2D Transformations

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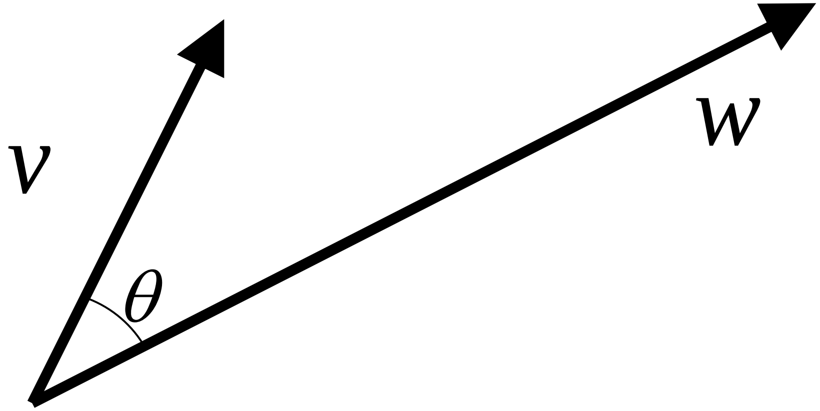
TAMU Computer Science

Coordinates

- Depend on a particular choice of origin (0,0) and special axes (x/y-axis)
- Analogous to assembly language
  - Necessary evil for computation
  - We'd like to think at a higher level

Review – Vector Operations

- Dot Product



Review – Vector Operations

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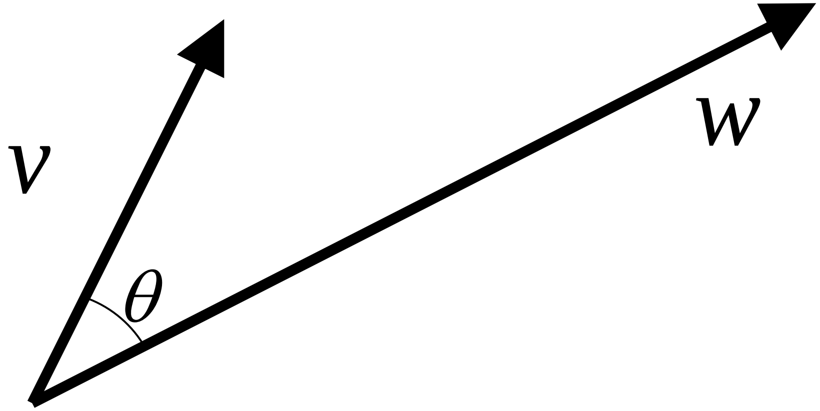

$$v \cdot w = |v||w|\cos(\theta)$$

Review – Vector Operations

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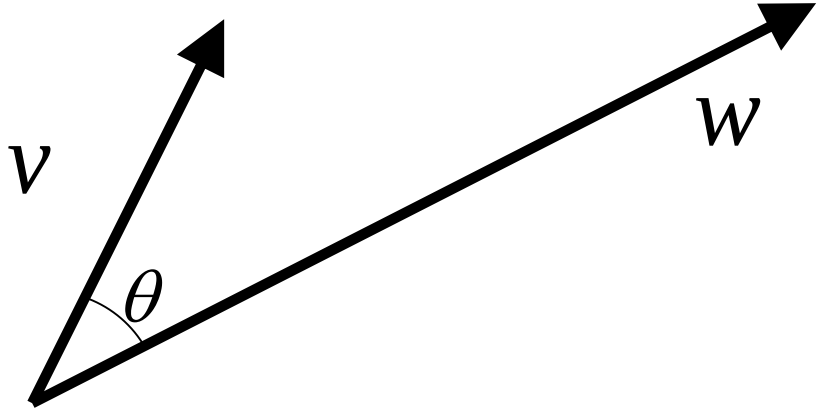

$$w \cdot v = |v||w|\cos(\theta)$$

Review – Vector Operations

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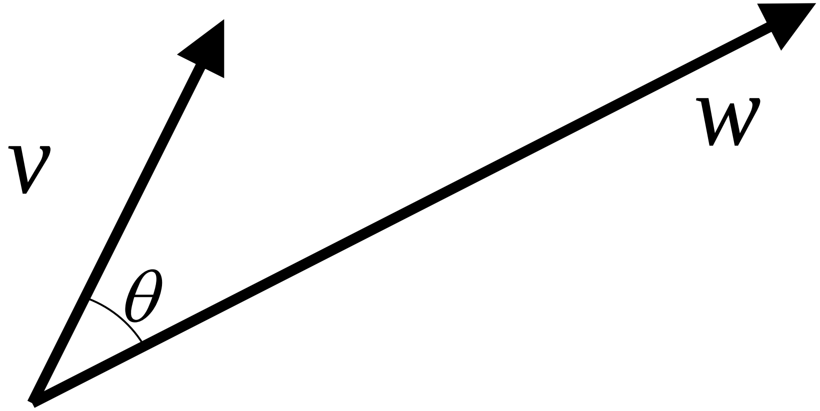

$$v \cdot v = |v|^2$$

Review – Vector Operations

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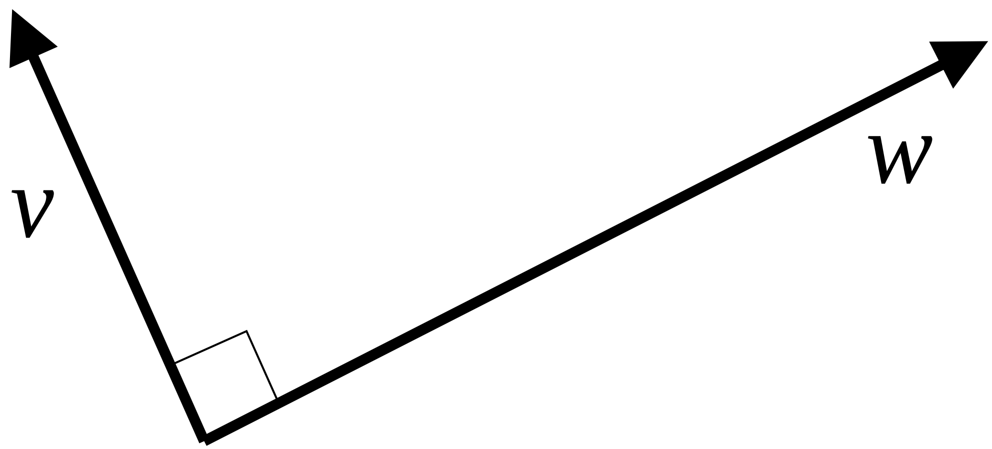
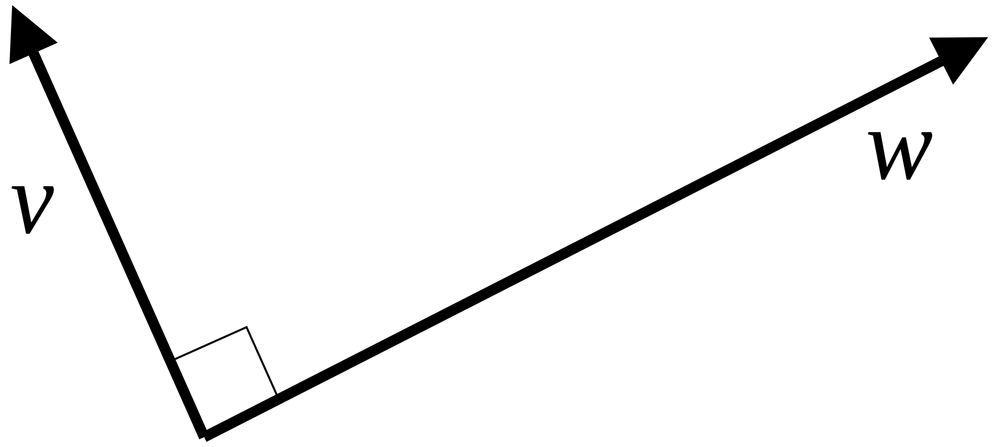


$$v \cdot w = |v||w|\cos(\theta)$$

Review – Vector Operations

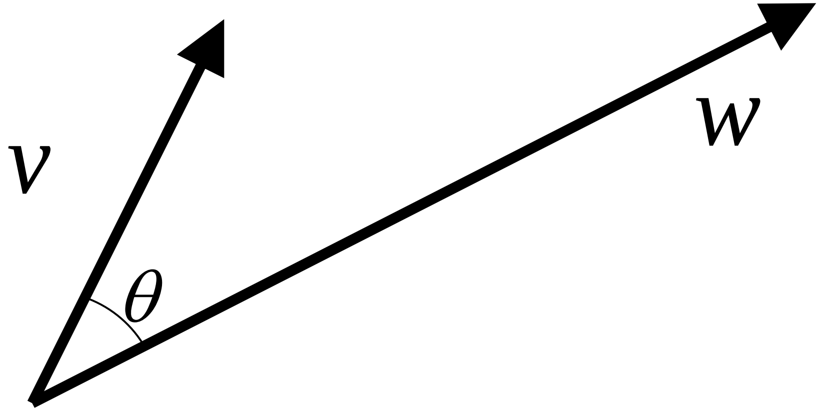
- Dot Product



$$v \cdot w = 0$$

Review – Vector Operations

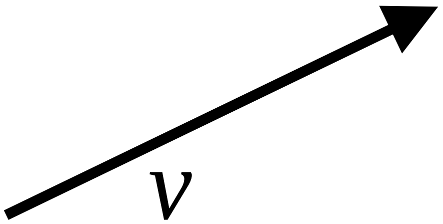
- Dot Product



$$v \cdot w = v_x w_x + v_y w_y$$

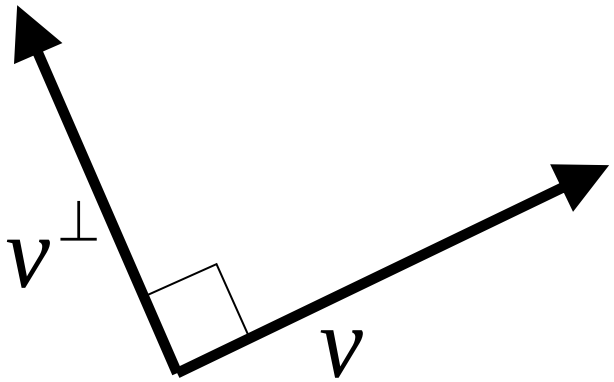
Review – Vector Operations

- 2D Cross Product



Review – Vector Operations

- 2D Cross Product



Review – Vector Operations

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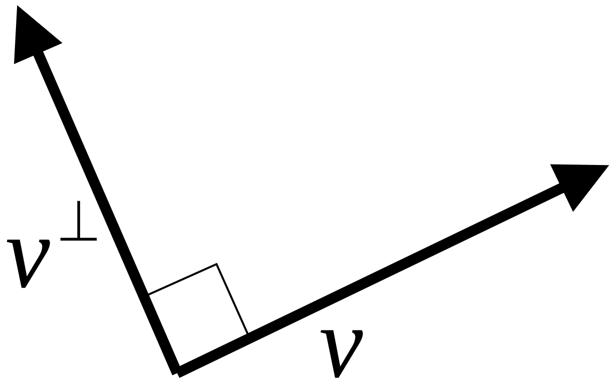
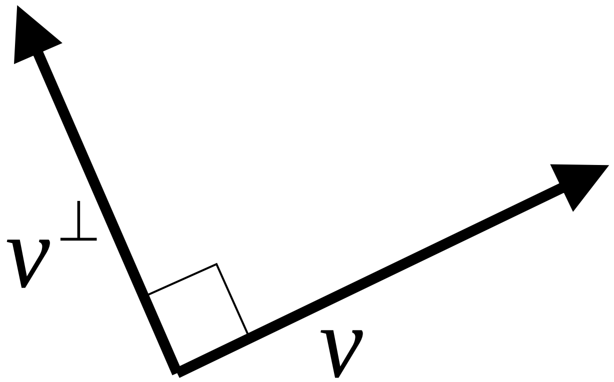


$$v^{\perp} = \begin{pmatrix} -v_y \\ v_x \end{pmatrix}$$

Review – Vector Operations

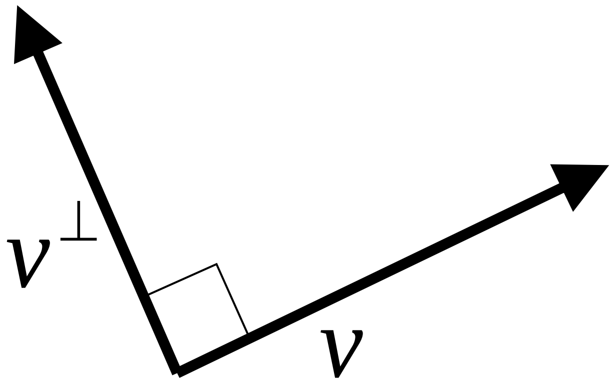
- 2D Cross Product



$$v \cdot v^\perp = v_x(-v_y) + v_y v_x = 0$$

Review – Vector Operations

- 2D Cross Product

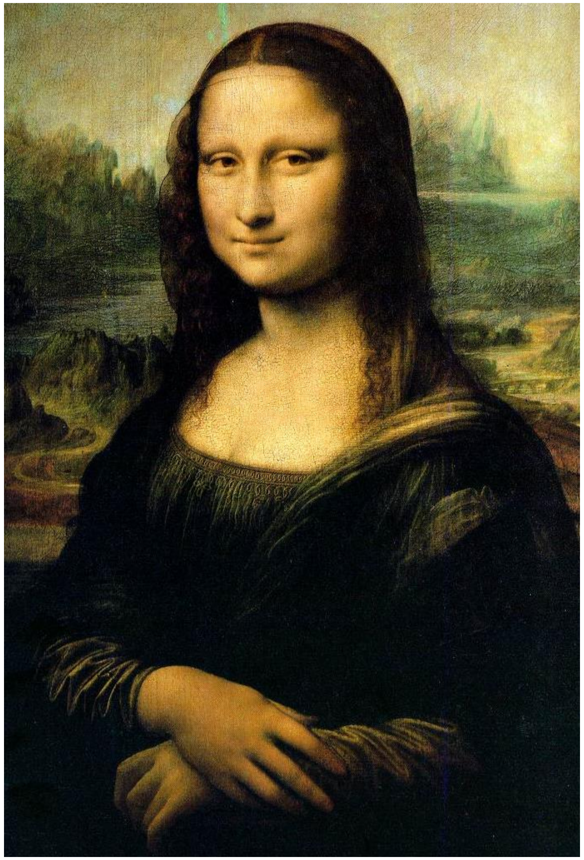


$$|v^{\perp}|^2 = (-v_y)^2 + v_x^{\,2} = |v|^2$$

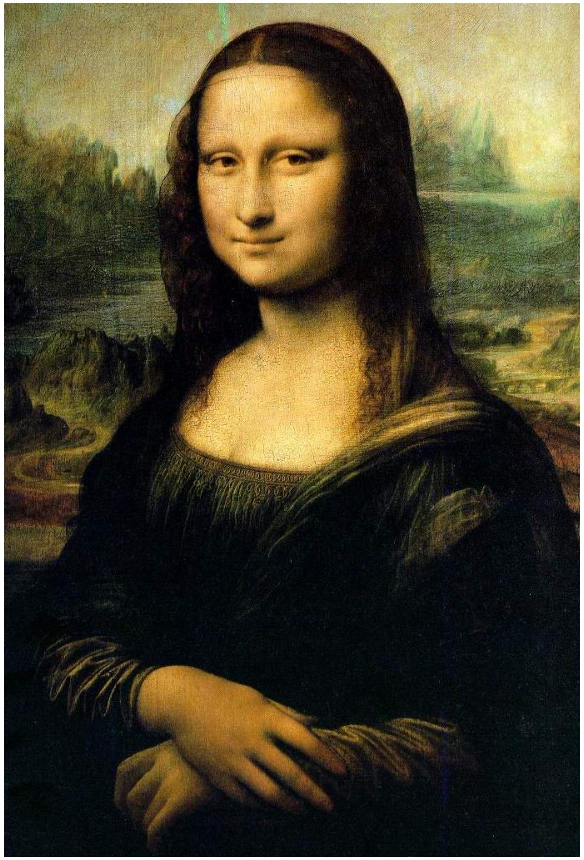
Types of Transformations

- Conformal
  - Preserves angles
  - Translation, Rotation, Uniform Scaling
- Affine
  - Represented by matrix multiplication
  - Translation, Rotation, Scaling, Shear

