

On C^2 Triangle/Quad Subdivision

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Abstract

In this paper we present a subdivision scheme for mixed triangle/quad meshes that is C^2 everywhere except for isolated, extraordinary points where the surface is C^1 . The rules that we describe are the same as Stam/Loop's scheme except that we perform an unzipping pass prior to subdivision. This simple modification improves the smoothness along the ordinary triangle/quad boundary from C^1 to C^2 and creates a scheme capable of subdividing arbitrary meshes. Finally, we end with a proof based on Levin/Levin's joint spectral radius calculation to show our scheme is indeed C^2 along the triangle/quad boundary.

1 Introduction

Subdivision has become a staple of the geometric modeling community allowing coarse, polygonal shapes to represent highly refined, smooth shapes with guaranteed continuity properties. Previously, there has been a dichotomy between polygonal primitives that subdivision schemes operate on. Two of the most popular subdivision schemes, Loop [Loop 1987] and Catmull-Clark [Catmull and Clark 1978], operate on triangle and quad meshes respectively.

1.1 Stam/Loop's Scheme

Recently, Stam and Loop [Stam and Loop 2003] introduced a generalization of Loop and Catmull-Clark subdivision that unifies these schemes together and operates on mixed triangle/quad surfaces. The subdivision scheme that they present reproduces Loop subdivision on the triangular portions of the mesh and Catmull-Clark subdivision on the quadrilateral polygons. Furthermore, the authors derive subdivision rules for extraordinary vertices composed of both quads and triangles where the subdivision scheme is C^1 .

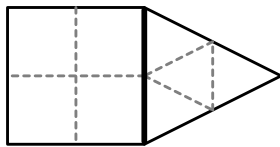


Figure 1: Linear subdivision for triangle/quad meshes. An ordinary triangle/quad configuration is introduced all along the boundary edge.

Stam/Loop created their generalization of triangle and quad subdivision by utilizing the fact that both Loop and Catmull-Clark subdivision can be written as linear subdivision followed by averaging [Zorin and Schröder 2001; Stam 2001; Warren and Weimer 2001]. For triangle/quad meshes, linear subdivision splits triangles into four new triangles and quads into four new quads. This process

introduces what Stam/Loop called an ordinary edge along the triangle/quad boundary where vertices are contained by two adjacent quads and three adjacent triangles (see figure 1).

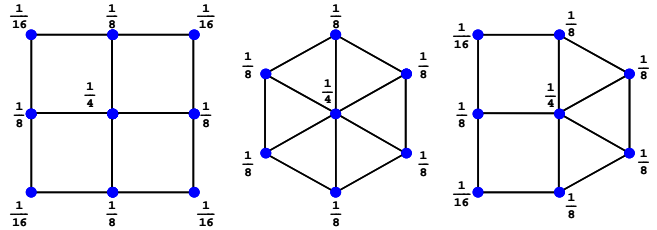


Figure 2: Averaging masks for Catmull-Clark (left), Loop (middle) and Triangle/Quad (right).

Once linear subdivision is complete, an averaging pass is applied to the mesh. Figure 2 shows the averaging mask for the ordinary case of Catmull-Clark and Loop subdivision. Stam/Loop noticed that the averaging masks for triangle and quad subdivision looked remarkably similar and hypothesized that the averaging mask for mixed triangle/quad surfaces at the ordinary boundary would simply be the mask shown on the right of figure 2. The authors then used this observation to generalize the averaging mask to arbitrary configurations of quads and triangles around a vertex. Finally, Stam/Loop show that their scheme is C^2 everywhere except for extraordinary points and the ordinary triangle/quad boundary where their scheme is C^1 .

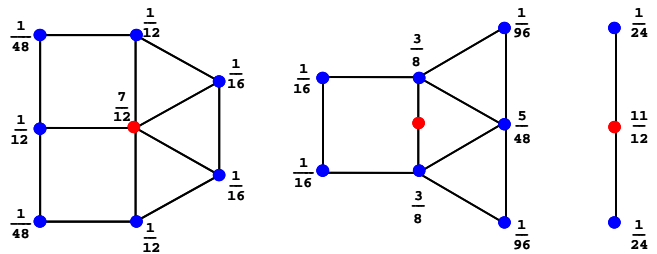


Figure 3: Levin/Levin's rules for the central edge (left and middle). Unzipping rule for triangular side (right).

