A Simple Class of Non-Linear Subdivision Schemes

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Etienne Vouga
Ron Goldman
Subdivision

- Set of rules $S$ that recursively act on a shape $p^0$
  \[ p^{k+1} = S(p^k) \]
- Converges to a smooth shape
Subdivision

- Set of rules $S$ that recursively act on a shape $p^0$

\[ p^\infty = S^\infty(p^0) \]

- Converges to a smooth shape
Linear Subdivision

- Locally can be written as matrix multiplication
  \[ p^{k+1} = M p^k \]
- Usually reproduce polynomials
- Easy to analyze
  - Sufficient conditions of continuity based on eigen-structure of \( M \) [Reif 95]
- Includes Catmull-Clark, Loop, Butterfly, etc…
Non-linear Subdivision

- Greater expression
  - Reproduce non-polynomial functions
    - circles [Sabin et al. 2005]
    - $p(x)e^{l(x)}$ [Micchelli 1996]
  - Preserve convexity [Floater et al. 1998]
  - Subdivision curves on manifolds
- Hard to analyze smoothness
Contributions

- Provide a simple class of non-linear subdivision schemes
  - Easy to analyze smoothness
  - Modification of linear subdivision schemes
  - Can reproduce interesting functions: trigonometrics, gaussians
- Applications to intersection calculations
Linear Subdivision Example

- Uniform B-splines [Lane, Reisenfeld 1980]
  - Doubling followed by mid-point averaging
  - Smoothness: $C^{n-1}$ ($n = \# \text{ of averaging steps}$)
  - Piecewise polynomial
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![Diagram of a piecewise polynomial curve]
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![Diagram of linear subdivision example]
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Simple Non-Linear Subdivision

- Replace mid-point with geometric mean
  \[ \frac{a + b}{2} \rightarrow \sqrt{ab} \]

- Is the curve smooth?

- What functions does this method reproduce?
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Functional Equations

- Find parametric midpoint of a function $F$
  
  $$F\left(\frac{x_0 + x_1}{2}\right) = G(F(x_0), F(x_1))$$

- Example: $L(x) = m \cdot x + b$
  
  $$L\left(\frac{x_0 + x_1}{2}\right) = \frac{L(x_0) + L(x_1)}{2}$$
Functional Equations

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- Example: $L(x) = mx + b$

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[Graph of a decreasing function with points plotted on it]
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Other Averaging Rules

<table>
<thead>
<tr>
<th>Function</th>
<th>Averaging Rule</th>
</tr>
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<tbody>
<tr>
<td>$F(x) = \sqrt{x}$</td>
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Non-linear Maps

- **Given**
  - $F$: 1-1 function on $\Omega \subseteq \mathbb{R}^n$
  - $S$: subdivision scheme
  - $\hat{S} = F \circ S \circ F^{-1}$

- **Then**
  - $\hat{S}^\infty = F \circ S^\infty \circ F^{-1}$
  - $S^\infty(p^0) = p^\infty \implies \hat{S}^\infty(F(p^0)) = F(p^\infty)$
Non-linear Maps

- **Given**
  - \( F: 1-1 \) function on \( \Omega \subseteq R^n \)
  - \( S = S_d \circ \ldots \circ S_2 \circ S_1 \): subdivision scheme
  - \( \hat{S} = F \circ S \circ F^{-1} \)

- **Then**
  - \( \hat{S} = (F \circ S_d \circ F^{-1}) \circ \ldots \circ (F \circ S_2 \circ F^{-1}) \circ (F \circ S_1 \circ F^{-1}) \)
Non-linear Maps Example

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\hat{S} = (F \circ S_d \circ F^{-1}) \circ \ldots \circ (F \circ S_2 \circ F^{-1}) \circ (F \circ S_1 \circ F^{-1})
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Lane-Reisenfeld

\[
S_1(p)_j = p \lfloor j/2 \rfloor
\]

\[
S_{i \neq 1}(p)_j = \frac{p_j + p_{j+1}}{2}
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\[
\hat{S}_1(p)_j = F(F^{-1}(p \lfloor j/2 \rfloor))
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\[
\hat{S}_{i \neq 1}(p)_j = F\left(\frac{F^{-1}(p_j) + F^{-1}(p_{j+1})}{2}\right)
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Non-linear Maps Example

\[ \hat{S} = (F \circ S_d \circ F^{-1}) \circ \ldots \circ (F \circ S_2 \circ F^{-1}) \circ (F \circ S_1 \circ F^{-1}) \]

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S_1(p)_j = p\left\lfloor \frac{j}{2} \right\rfloor \\
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F(x) = e^x \\
\hat{S}_1(p)_j = p\left\lfloor \frac{j}{2} \right\rfloor \\
\hat{S}_{i \neq 1}(p)_j = \sqrt{p_j p_{j+1}}
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Smoothness and Interpolation

- Given
  - $F$: 1-1 function on $\Omega \subseteq R^n$
  - $S$: subdivision scheme
  - $\hat{S} = F \circ S \circ F^{-1}$

- Then
  - $S^\infty(p^0): C^k$ & $F:C^n \Rightarrow \hat{S}^\infty(\hat{p}^0): C^{\min(k,n)}$
  - $S$: interpolatory $\Rightarrow \hat{S}$: interpolatory
Example

Four-Point [Dyn et al. 1987]
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Mobius Transform
Geometric Properties

- Properties: convex-hull, variation diminishing

Linear

\[ F(z) = e^z \]
Geometric Interpretation

- Modify geodesics so that the properties hold

\[ D(\hat{P}, \hat{Q}) = \text{Dist}_{\text{Euclidean}}(F^{-1}(\hat{P}), F^{-1}(\hat{Q})) \]
Geometric Interpretation

- A set $C$ is convex w.r.t. the geodesics $G$ if the geodesic connecting any two points in $C$ lies completely within $C$
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Intersection

1) If convex hulls of the control points do not intersect, then the curves do not intersect

2) If each curve is approximately a straight line, intersect those lines; else subdivide
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- Non-linear hulls may be curved and difficult to compute
- If $F'(t)$ is monotonic, we can compute a simple piecewise linear approximation
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Future Work

- Other types of averaging rules (non-analytic)
  - lofting curve networks
- Extensions to surfaces
  - Extraordinary points
- Slowing varying non-linear maps