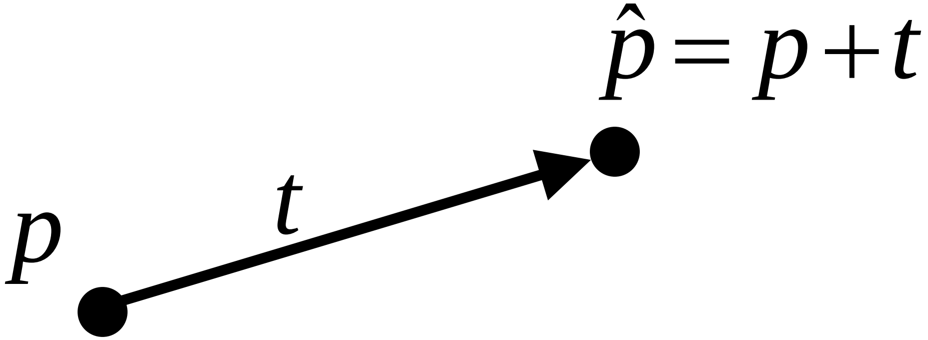
Translation



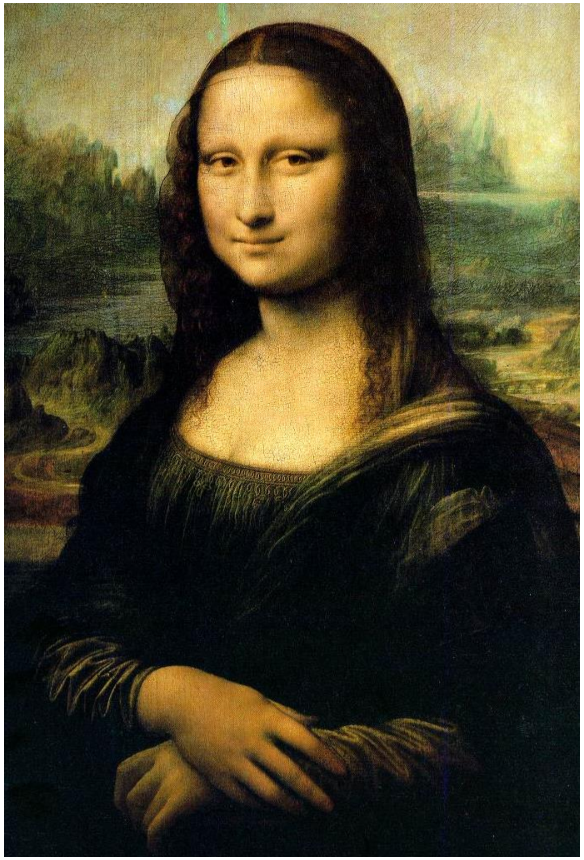
Translation



Translation



Uniform Scaling



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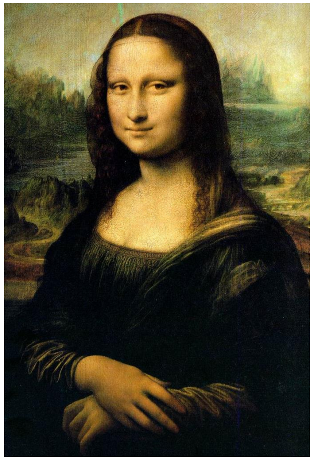

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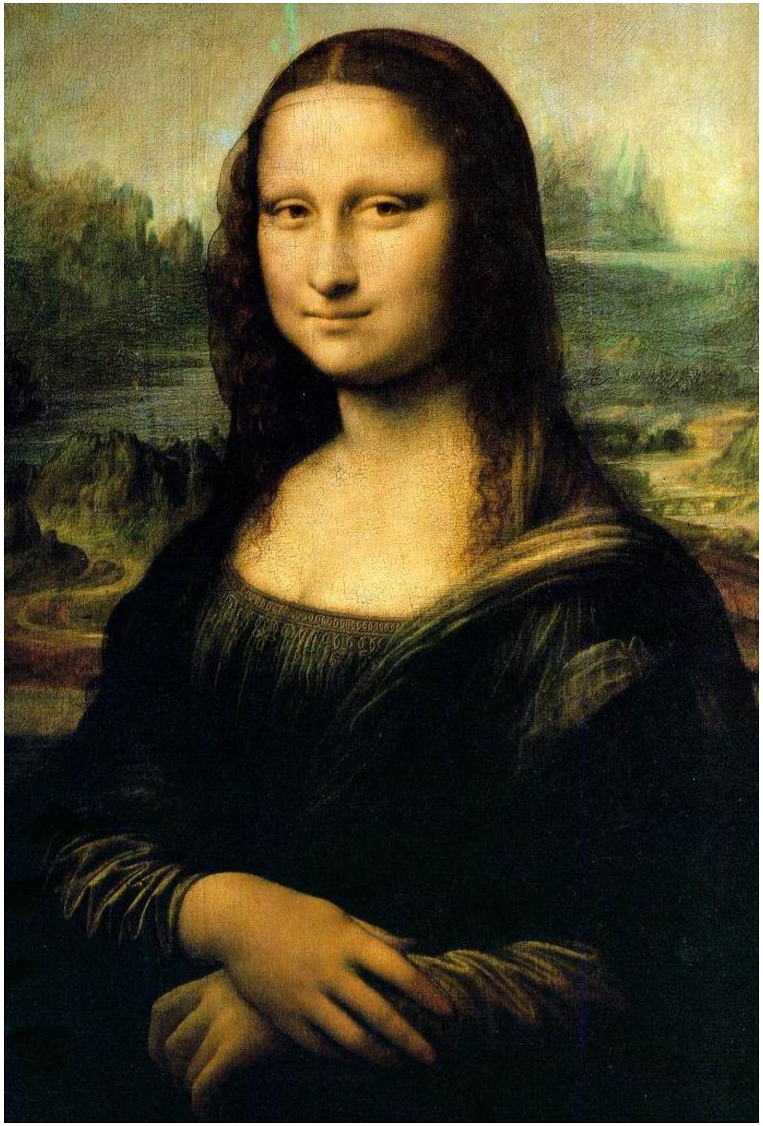

Uniform Scaling

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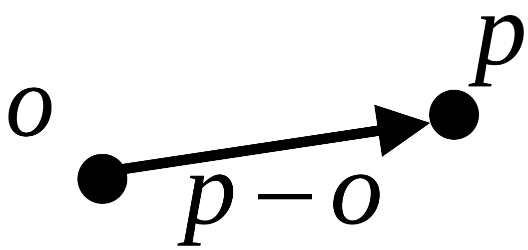

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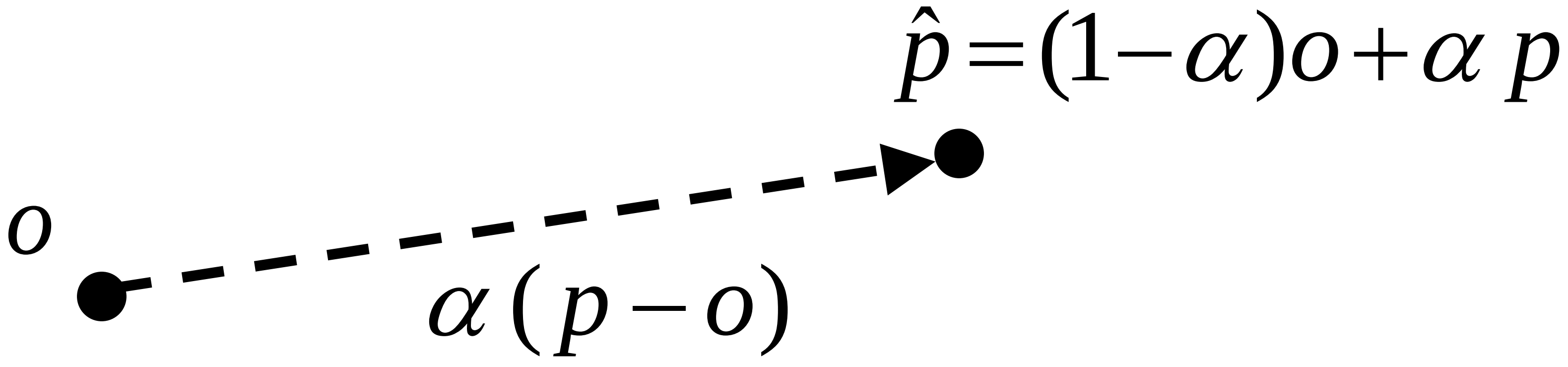
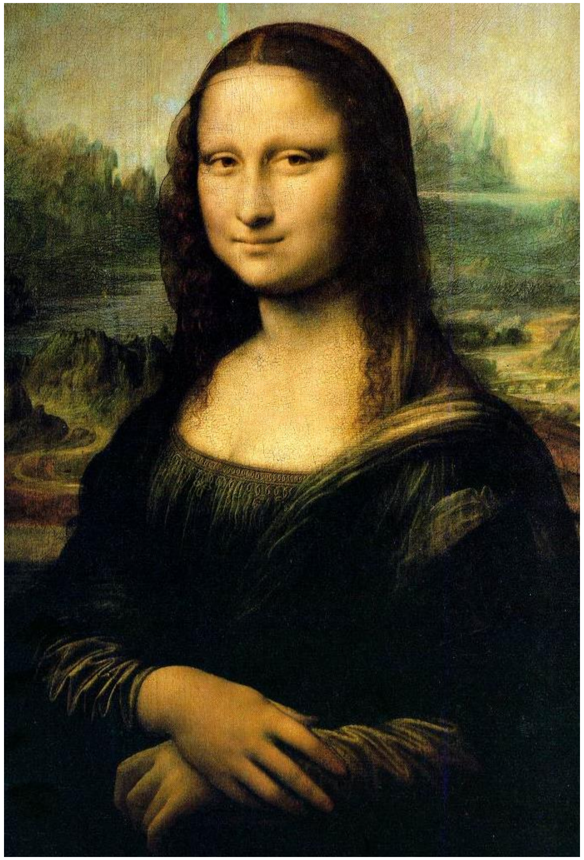


Non-Uniform Scaling



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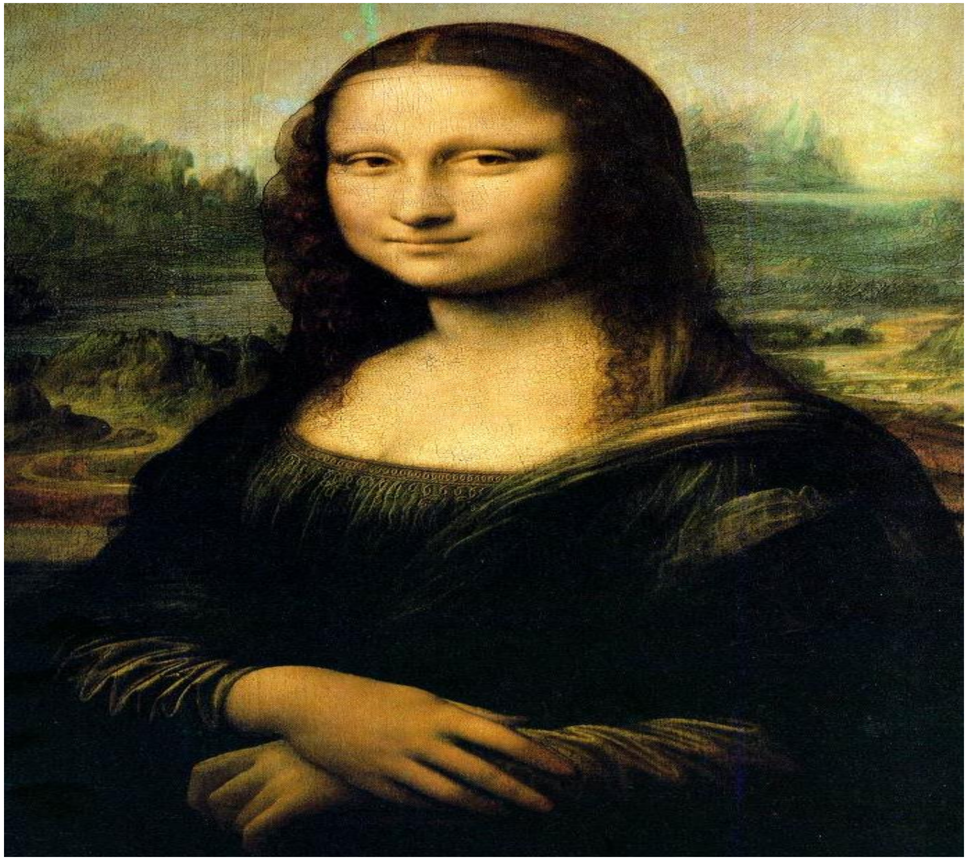

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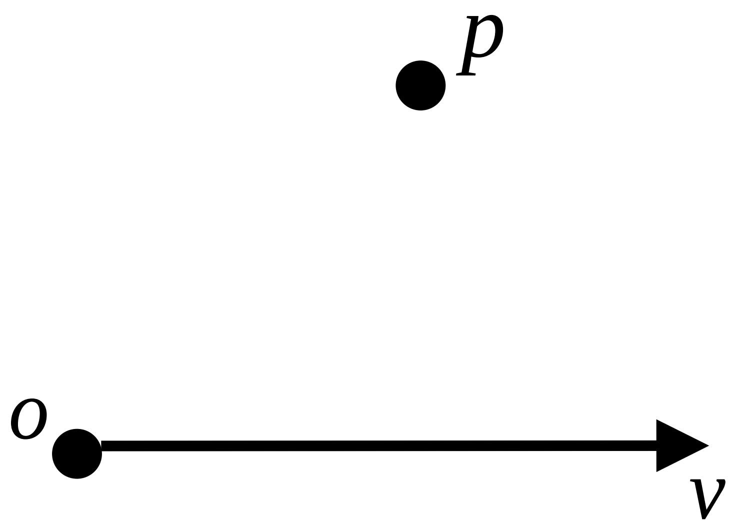