1.2 Levin/Levin's Scheme

To remedy this smoothness problem along ordinary triangle/quad edges, Levin and Levin [Levin and Levin 2003] introduced a set of modified rules along the triangle/quad boundary shown in figure 3. The authors also present the concept of an "unzipping" mask shown in figure 3 (right). Prior to subdivision, points along the regular triangle/quad boundary are replicated; one set of vertices for the quadrilateral polygons and another for the triangular

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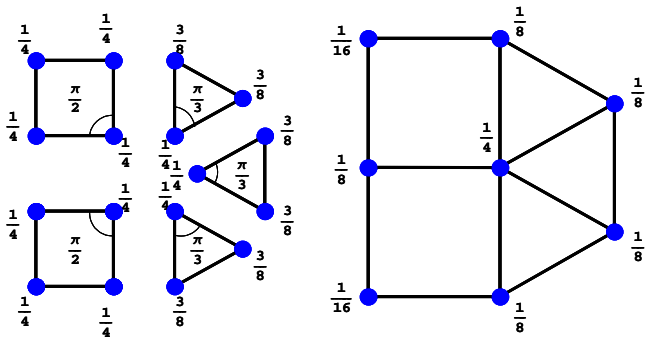


Figure 4: Subdivision for triangle/quad meshes as centroid averaging. Centroids of each type of polygon weighted by the angle spanned in the ordinary configuration (left). Averaging rule at ordinary boundary formed from centroids (right).

polygons. This replication essentially “unzippers” the mesh into disjoint pieces consisting of only triangles or only quads. The replicated vertices for the quadrilateral polygons retain their original positions; however, the vertices along the boundary for the triangular polygons have the mask in figure 3 (right) applied to them. When subdivision is performed, the new vertices on the quadrilateral portions of the mesh use only the quadrilateral vertices while new vertices on the triangular portions of the mesh use only triangular vertices. The vertices actually on the triangle/quad boundary use only the original vertices of the mesh.

Levin/Levin then prove that these modified rules generate a surface that is C^2 across the triangle/quad boundary. As part of their proof, the authors present a sufficient test for C^2 smoothness based on a joint spectral radius calculation between two subdivision matrices and show that their modifications generate a C^2 subdivision scheme. However, this subdivision scheme can be difficult to apply in practice due to the special rules introduced along the triangle/quad boundary, which also have larger support than the C^1 rules and do not readily fit into the averaging subdivision framework.

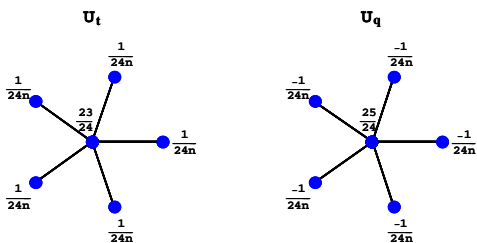


Figure 5: Unzipping mask for the vertices part of the triangle/quad boundary. n is the number of edges incident to the vertex that are part of the network of triangle/quad boundary edges.

2 The Unified Subdivision Scheme

Our implementation of triangle/quad subdivision uses the centroid averaging approach described by Warren and Schaefer [Warren and Schaefer 2003]. In that method the authors derive subdivision rules for arbitrary configurations of triangles and quads as a weighted average of centroids of polygons. For instance, figure 4 illustrates

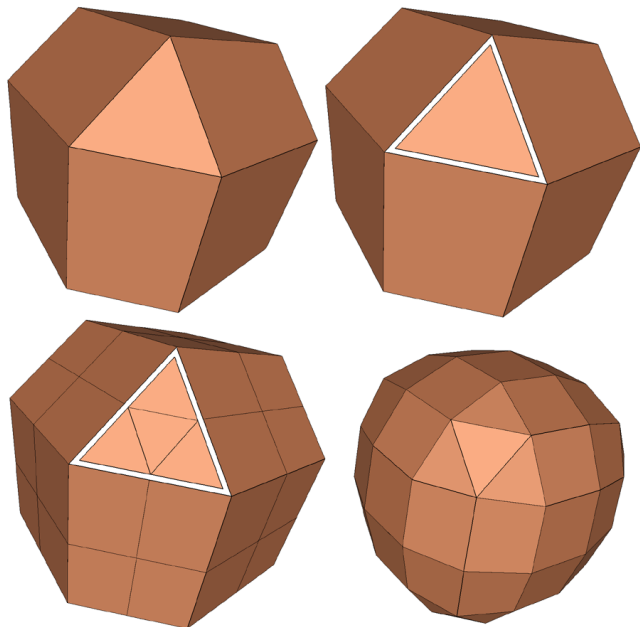


Figure 6: Initial shape (upper left). Unzipped shape (upper right). Linear subdivision (bottom left). Averaging pass zippers mesh back together (bottom right).

the centroids and relative weightings of those centroids that generate the averaging mask of Stam/Loop for the ordinary triangle/quad boundary.

