\frac{Av(F(x),F(y+h))-Av(F(x),F(y))}{F(y+h)-F(y)}}.$$

But by Corollary 3.6,

$$\begin{aligned}Av(F(x+h), F(y)) - Av(F(x), F(y)) &= F\left(\frac{x+y+h}{2}\right) - F\left(\frac{x+y}{2}\right)\\ Av(F(x), F(y+h)) - Av(F(x), F(y)) &= F\left(\frac{x+y+h}{2}\right) - F\left(\frac{x+y}{2}\right).\end{aligned}$$

Therefore

$$\frac{Av^{(1,0)}(F(x),F(y))}{Av^{(0,1)}(F(x),F(y))} = \lim_{h\to 0}\frac{F(y+h)-F(y)}{F(x+h)-F(x)} = \lim_{h\to 0}\frac{\frac{F(y+h)-F(y)}{h}}{\frac{F(x+h)-F(x)}{h}},$$

so

$$F'(y) = \lim_{h\to 0}\frac{F(y+h)-F(y)}{h} = \frac{Av^{(1,0)}(F(x),F(y))}{Av^{(0,1)}(F(x),F(y))}F'(x).$$

Thus if F is differentiable at one point, then F is differentiable at every point. □

Theorem 3.9. *If F is a monotonic function, then F is differentiable almost everywhere.*

Proof. See [1, Theorem 39.9, page 375]. □

Corollary 3.10. *Let F be a function generated by a C^1 averaging rule Av. Then F is also C^1.*

Proof. This result is an immediate consequence of Proposition 3.8 and Theorem 3.9. □

Corollary 3.11. *Let F be a function generated by a C^1 averaging rule Av. Then the slopes of the piecewise linear functions $F_k(x)$ converge to the slope of $F(x)$.*

Proof. Consider first a dyadic point d. We can compute the slope of F at d by approaching d along a sequence of dyadic points $d_1, d_2, \ldots$. But the points $d_1, d_2, \ldots$ can be used to calculate the slope at d of the functions F_k, since for sufficiently large values of k, F and F_k agree at dyadic points. Thus the result is valid at the dyadic points. Now the result follows at the non-dyadic points, since F is C^1 and the dyadic points are dense in the reals. □

3.3 C^2

To simplify our notation, we shall write $Av^{(i,j)}(a,b)$ in place of $\frac{\partial^{i+j}Av}{\partial x^i \partial y^j}(a,b)$.

Proposition 3.12. *Let F be a function generated by a C^2 averaging rule Av. Then F is also C^2. Moreover,*

$$F''(x) = 4Av^{(1,1)}(F(x),F(x))(F'(x))^2. \tag{3.8}$$

Proof. From Corollary 3.6 F satisfies the functional equation

$$F\left(\frac{x+y}{2}\right) = Av(F(x),F(y)).$$

Differentiating both sides with respect to x by the chain rule yields

$$\frac{1}{2}F'\left(\frac{x+y}{2}\right) = Av^{(1,0)}(F(x),F(y))F'(x).$$

Let

$$R(y) = Av^{(1,0)}(F(x),F(y))F'(x).$$

Then $F'\left(\frac{x+y}{2}\right)$ is differentiable with respect to y if and only if $R(y)$ is differentiable with respect to y. But since by assumption Av is C^2, it follows by the chain rule that

$$R'(y) = Av^{(1,1)}(F(x),F(y))F'(x)F'(y).$$

Hence $R'(y)$ exists and is continuous. Therefore $F'\left(\frac{x+y}{2}\right)$ is differentiable with respect to y, and

$$\frac{1}{4}F''\left(\frac{x+y}{2}\right) = R'(y) = Av^{(1,1)}(F(x),F(y))F'(x)F'(y).$$

This result is true for all x, y. Setting $y = x$ yields

$$F''(x) = 4Av^{(1,1)}(F(x),F(x))(F'(x))^2.$$

□

3.4 C^k

Once we have proved that if an averaging rule Av is C^k, then the functions F generated by Av are also C^k when $k = 2$, the general result for arbitrary k follows by a simple induction on k because Equation 3.8 provides us with an explicit formula for $F''(x)$ in terms of the partial derivatives of Av and $F'(x)$.

Proposition 3.13. *Let F be a function generated by a C^k averaging rule Av. Then F is also C^k. Moreover, if $k \geq 2$, then there is a polynomial P_k in the variables $Av^{(i,j)}(F(x), F(x)), F^{(h)}(x)$, where $i, j \geq 1$, $i + j \leq k$, and $1 \leq h \leq k - 1$ such that*

$$F^{(k)}(x) = P_k(Av^{(i,j)}(F(x), F(x)), F^{(h)}(x)).$$

Proof. By induction on k. We have already proved this result for $k = 0, 1, 2$. Suppose then that the result is true for some value of $k \geq 2$. We shall prove that the result is also true for $k + 1$. By the inductive hypothesis,