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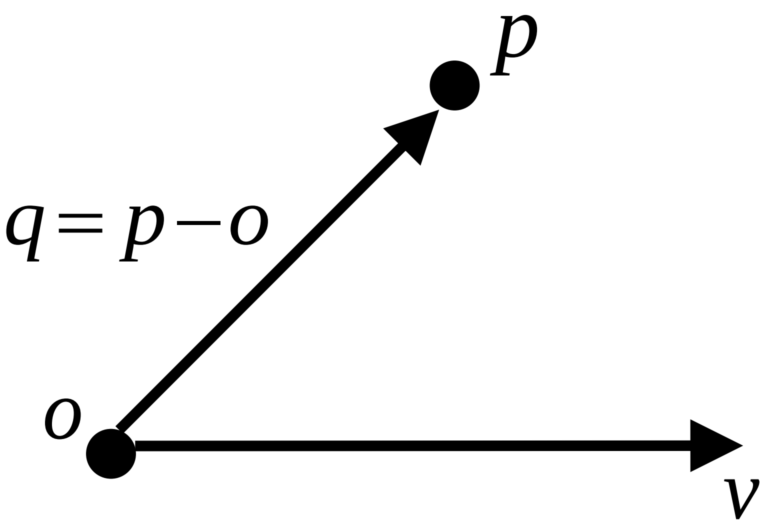

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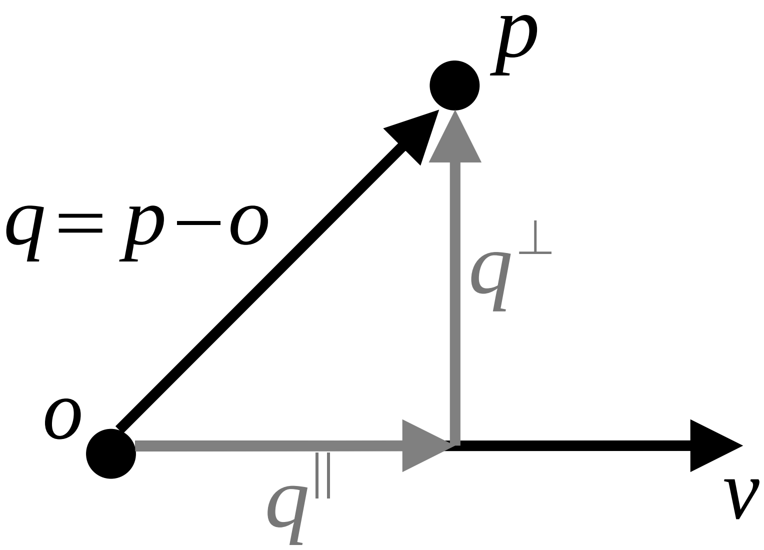

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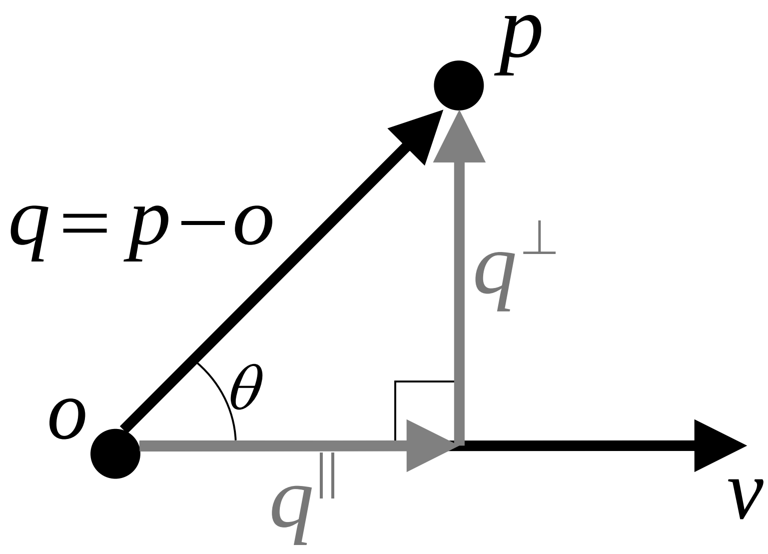

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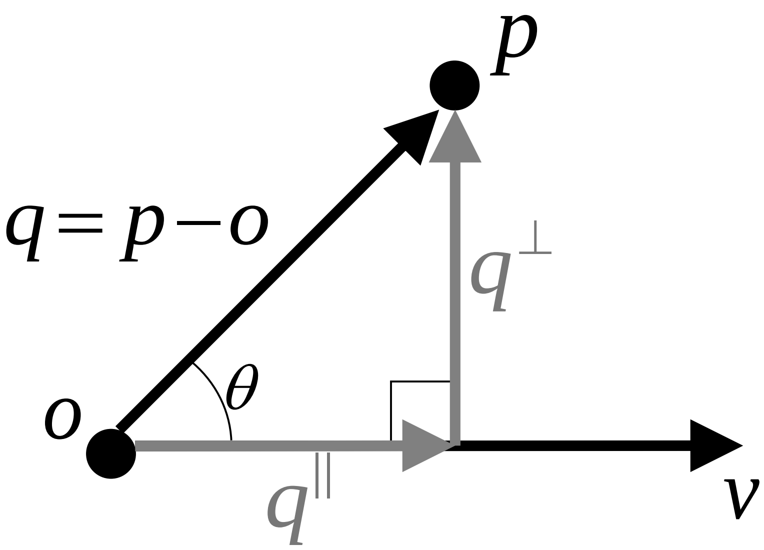
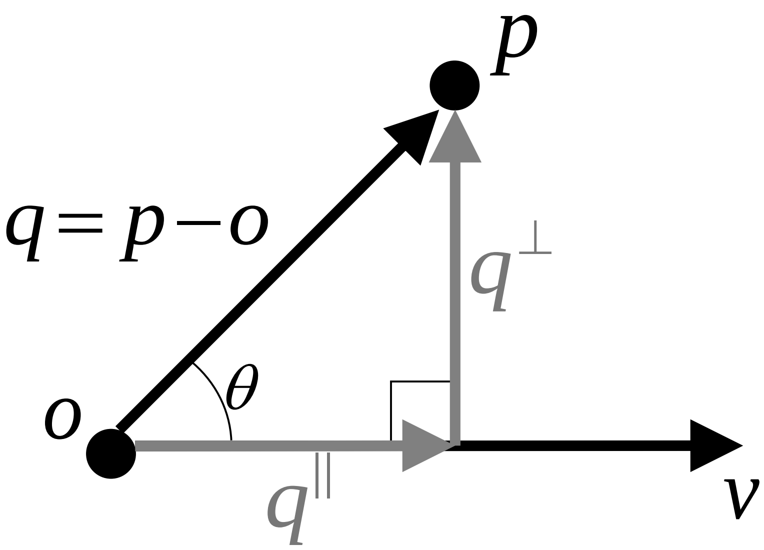