Like Levin/Levin, we utilize an unzipping mask during subdivision. However, our unzipping mask differs from Levin/Levin’s choice and is shown in figure 5. We have also extended our unzipping masks to arbitrary configurations of edges part of the triangle/quad boundary, which allows us to subdivide a greater variety of surfaces.

Prior to subdivision, we identify edges on the surface contained by both triangles and quads. These edges define a network of curves on the surface. Then we apply the unzipping masks (U_t, U_q) to this curve network to generate separate triangle and quad vertices along the triangle/quad boundary (we also designate vertices contained completely by triangles or completely by quads to be triangle and quad vertices respectively). Next, we apply linear subdivision and averaging to the resulting points. Our only modification that we make to Warren and Schaefer’s scheme is that we require that each centroid is calculated using vertices only of the same type as the polygon; that is, triangle centroids are calculated using only triangle vertices and, similarly, quad centroids are calculated using only quad vertices. This small modification generates surfaces that are C^2 across the ordinary triangle/quad boundary, which we prove in section 3. Furthermore, these changes also extend the subdivision scheme to arbitrary surfaces such as non-manifold surfaces.

The entire subdivision process is depicted in figure 6. Starting with an initial shape, we first unzip the surface into disjoint pieces consisting of entirely triangles or entirely quads by applying the masks in figure 5. Next, we perform linear subdivision on the separate pieces. Finally, we close the surface back together by performing averaging, which completes one round of subdivision.

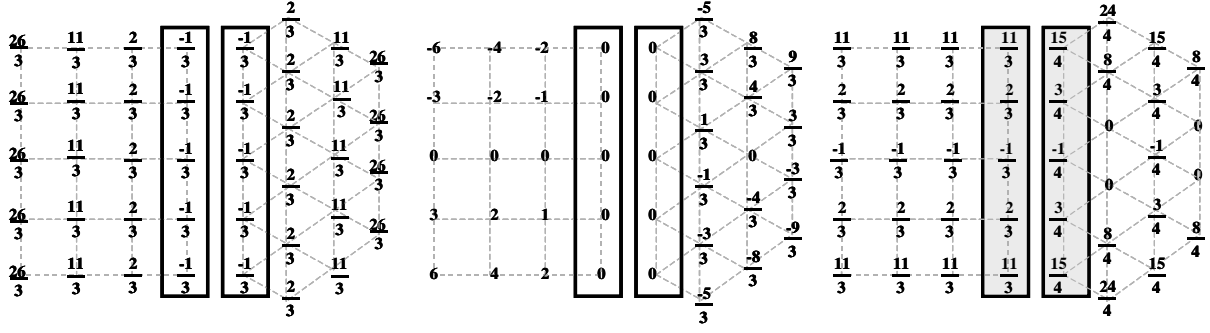


Figure 7: Coefficients of the Box spline reproducing x^2 (left), xy (middle) and y^2 (right). The boundary vector for the quadrilateral and triangular side are highlighted.

3 Analysis

3.1 Necessary Conditions

Given an ordinary triangle/quad boundary (shown in figure 7) we define S to be the subdivision matrix for Stam/Loop's scheme formed by centroid averaging. For the subdivision scheme to be C^2 in the functional sense, S must satisfy

$$S z_i = \lambda_i z_i \quad (1)$$

where $\lambda_i = 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$ and z_i are the corresponding eigenvectors producing the polynomials $1, x, y, x^2, xy, y^2$ [Warren and Weimer 2001]. While S satisfies equation 1 for $i = 1 \dots 5$, S does not for z_6 (corresponding to y^2). Figure 7 shows the coefficients of the vertices that reproduce the quadratic polynomials x^2, xy and y^2 over the triangular and quadrilateral portions of the mesh. Notice that the coefficients for y^2 do not agree at the boundary so S cannot possibly be C^2 at the boundary.