$$F^{(k)}(x) = P_k(Av^{(i,j)}(F(x), F(x)), F^{(h)}(x)).$$

Let

$$R(x) = P_k(Av^{(i,j)}(F(x), F(x)), F^{(h)}(x)).$$

To show that $F^{(k+1)}(x)$ exists and is continuous, we need only show that $R'(x)$ exists and is continuous. But since P_k is a polynomial depending only on the variables $Av^{(i,j)}(F(x), F(x)), F^{(h)}(x)$, where $i, j \geq 1$, $i + j \leq k$, and $1 \leq h \leq k - 1$, we can differentiate $R(x)$ using only the chain rule and the product rule to get a polynomial P_{k+1} depending only on the variables $Av^{(i,j)}(F(x), F(x)), F^{(h)}(x)$, where $i, j \geq 1$, $i + j \leq k + 1$, and $1 \leq h \leq k$. Since by assumption Av is C^{k+1}, the functions $Av^{(i,j)}(F(x), F(x))$, where $i, j \geq 1$ and $i + j \leq k + 1$ exist and are continuous. Similarly, by the inductive hypothesis, the functions $F^{(h)}(x)$, $1 \leq h \leq k$, exist and are continuous. Hence $R'(x)$ exists and is continuous, so $F^{(k+1)}(x)$ exists and is continuous. □

Corollary 3.14. *Let F be a function generated by a C^k averaging rule Av. Then F is also C^k. Moreover, if $k \geq 2$, then there is a polynomial P_k in the variables $Av^{(i,j)}(F(x), F(x)), F'(x)$, where $i, j \geq 1$ and $i + j \leq k$ such that*

$$F^{(k)}(x) = P_k(Av^{(i,j)}(F(x), F(x)), F'(x)).$$

That is, we only need the first derivative of F along with the partial derivatives of Av at F to compute $F^{(k)}(x)$ for $k \geq 2$.

Proof. This result follows immediately from Proposition 3.13. □

Despite Corollary 3.14, it is not so simple to calculate $F^{(k)}(x)$ explicitly, even if we have explicit formulas for Av and for the partial derivatives of Av. In general, the only way to calculate $F(x)$ is to apply subdivision to compute the piecewise linear functions F_k that approach F in the limit and then to use the values of $F_k(x)$ to approximate $F(x)$. To calculate $F'(x)$, we can compute the piecewise linear functions F_k and use the slopes of these functions to approximate the derivative of F at each point. Alternatively, we can calculate the derivative $F'(x)$ at one point x by using the slopes of the functions F_k at x and then use Equation 3.7,

$$F'(y) = \frac{Av^{(1,0)}(F(x), F(y))}{Av^{(0,1)}(F(x), F(y))} F'(x),$$

to calculate $F'(y)$ at an arbitrary point y.

4 Summary and Conclusion

As a consequence of Theorem 3.7, Propositions 3.8 and 3.12, and Corollaries 3.10, 3.11, and 3.14, we have now proved the following general theorem:

Theorem 4.1. *Let Av be a C^k averaging rule, and let F_k be the piecewise linear functions generated by two point interpolatory subdivision from the averaging rule Av. Then the functions F_k converge uniformly to a C^k function F, and if $k \geq 1$ the slopes of F_k converge to the derivative of F. Moreover, F satisfies the following functional equations:*

$$F\left(\frac{x+y}{2}\right) = Av(F(x), F(y)) \qquad k \geq 0$$

$$F'(y) = \frac{Av^{(1,0)}(F(x),F(y))}{Av^{(0,1)}(F(x),F(y))}F'(x) \qquad k \geq 1$$

$$F''(x) = 4Av^{(1,1)}(F(x), F(x))(F'(x))^2 \qquad k \geq 2$$

$$F^{(k)}(x) = P_k(Av^{(i,j)}(F(x), F(x)), F'(x)) \qquad k \geq 3$$

where P_k is a polynomial in the variables $Av^{(i,j)}(F(x), F(x)), F'(x)$ for $i, j \geq 1$ and $i + j \leq k$.

The only difficult cases to prove are when Av is either C^0 or C^1. The reason that these cases are so hard is that we do not have an explicit expression for F in terms of Av; all we have is the functional equation:

$$F\left(\frac{x+y}{2}\right) = Av(F(x), F(y))$$

and even this equation must be derived when Av is C^0. Therefore we need to use some tricky arguments to prove that F is C^0 or C^1 when Av is C^0 or C^1. But once we establish that F is C^1 we have the functional equation

$$\frac{1}{2}F'\left(\frac{x+y}{2}\right) = A^{(1,0)}(F(x), F(x))F'(x)$$

and the proof that F is C^k when Av is C^k follows rather easily by induction on k.

As a consequence of the fact that when Av is C^k then F is C^k, we now have the following result, which is the ultimate goal of this paper:

Theorem 4.2. *Let S be a subdivision algorithm based on linear averaging and let $\overline{S}$ be the same subdivision procedure where the linear averaging rule $A(a, b) = (a + b)/2$ is replaced everywhere in the algorithm by a nonlinear averaging rule $Av(a, b)$. If the functions generated by S are C^n and the averaging rule Av is C^m, then the functions generated by $\overline{S}$ are C^k, where $k = \min(m, n)$.*

Proof. This result is an immediate consequence of Theorem 2.2 and Theorem 4.1. □

References

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[2] Hardy, G.H., Littlewood, J.E., and Polya, G. (1934), *Inequalities*, Cambridge University Press.

[3] Schaefer, S., Vouga, E., and Goldman, R. (2006), Nonlinear Subdivision through Nonlinear Averages, submitted to *CAGD*.