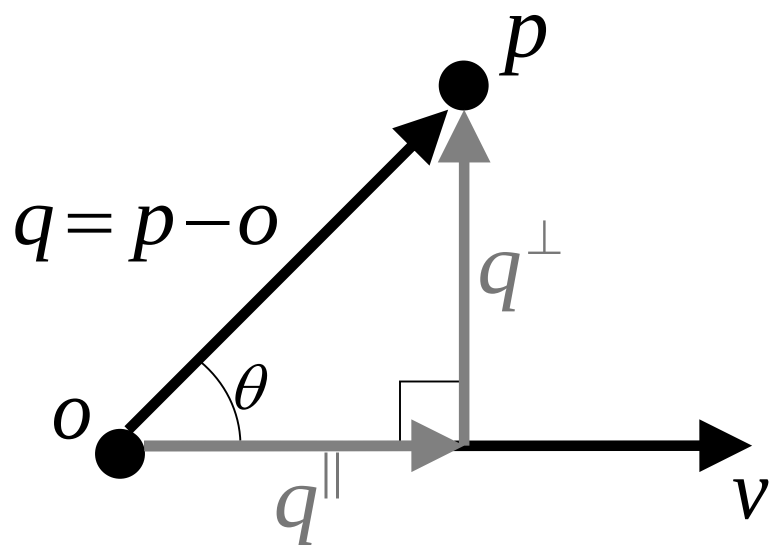
$$q^{\parallel} = v\,|q|\cos(\theta)$$

Non-Uniform Scaling



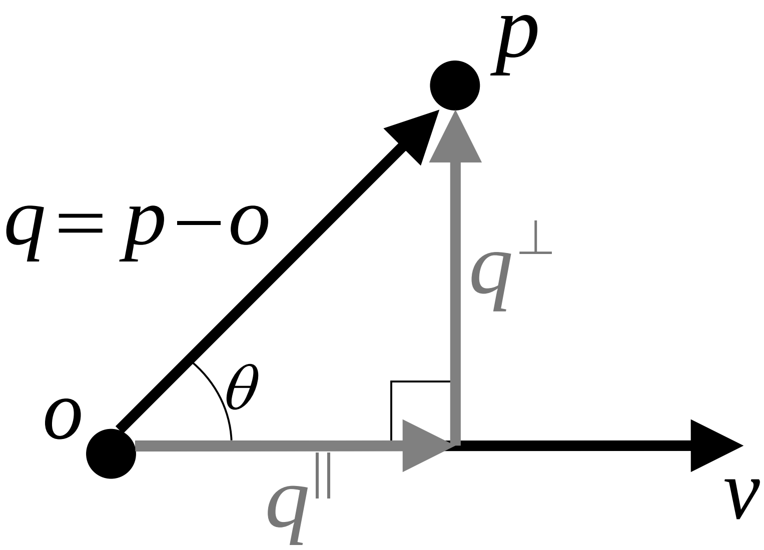
$$q^{\parallel} = (v \cdot q)v$$

Non-Uniform Scaling



$$q^{\parallel} = (v \cdot q)v$$
$$q^{\perp} = q - q^{\parallel}$$

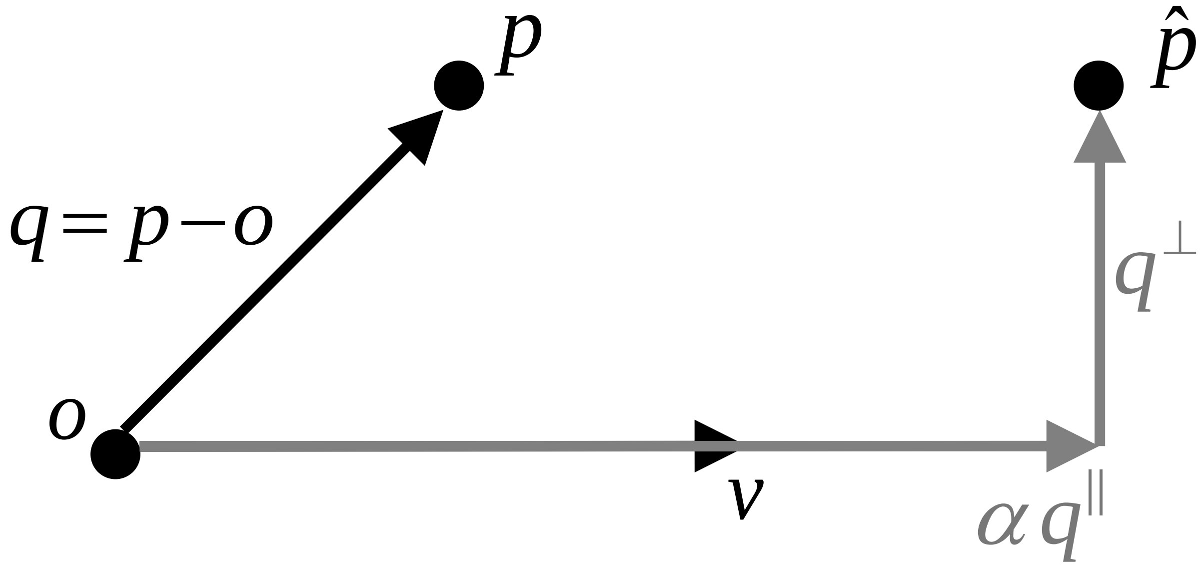
Non-Uniform Scaling



$$q^{\parallel} = (v \cdot q)v$$

$$q^{\perp} = q - (v \cdot q)v$$

Non-Uniform Scaling

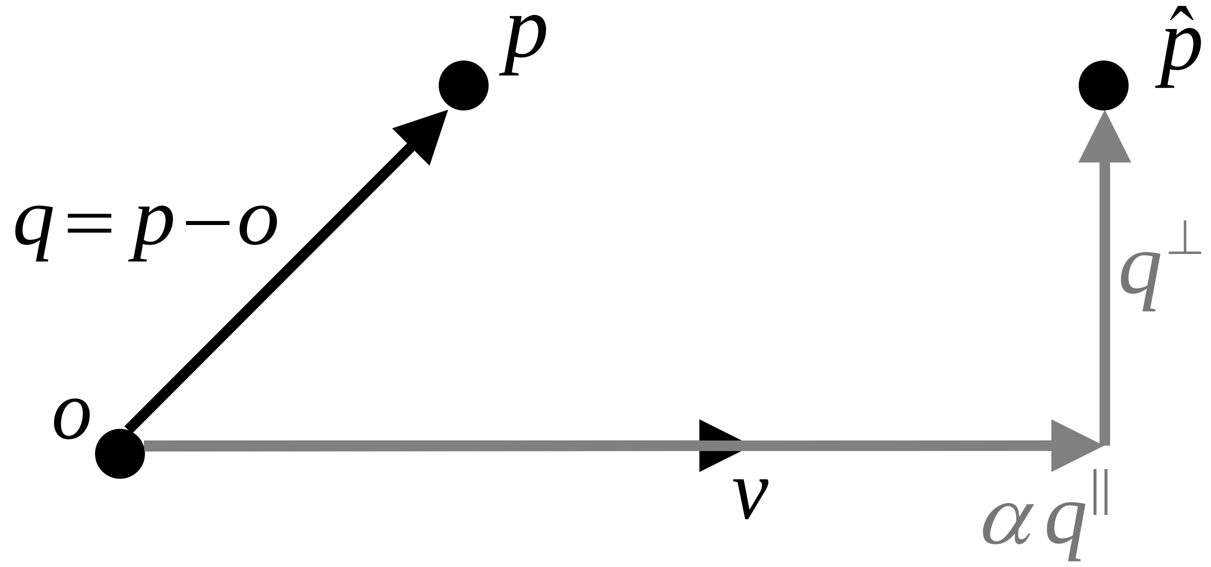


$$q^{\parallel} = (v \cdot q)v$$

$$q^{\perp} = q - (v \cdot q)v$$

$$\hat{p} = o + \alpha q^{\parallel} + q^{\perp}$$

Non-Uniform Scaling

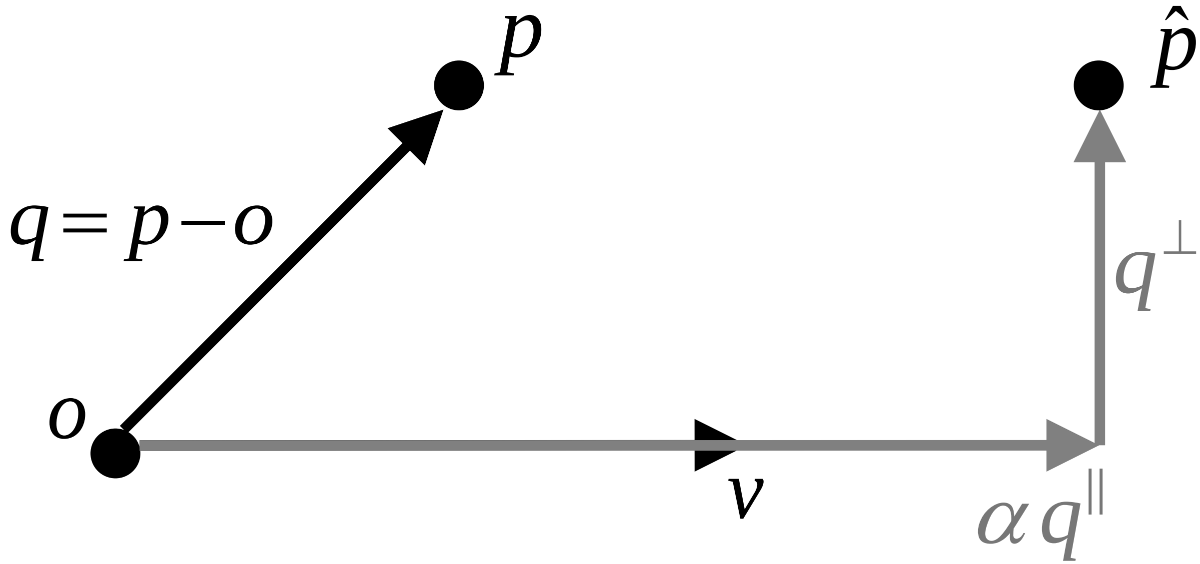


$$q^{\parallel} = (v \cdot q)v$$

$$q^{\perp} = q - (v \cdot q)v$$

$$\hat{p} = o + q + (\alpha - 1)(v \cdot q)v$$

Non-Uniform Scaling



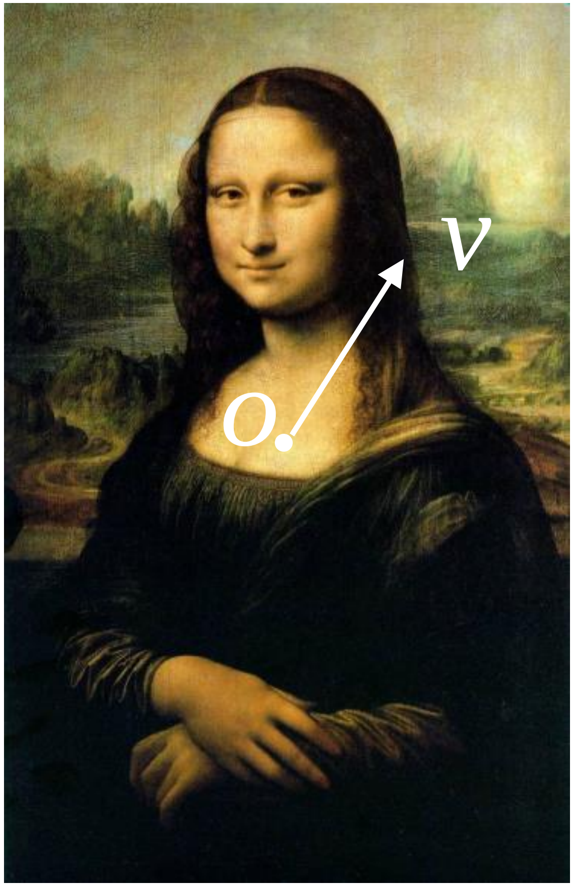
$$q^{\parallel} = (v \cdot q)v$$

$$q^{\perp} = q - (v \cdot q)v$$

$$\hat{p} = p + (\alpha - 1)(v \cdot (p - o))v$$

Non-Uniform Scaling

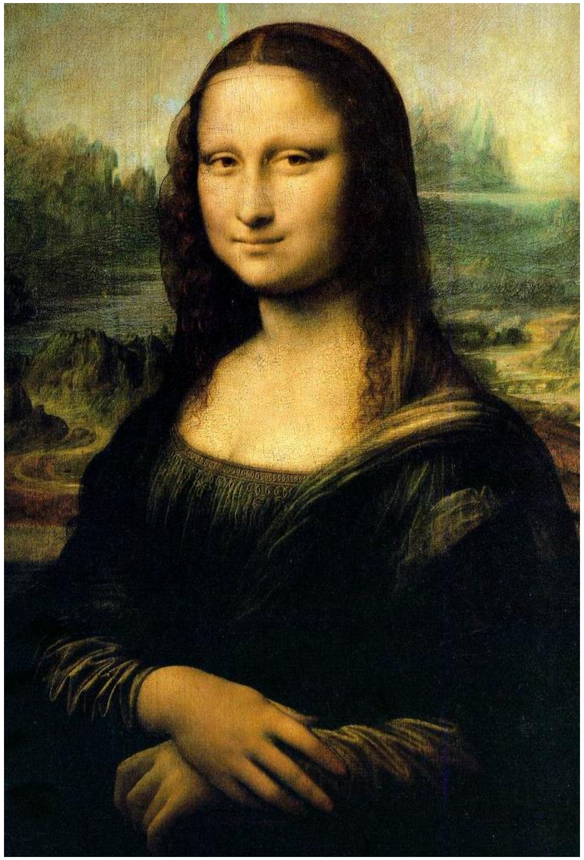
Non-Uniform Scaling



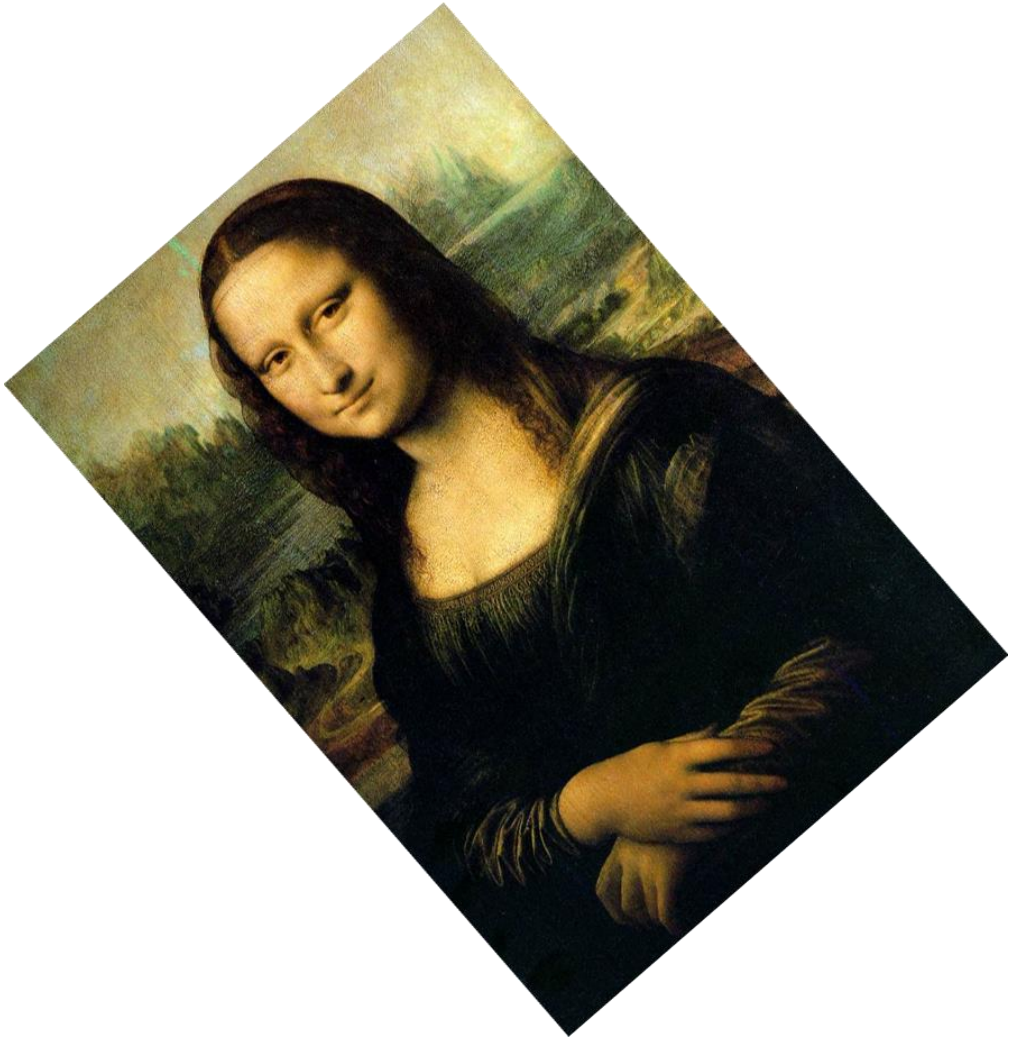
Non-Uniform Scaling

Not the same as scaling along x/y axis separately!!!

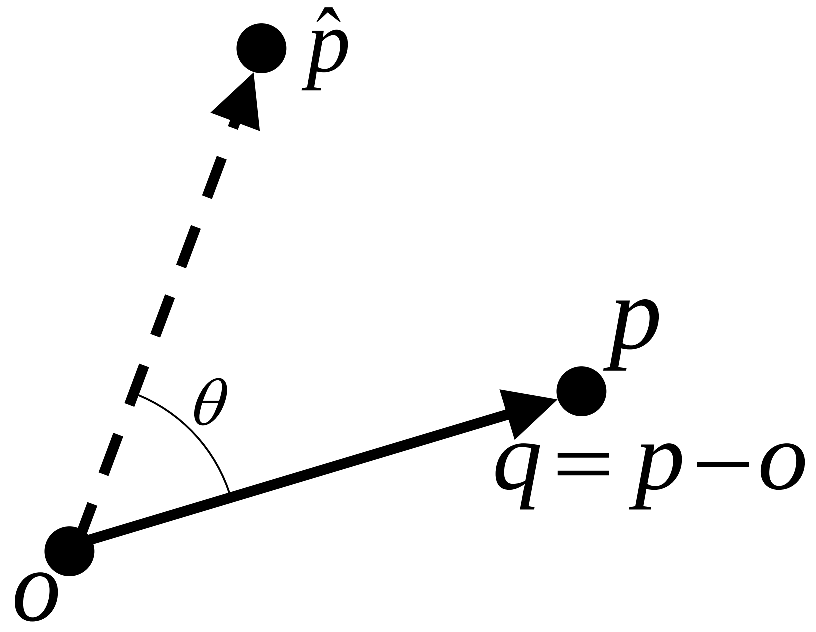
Rotation



Rotation



Rotation



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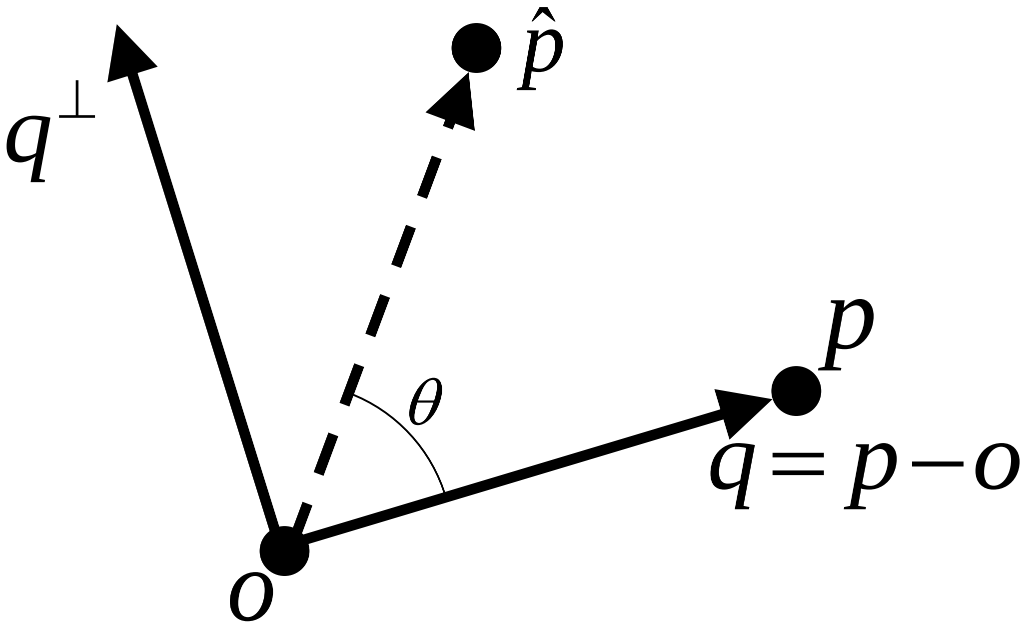
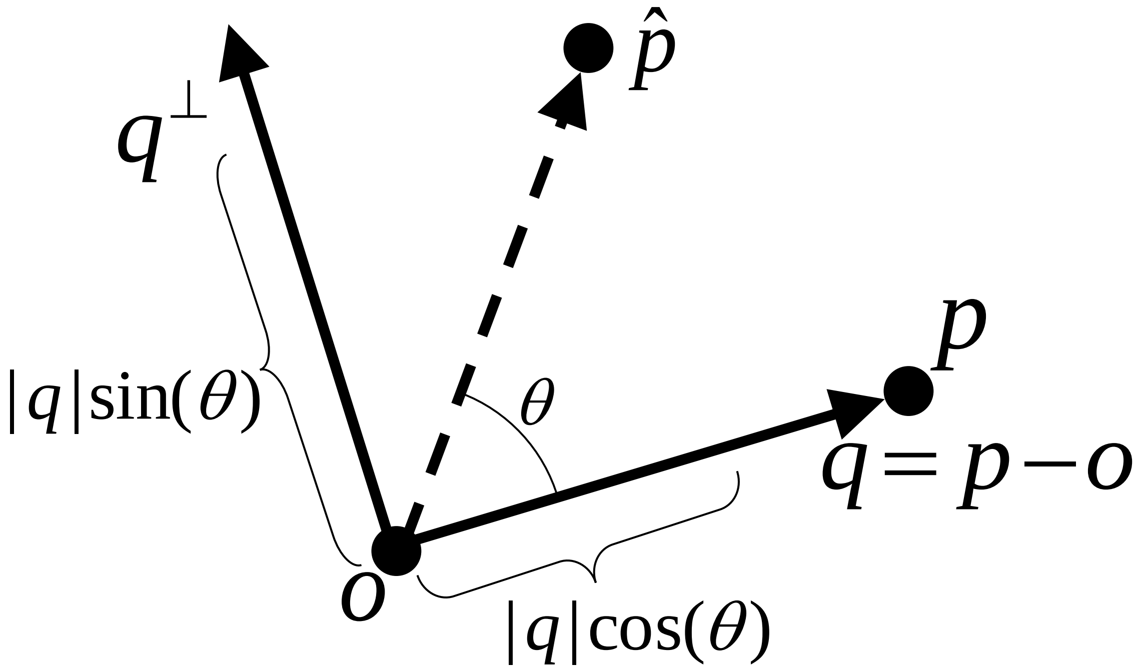
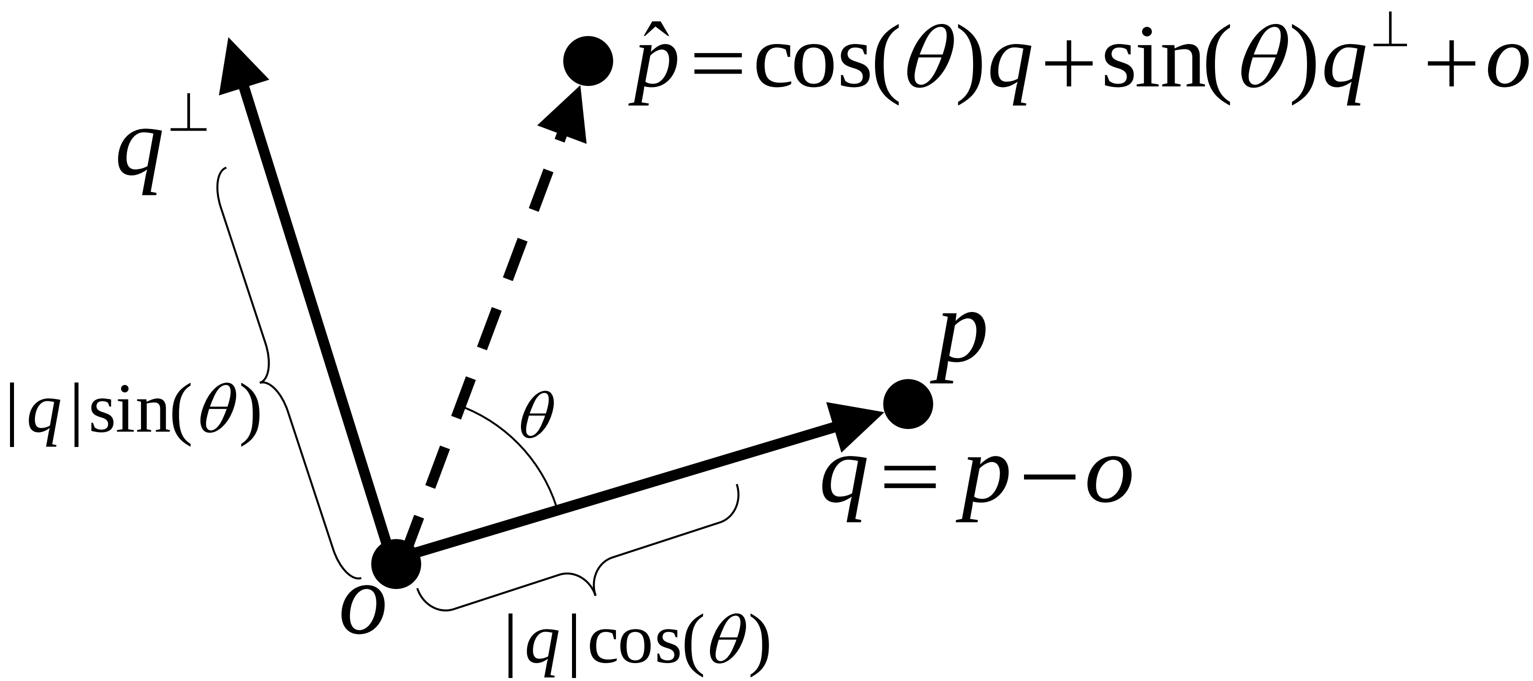