Our goal is to construct a new subdivision scheme \hat{S} such that \hat{S} satisfies equation 1 for $i = 1 \dots 5$ and that $\hat{S} v = \frac{1}{4} v$ where v is a new eigenvector corresponding to y^2 . To analyze the case of y^2 further, let v_t be the coefficients that reproduce y^2 on the triangle vertices and the boundary, but zero on the quadrilateral vertices. Similarly, let v_q be the coefficients reproducing y^2 on the quadrilateral vertices and the boundary, but zero on the triangle vertices. We define our new eigenvector v to be of the form

$$v = \begin{cases} v_t & \text{triangle vertices} \\ \alpha v_t + (1 - \alpha) v_q & \text{boundary} \\ v_q & \text{quad vertices} \end{cases} \quad (2)$$

We now construct unzipping matrices U_t and U_q such that

$$U_t v = \begin{cases} v_t & \text{triangle vertices and boundary} \\ 0 & \text{quad vertices} \end{cases}$$

$$U_q v = \begin{cases} 0 & \text{triangle vertices} \\ v_q & \text{boundary and quad vertices} \end{cases}$$

Using equation 2 we solve for the unzipping matrices as

$$U_t = \begin{cases} 1 & \text{triangle vertices} \\ (\frac{1-\alpha}{24}, \frac{11+\alpha}{12}, \frac{1-\alpha}{24}) & \text{boundary} \\ 0 & \text{quad vertices} \end{cases}$$

$$U_q = \begin{cases} 0 & \text{triangle vertices} \\ (\frac{-\alpha}{24}, \frac{12+\alpha}{12}, \frac{-\alpha}{24}) & \text{boundary} \\ 1 & \text{quad vertices} \end{cases}$$

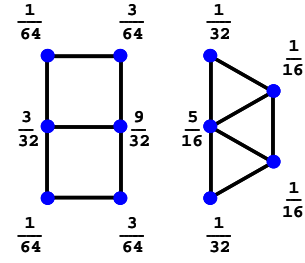


Figure 8: Mask calculated through centroid averaging on each side of the triangle/quad boundary. The mask is exactly half of the regular mask for triangular or quadrilateral surfaces.

Notice that Levin/Levin's choice of the unzipping mask corresponds to $\alpha = 0$. Levin/Levin's special boundary rules were then chosen to satisfy equation 1 for that particular choice of α .

With these definitions we now partition S into the form $S = S_t + S_q$ where S_t and S_q are formed by centroid averaging on the triangle and quad portions of the mesh respectively. S_t and S_q then satisfy

$$S_t v_t = \begin{cases} \frac{1}{4} v_t & \text{triangle vertices} \\ \frac{1}{8} v_t & \text{boundary} \\ 0 & \text{quad vertices} \end{cases}$$

$$S_q v_q = \begin{cases} 0 & \text{triangle vertices} \\ \frac{1}{8} v_q & \text{boundary} \\ \frac{1}{4} v_q & \text{quad vertices} \end{cases}$$

Notice that, on the boundary, S_t and S_q produce $\frac{1}{8} v_t$ and $\frac{1}{8} v_q$ respectively because the subdivision matrices contain half-masks (shown in figure 8) formed from centroid averaging.

Our subdivision scheme \hat{S} is then represented as

$$\hat{S} = S_t U_t + S_q U_q.$$

Applying \hat{S} to v yields

$$\hat{S} v = \begin{aligned} & S_t U_t v + S_q U_q v \\ &= \begin{cases} \frac{1}{4} v_t & \text{triangle vertices} \\ \frac{1}{8} v_t + \frac{1}{8} v_q & \text{boundary} \\ \frac{1}{4} v_q & \text{quad vertices} \end{cases} \end{aligned}$$

The final piecewise definition corresponds to exactly $\frac{1}{4} v$. Since U_t and U_q do not modify the boundary for the eigenvectors z_i for

$i = 1 \dots 5$ and $\hat{S}v = \frac{1}{4}v$, \hat{S} satisfies the necessary conditions for C^2 continuity at the boundary.

3.2 Sufficient Conditions

To analyze the smoothness of the subdivision scheme that we present, we use a sufficient test described by Levin/Levin [Levin and Levin 2003]. This smoothness test requires that the subdivision scheme is C^2 away from the boundary edge and that the subdivision matrix for a point on the boundary satisfies the necessary conditions from section 3.1. Furthermore, the subdivision scheme along the edge must satisfy a joint spectral radius condition.