Rotation

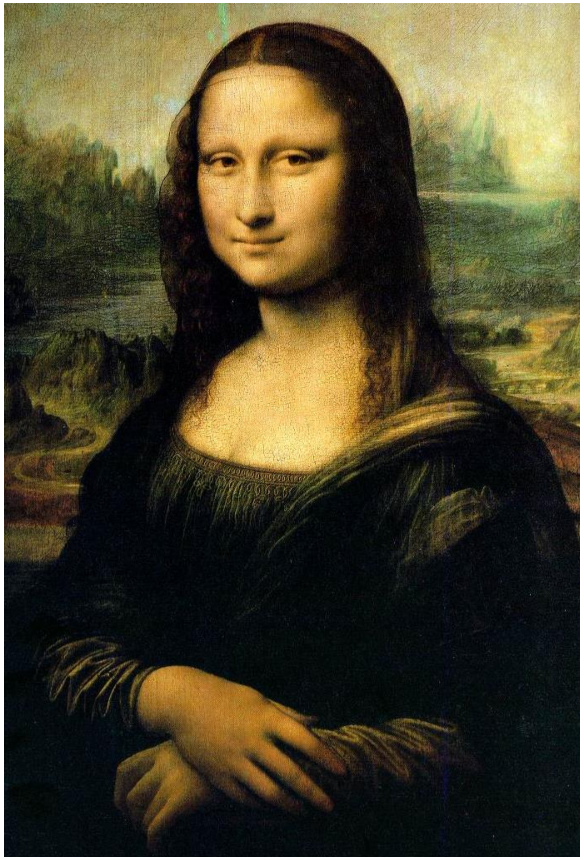


Rotation

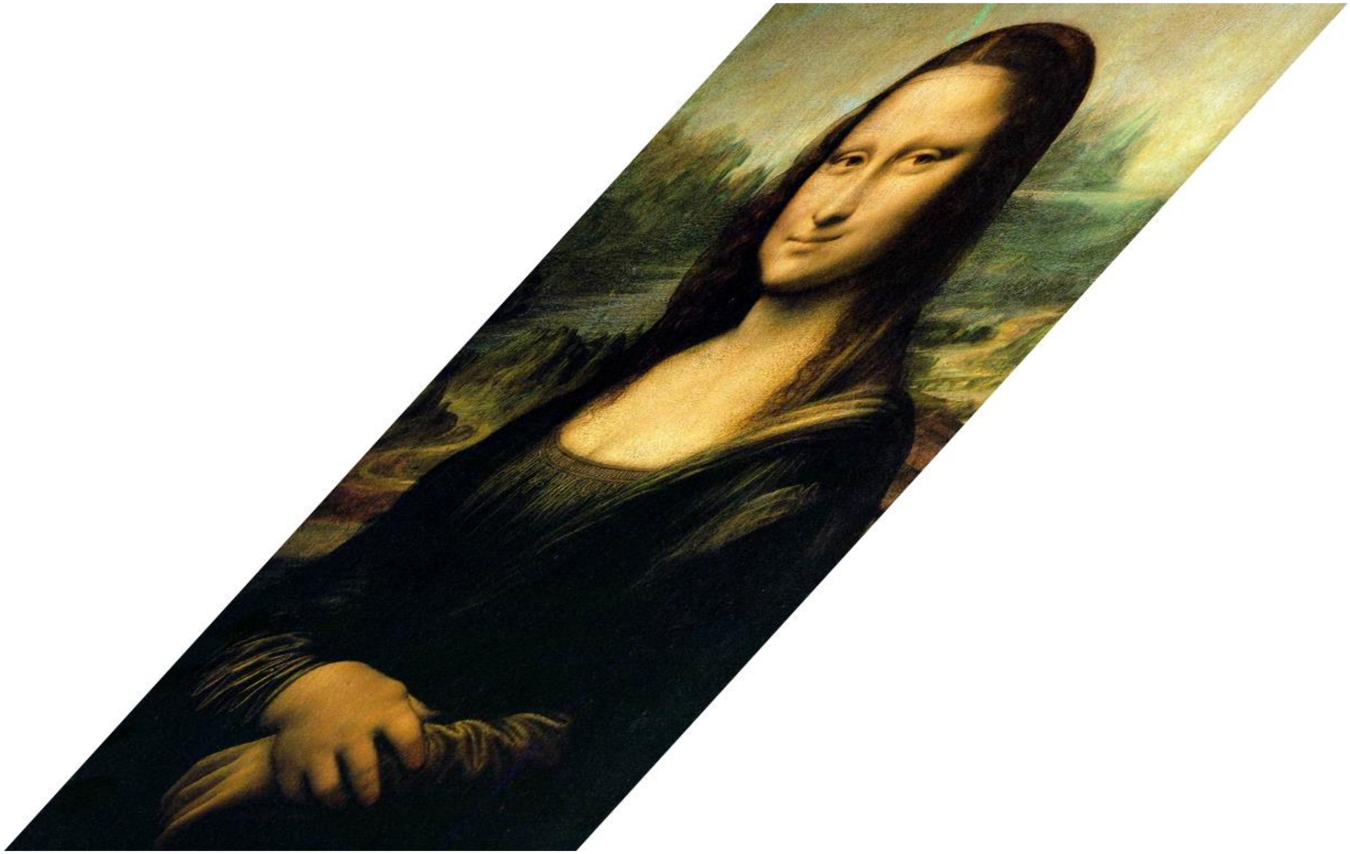


$\hat{p} = \cos(\theta)q + \sin(\theta)q^{\perp} + o$
$q^{\perp}$
p
$|q|\sin(\theta)$
θ
$q = p - o$
o
$|q|\cos(\theta)$

Shear



Shear



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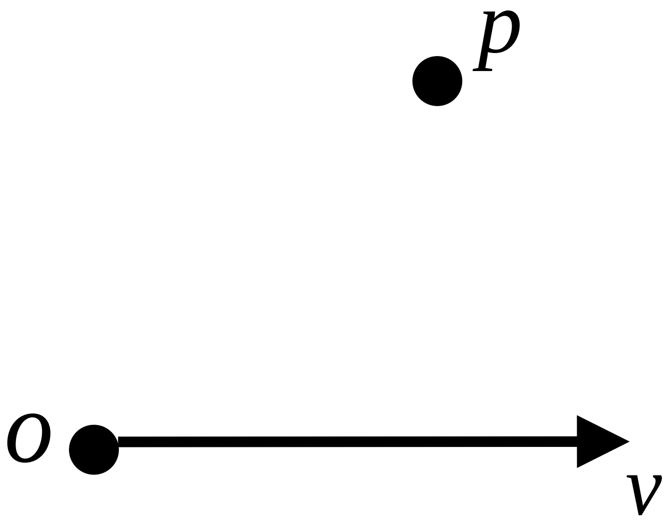

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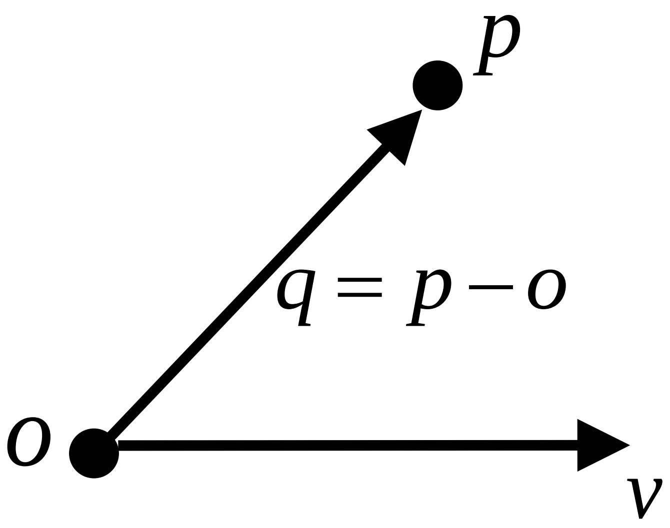

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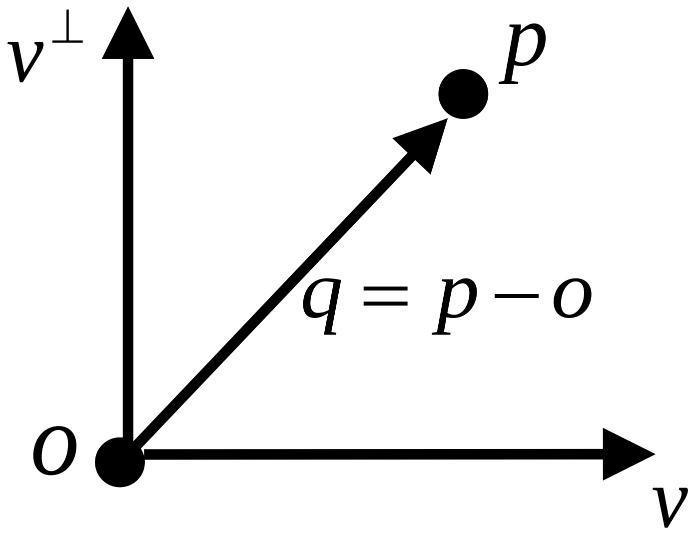

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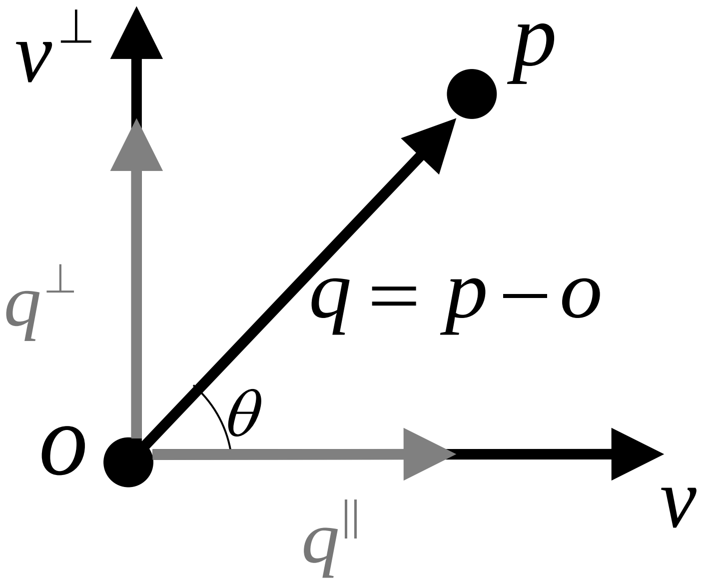

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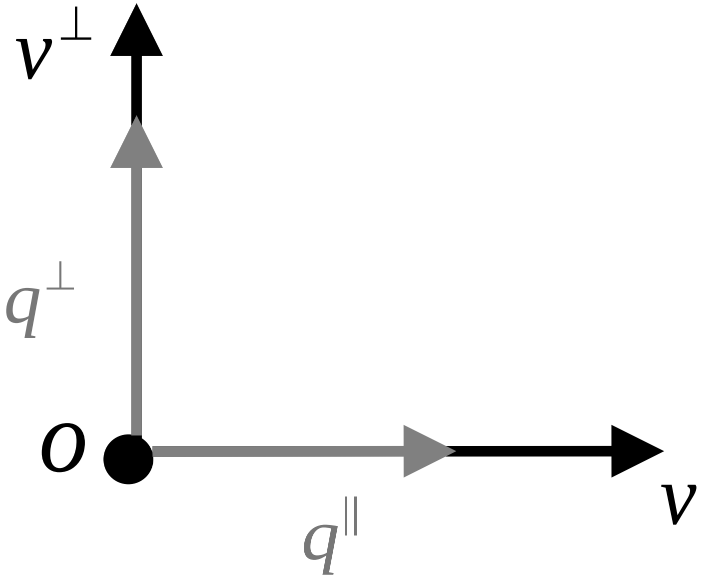

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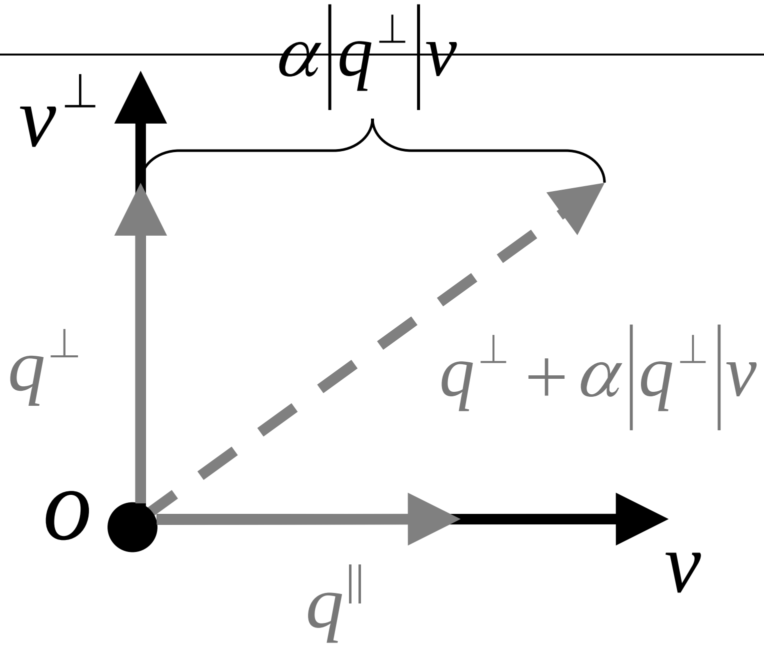

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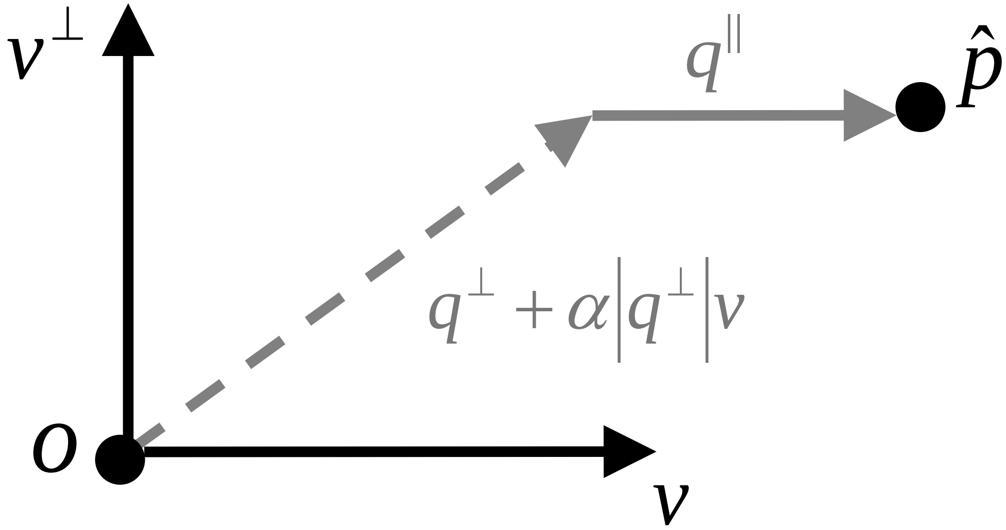

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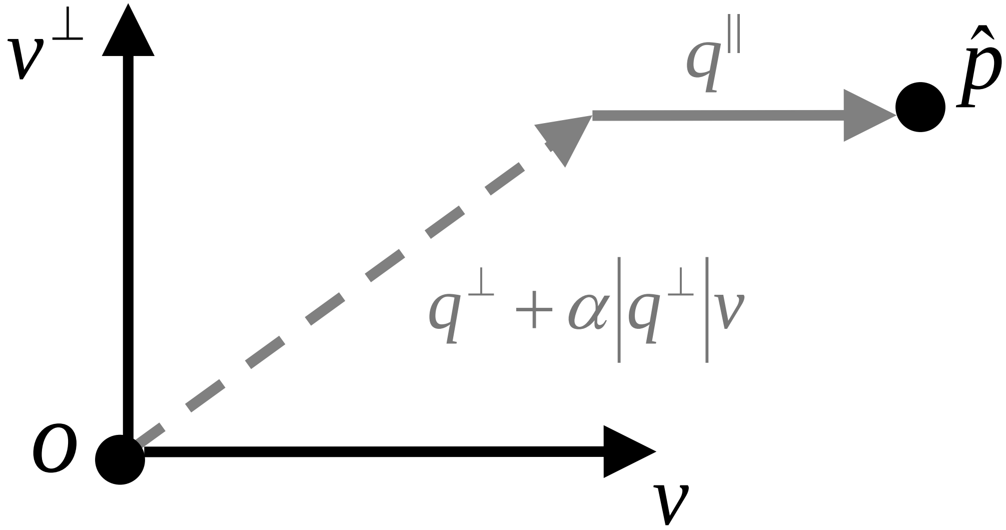
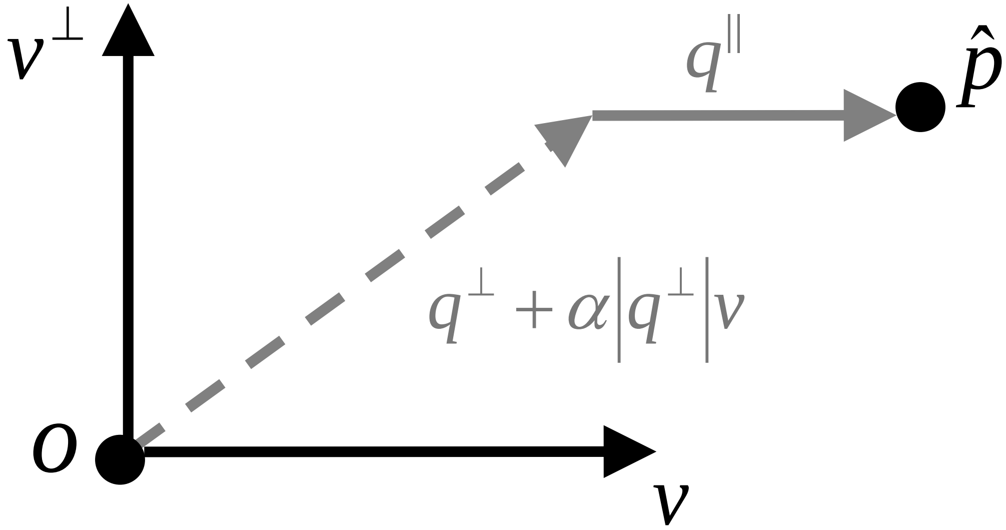


$$\hat{p} = o + q + \alpha|q^{\perp}|v$$

Shear



$$\hat{p} = p + \alpha(v^{\perp} \cdot q)v$$

Transformations as Matrices

- Compact representation of all affine transformations
- Allows multiple transformations to be represented as a single matrix
- Requires coordinates…. ☹

$$\hat{p} = Mp + t$$

Transformations as Matrices

- Compact representation of all affine transformations
- Allows multiple transformations to be represented as a single matrix
- Requires coordinates…. ☹

$$\begin{pmatrix} \hat{p}_x \\ \hat{p}_y \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

Transformations as Matrices

- Compact representation of all affine transformations
- Allows multiple transformations to be represented as a single matrix
- Requires coordinates…. ☹

$$\begin{pmatrix} \hat{p}_x \\ \hat{p}_y \\ 1 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & t_x \\ M_{21} & M_{22} & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix}$$

Transformations as Matrices

- Compact representation of all affine transformations
- Allows multiple transformations to be represented as a single matrix
- Requires coordinates…. ☹

$$\begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ 0 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & t_x \\ M_{21} & M_{22} & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ 0 \end{pmatrix}$$

Review – Vector Operations

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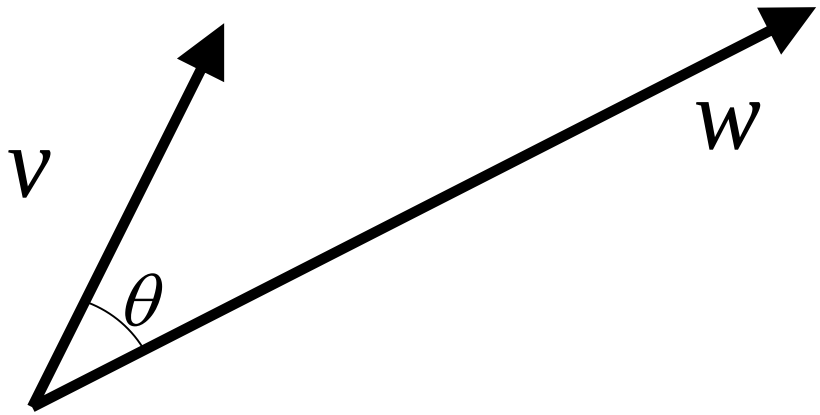
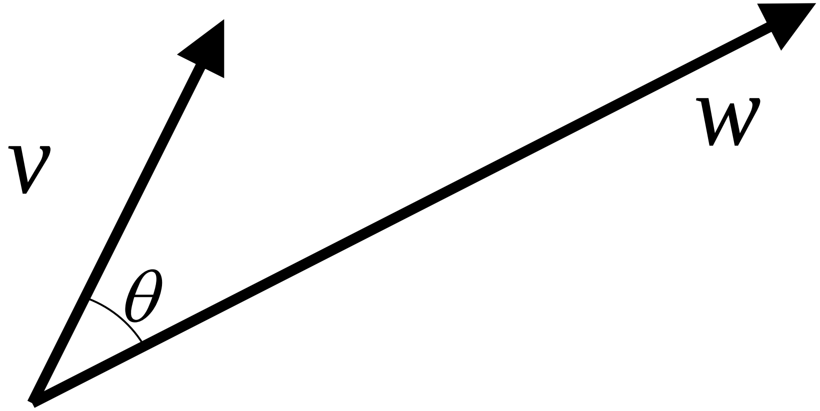


$$v \cdot w = v_x w_x + v_y w_y$$

Review – Vector Operations

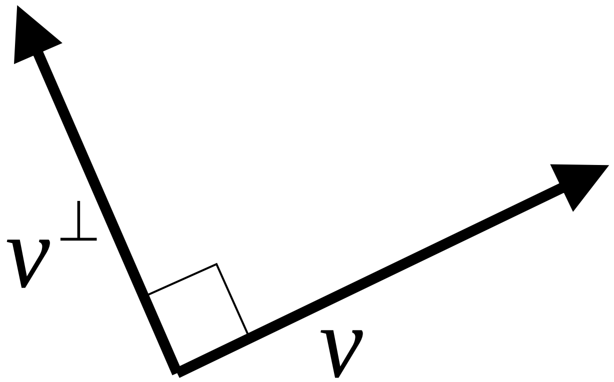
- Dot Product



$$v \cdot w = v^T w = \begin{pmatrix} v_x & v_y \end{pmatrix} \begin{pmatrix} w_x \\ w_y \end{pmatrix}$$

Review – Vector Operations

- 2D Cross Product



$$v^{\perp} = \begin{pmatrix} -v_y \\ v_x \end{pmatrix}$$

Review – Vector Operations

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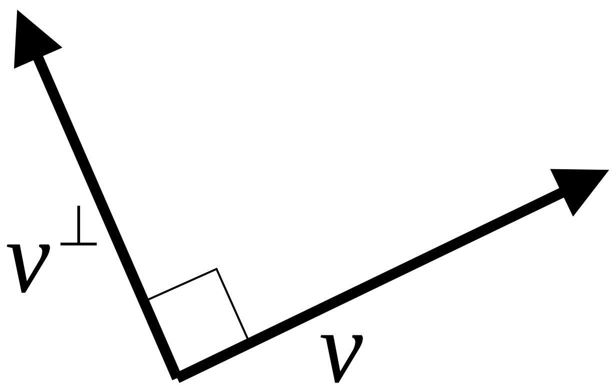
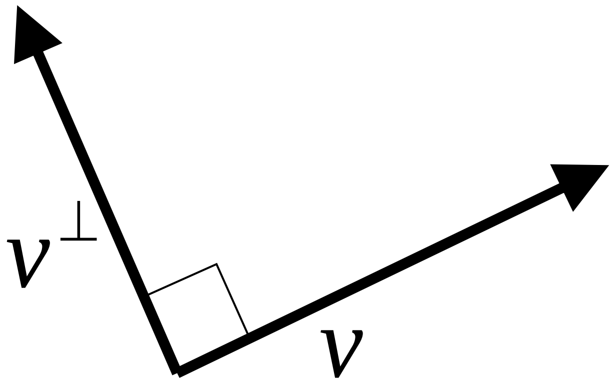


$$-^{\perp} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

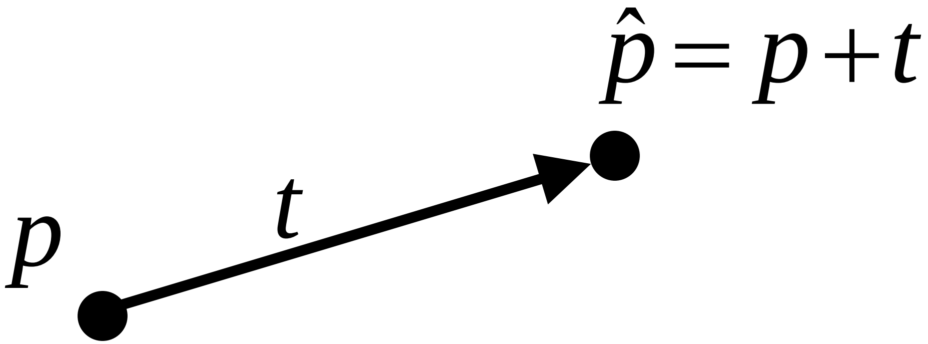
Review – Vector Operations

- 2D Cross Product



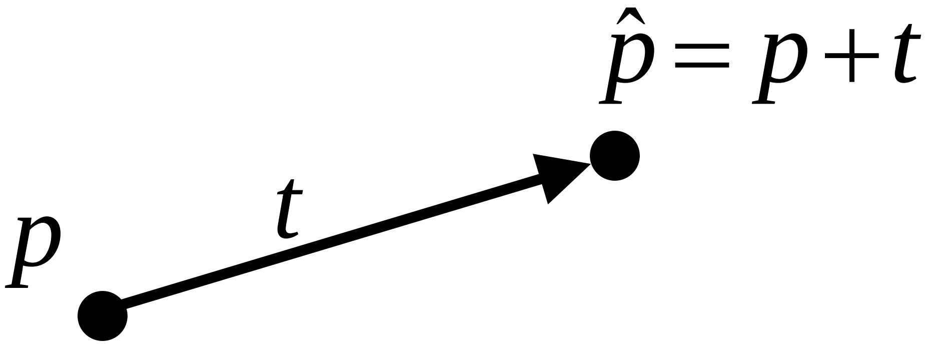
$$-^{\perp}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -v_y \\ v_x \end{pmatrix} = v^{\perp}$$

Translation



$$\begin{pmatrix} I & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ 1 \end{pmatrix} = \begin{pmatrix} p+t \\ 1 \end{pmatrix}$$

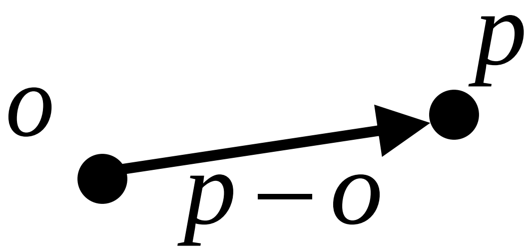
Translation



$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} = \begin{pmatrix} p_x + t_x \\ p_y + t_y \\ 1 \end{pmatrix}$$