To perform the joint spectral radius test, we require two subdivision matrices (A and B) that map an edge L on the boundary to two smaller edges (L_1 and L_2) after one round of subdivision. The matrices A and B should contain all of the vertices that influence the surface over the edges L_1 and L_2 . Next, we find a diagonalizing matrix W such that

$$\begin{aligned} W^{-1}AW &= \begin{pmatrix} \Lambda & C_0 \\ 0 & Y_0 \end{pmatrix} \\ W^{-1}BW &= \begin{pmatrix} \theta & C_1 \\ 0 & Y_1 \end{pmatrix} \end{aligned} \quad (3)$$

where Λ is a diagonal matrix with the entries $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$ and θ is an upper-triangular matrix with the same diagonal entries as Λ . Finally, we use Y_0 and Y_1 to compute

$$\rho^{[k]}(Y_0, Y_1) = (\text{Max}\|Y_{\varepsilon_k} Y_{\varepsilon_{k-1}} \dots Y_{\varepsilon_1}\|_{\infty})^{\frac{1}{k}} \text{ where } \varepsilon_i \in \{0, 1\}.$$

According to Levin/Levin, if there exists a k such that $\rho^{[k]} < \frac{1}{4}$, then the subdivision scheme is C^2 at the boundary.

The obvious choice for constructing the matrix W is to simply use all of the eigenvectors of A . However, this approach can be numerically unstable if the matrix has small eigenvalues. Levin/Levin suggest that W be formed from the right eigenvectors associated with the eigenvalues from Λ and a basis of the null space from the corresponding left eigenvectors. Since symbolic math packages such as *Mathematica* can generate the eigenvectors corresponding to Λ exactly, this method yields a numerically stable method for creating W .

While Levin/Levin’s approach leads to a matrix W satisfying equation 3, we found that the rate of convergence in the spectral radius calculation was slow for our subdivision scheme. Instead, we form a diagonalizing matrix W using the right eigenvectors corresponding to the eigenvalues in Λ and the null space of those vectors. In our experience, we found that the matrix W created in this fashion yields a matrix satisfying equation 3 and generates faster convergence in the joint spectral radius calculation.

When applying the spectral radius technique to our subdivision scheme, we calculated $\rho^{[17]} = 0.172878$. Since $\rho^{[17]} < \frac{1}{4}$ and our scheme satisfies the necessary conditions for polynomial generation, we conclude that our subdivision scheme is C^2 at triangle/quad boundaries. Figure 9 shows a curvature plot of a highly subdivided model from figure 6. Notice the color discontinuity at the triangle/quad boundary in Stam/Loop’s scheme where as our modification generates continuous curvature at the boundary.

4 Conclusion

We have presented a subdivision scheme for mixed triangle/quad surfaces that is C^2 everywhere except for isolated, extraordinary vertices where the scheme is C^1 . The subdivision scheme itself is

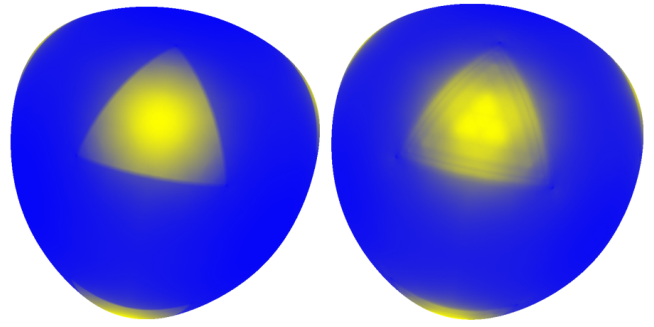


Figure 9: Curvature plots of the finely subdivided shape from figure 6 for Stam/Loop’s scheme (bottom left) and our modified scheme (bottom right).

the same as Stam/Loop’s triangle/quad scheme except that we perform an unzipping pass before subdivision. Our choice of the unzipping mask does not yield special rules in the implementation and lends itself to real-world applications as the method is very easy to code. Furthermore, our subdivision scheme provides rules for handling arbitrary triangle/quad surfaces including non-manifold surfaces. Finally, we applied Levin/Levin’s sufficiency test for C^2 smoothness to prove our modification to the subdivision produces C^2 surfaces at the triangle/quad boundary.

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