Uniform Scaling

Uniform Scaling



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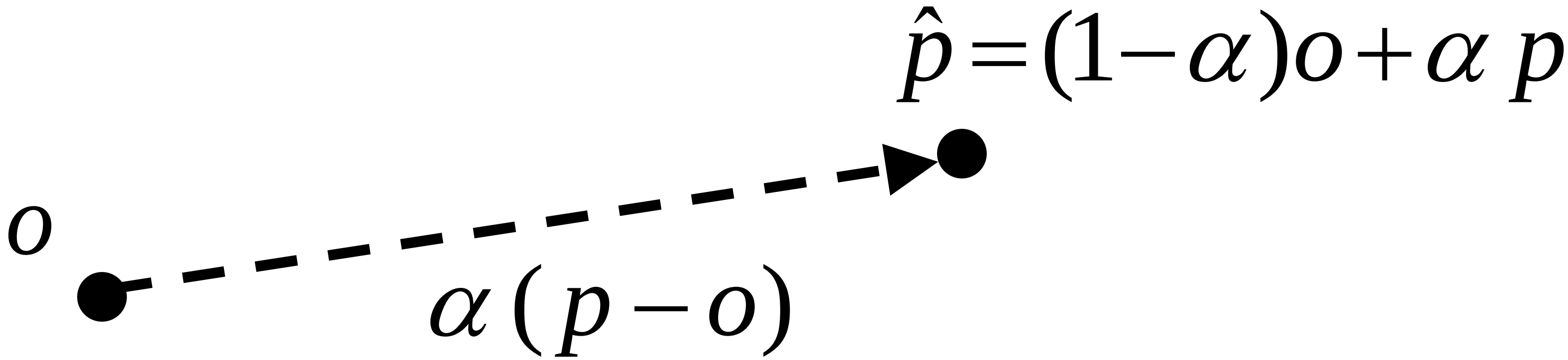
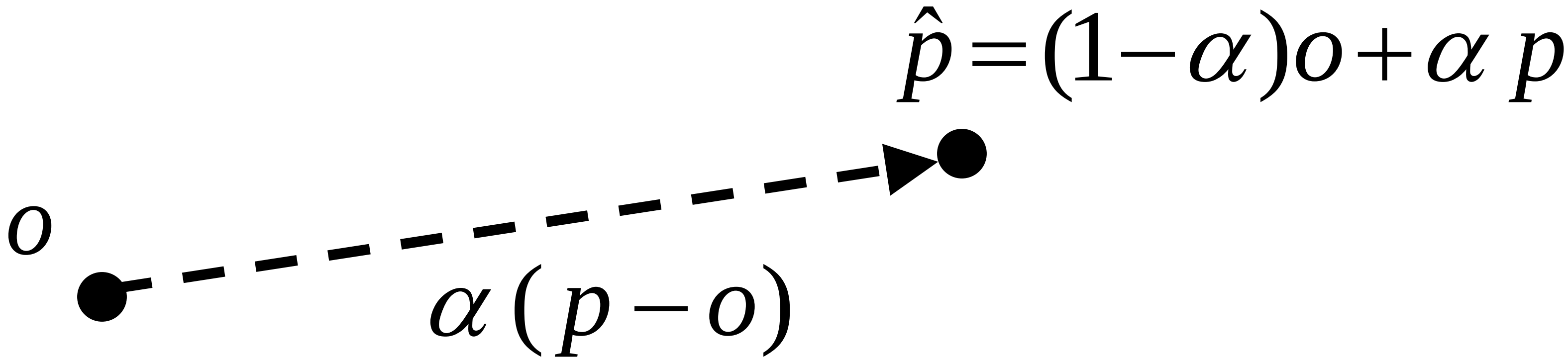


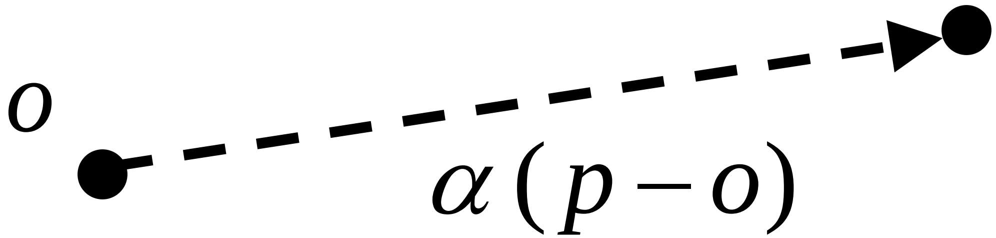
Uniform Scaling



$$\begin{pmatrix} \alpha I & (1-\alpha)o \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha p + (1-\alpha)o \\ 1 \end{pmatrix}$$

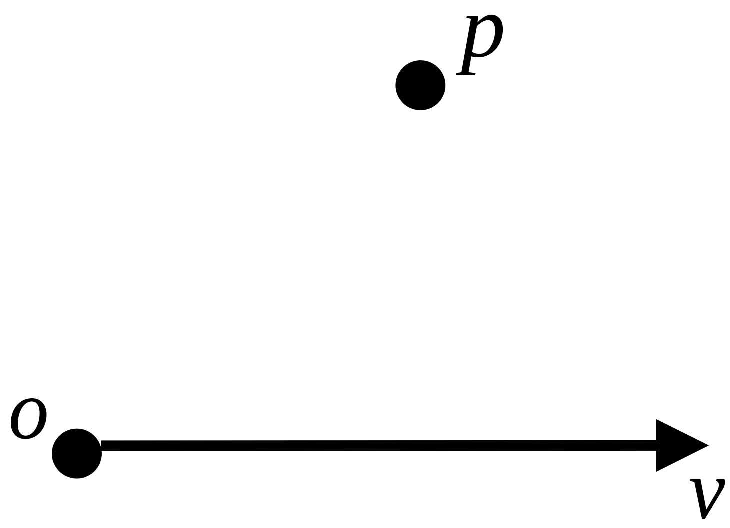
Uniform Scaling

$$\hat{p} = (1-\alpha)o + \alpha\, p$$

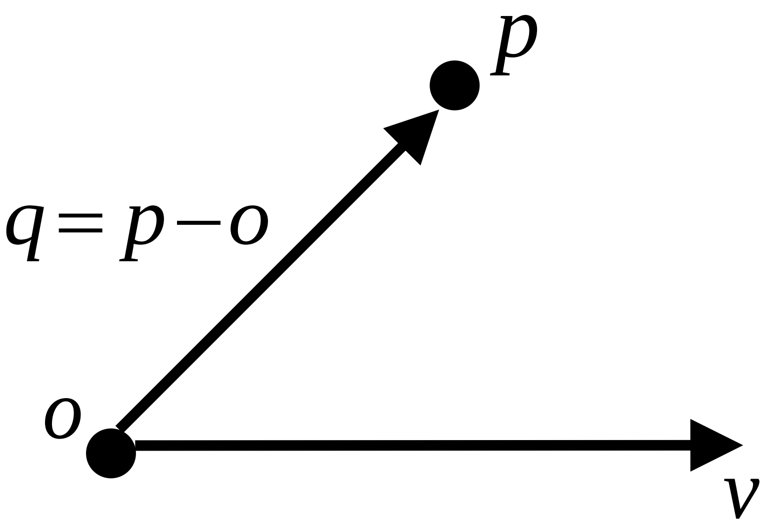


$$\begin{pmatrix} \alpha & 0 & (1-\alpha)o_x \\ 0 & \alpha & (1-\alpha)o_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha\, p_x + (1-\alpha)o_x \\ \alpha\, p_y + (1-\alpha)o_y \\ 1 \end{pmatrix}$$

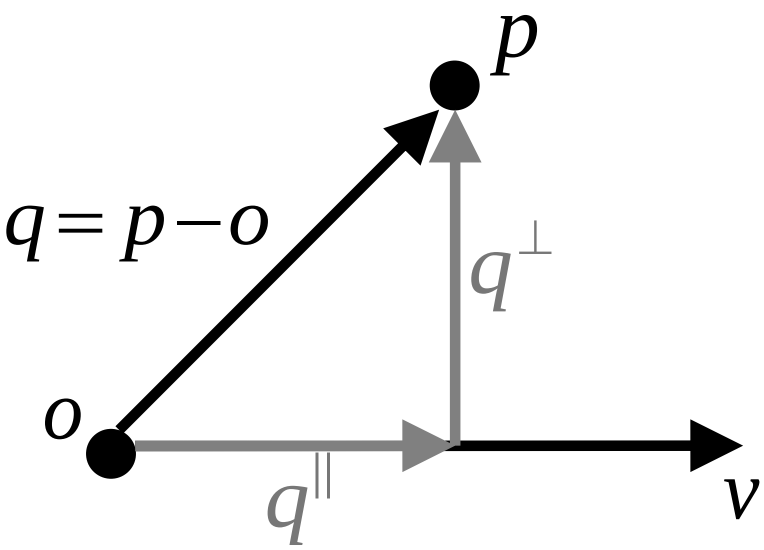
Non-Uniform Scaling



Non-Uniform Scaling

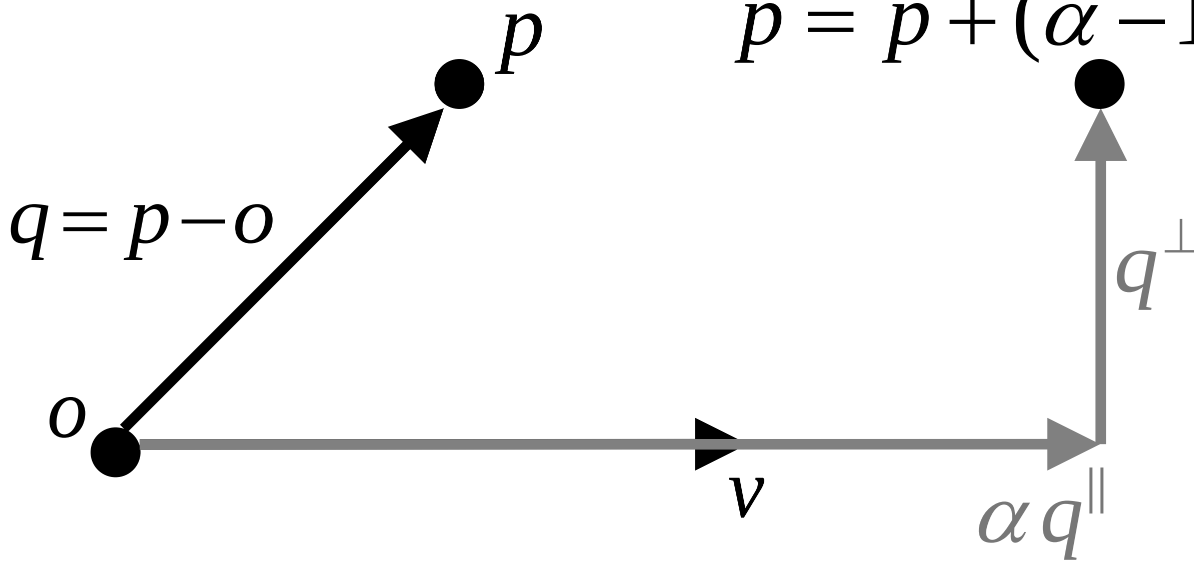


Non-Uniform Scaling



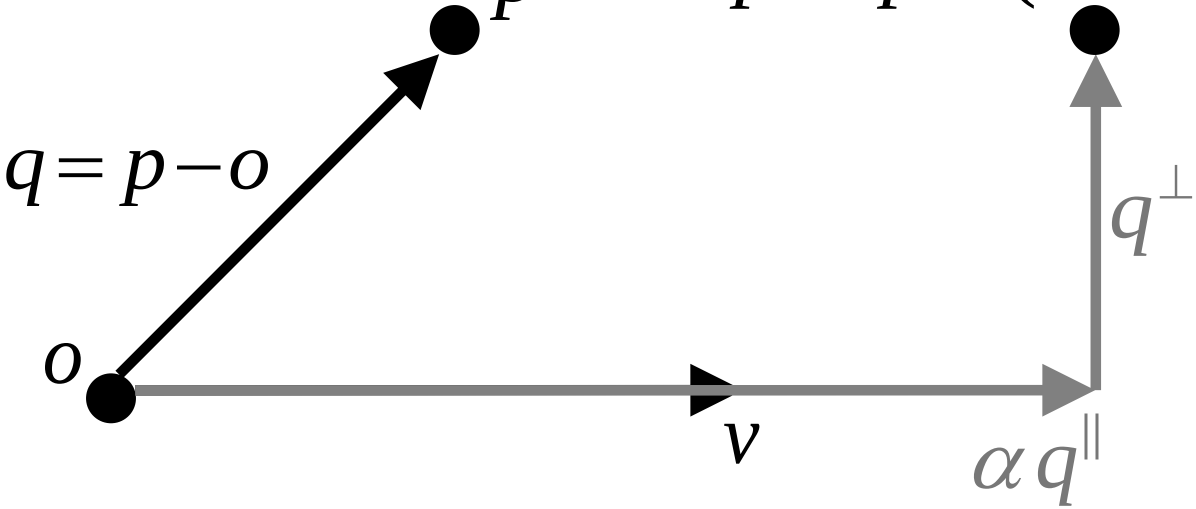
Non-Uniform Scaling

$$\hat{p} = p + (\alpha - 1)(v \cdot (p - o))v$$



Non-Uniform Scaling

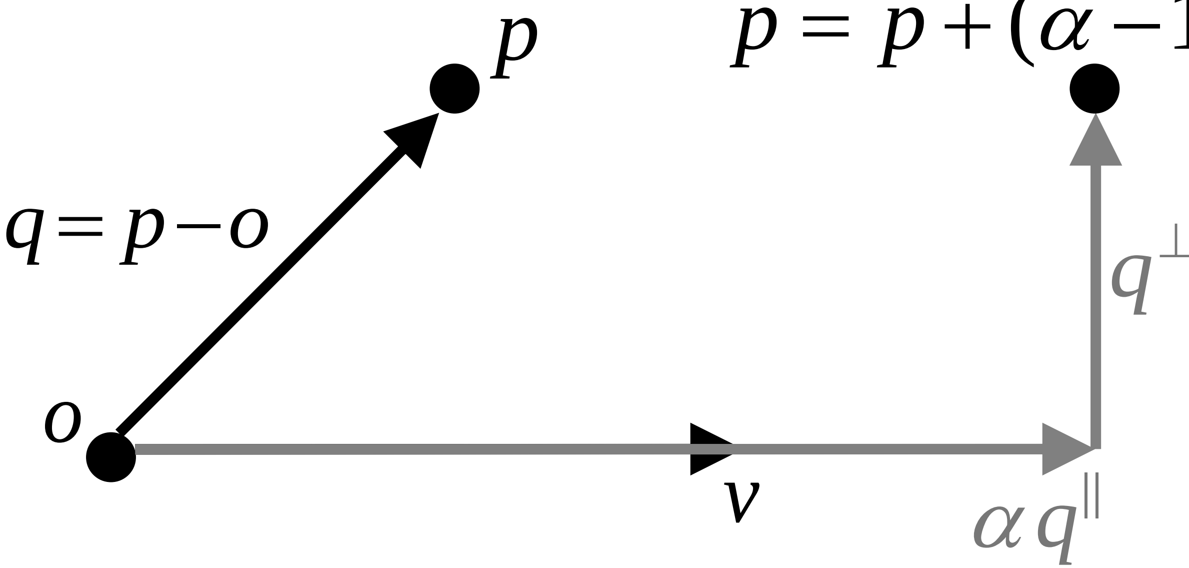
$$\hat{p} = p + (\alpha - 1)(v \cdot (p - o))v$$



$$\begin{pmatrix} I + (\alpha - 1)vv^T & (1-\alpha)vv^T o \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ 1 \end{pmatrix} = \begin{pmatrix} \hat{p} \\ 1 \end{pmatrix}$$

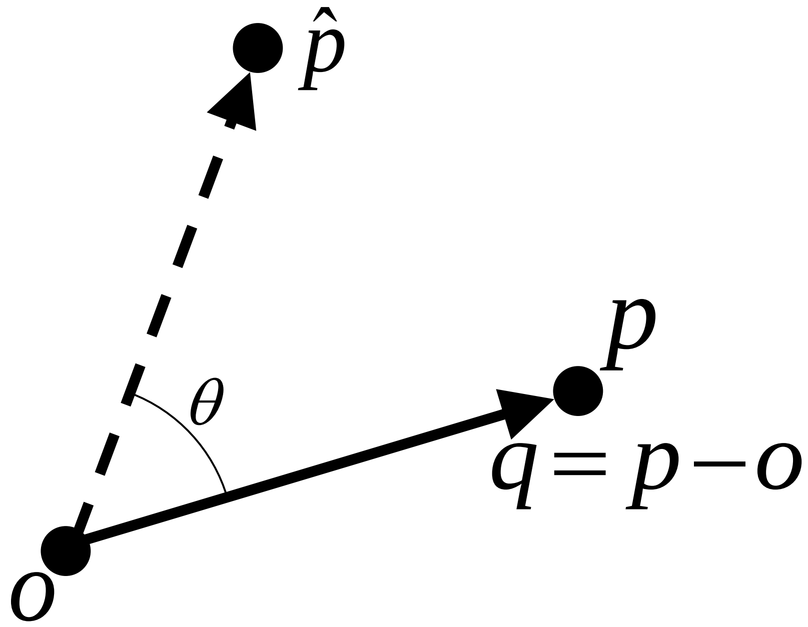
Non-Uniform Scaling

$$\hat{p} = p + (\alpha - 1)(v \cdot (p - o))v$$

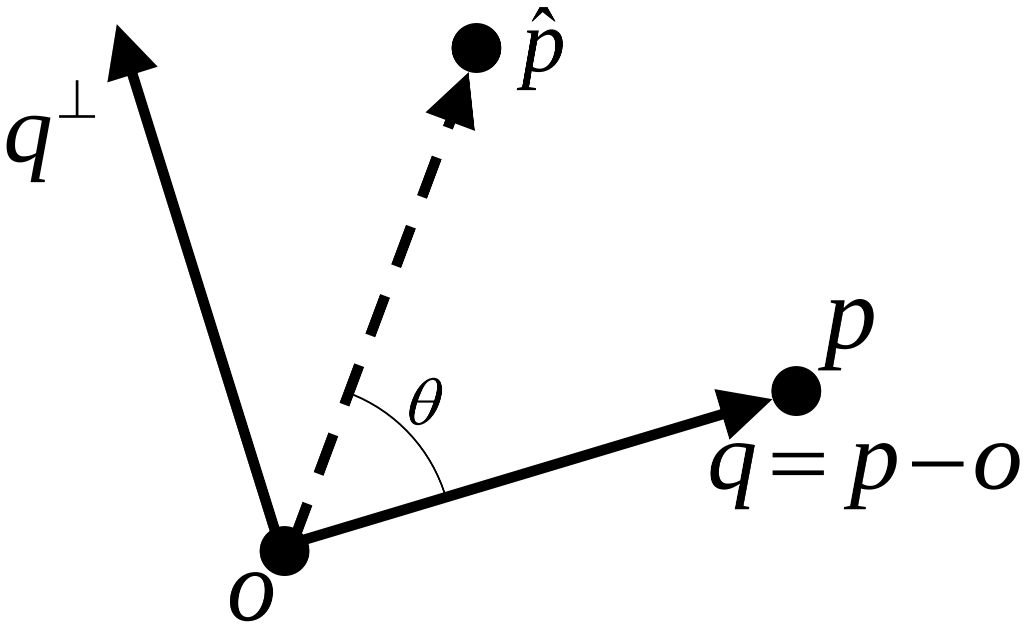


$$\begin{pmatrix} 1+(\alpha-1)v_x^{\ 2} & (\alpha-1)v_x v_y & v_x(o_x v_x + o_y v_y)(1-\alpha) \\ (\alpha-1)v_x v_y & 1+(\alpha-1)v_y^{\ 2} & v_y(o_x v_x + o_y v_y)(1-\alpha) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} = \begin{pmatrix} \hat{p}_x \\ \hat{p}_y \\ 1 \end{pmatrix}$$

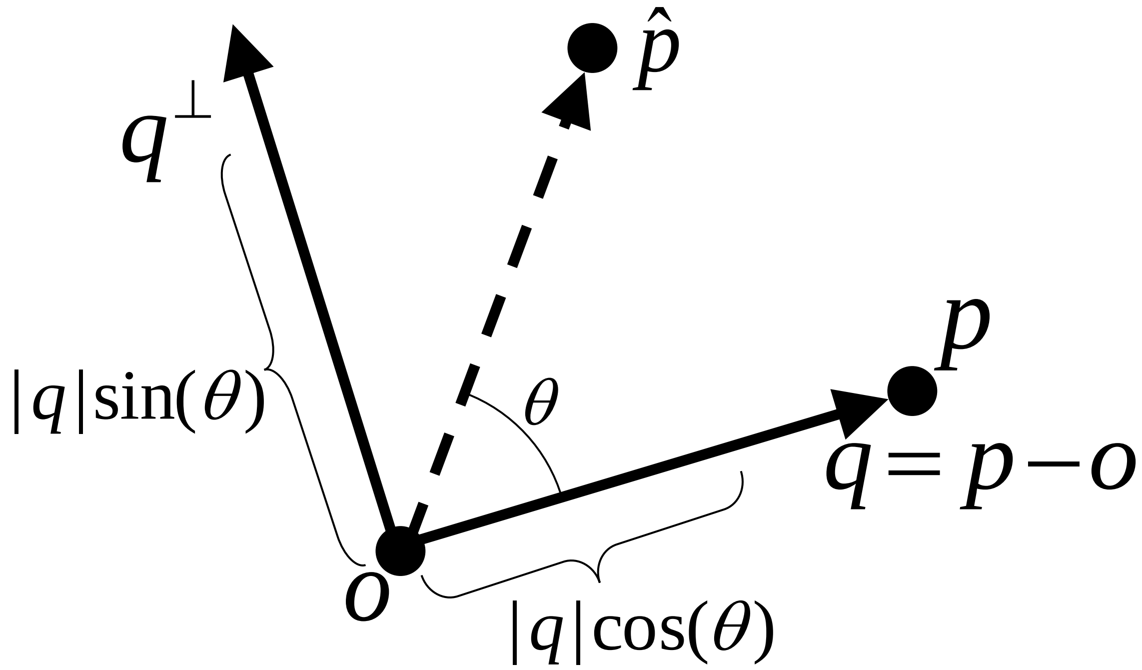
Rotation



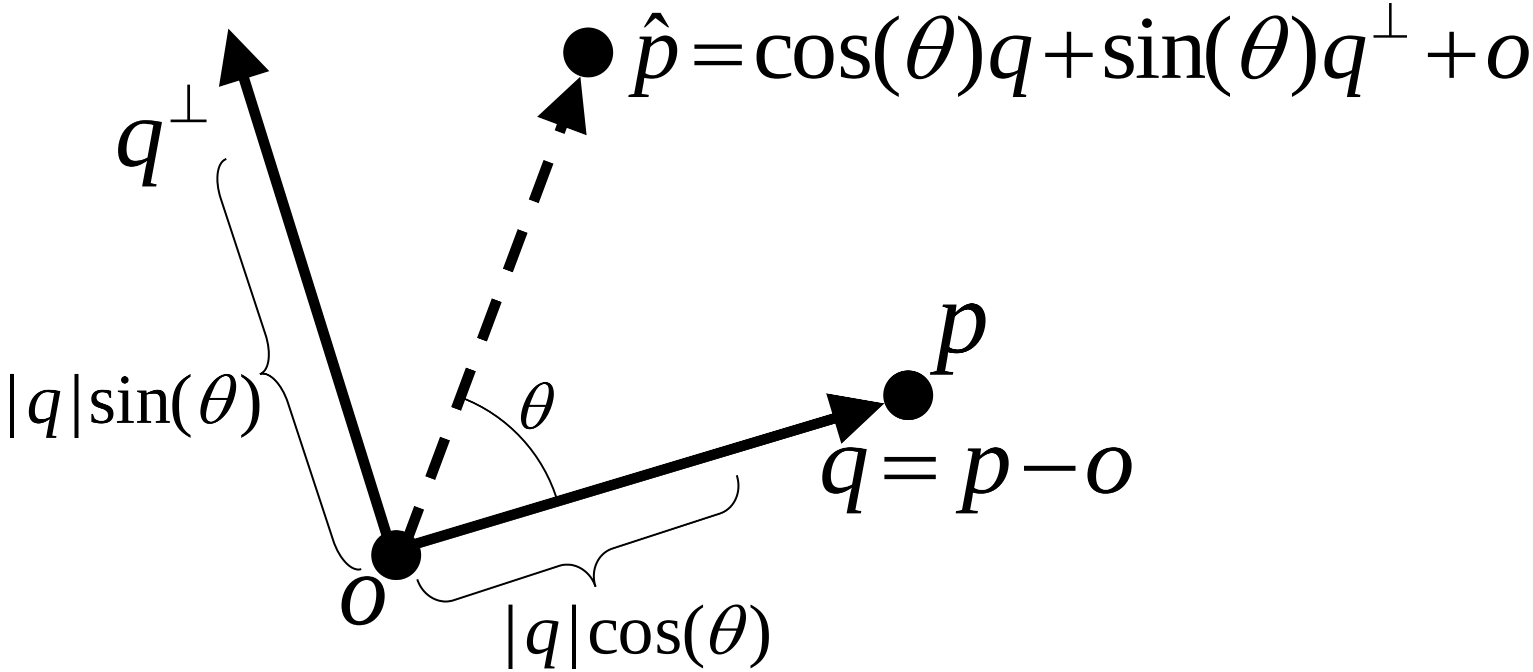
Rotation



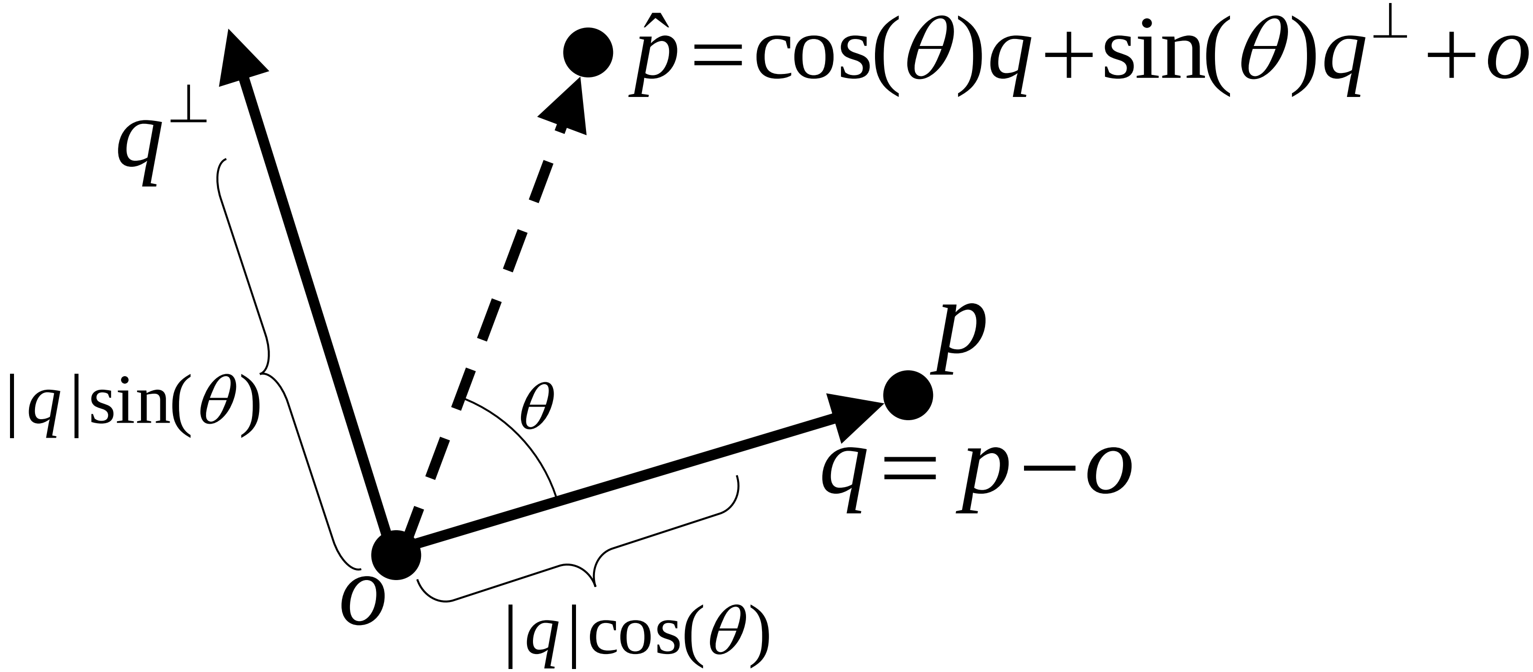
Rotation



Rotation

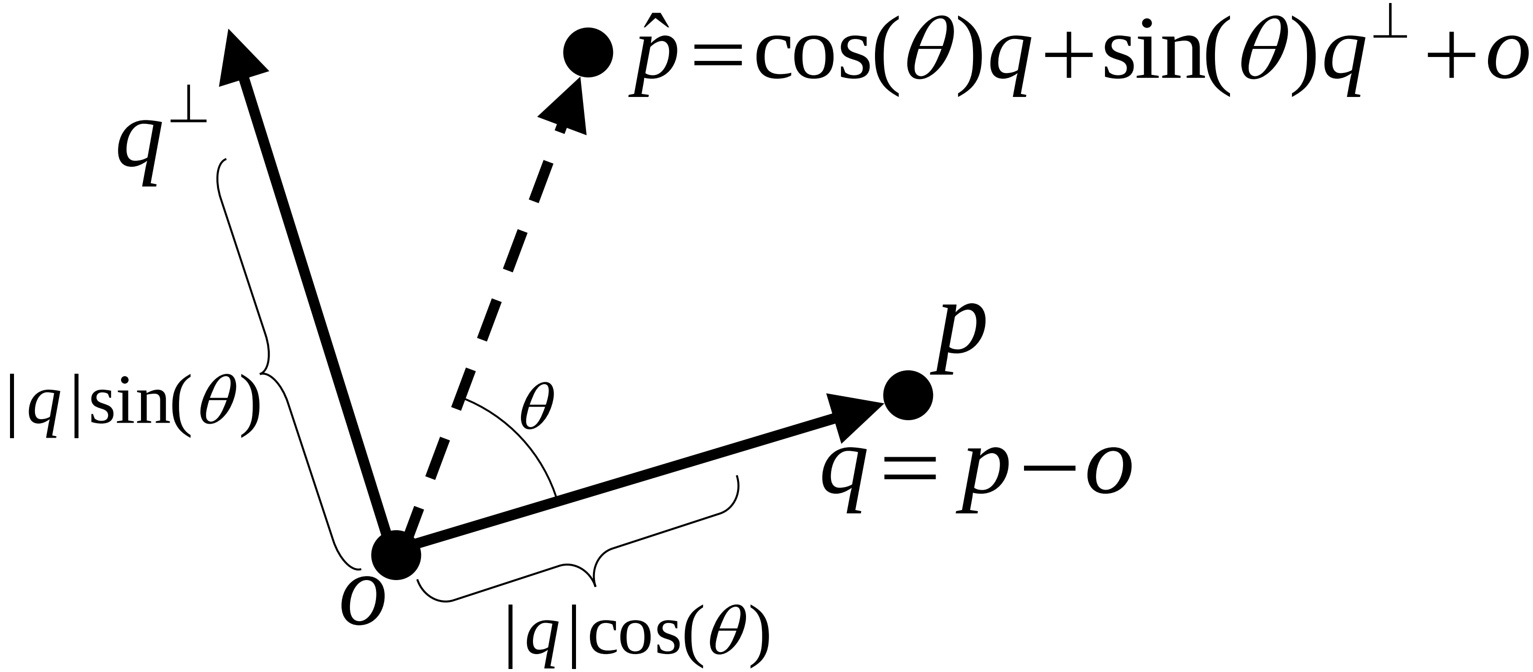

$\hat{p} = \cos(\theta) q + \sin(\theta) q^{\perp} + o$
$q^{\perp}$
p
$|q|\sin(\theta)$
θ
$q = p - o$
o
$|q|\cos(\theta)$

Rotation



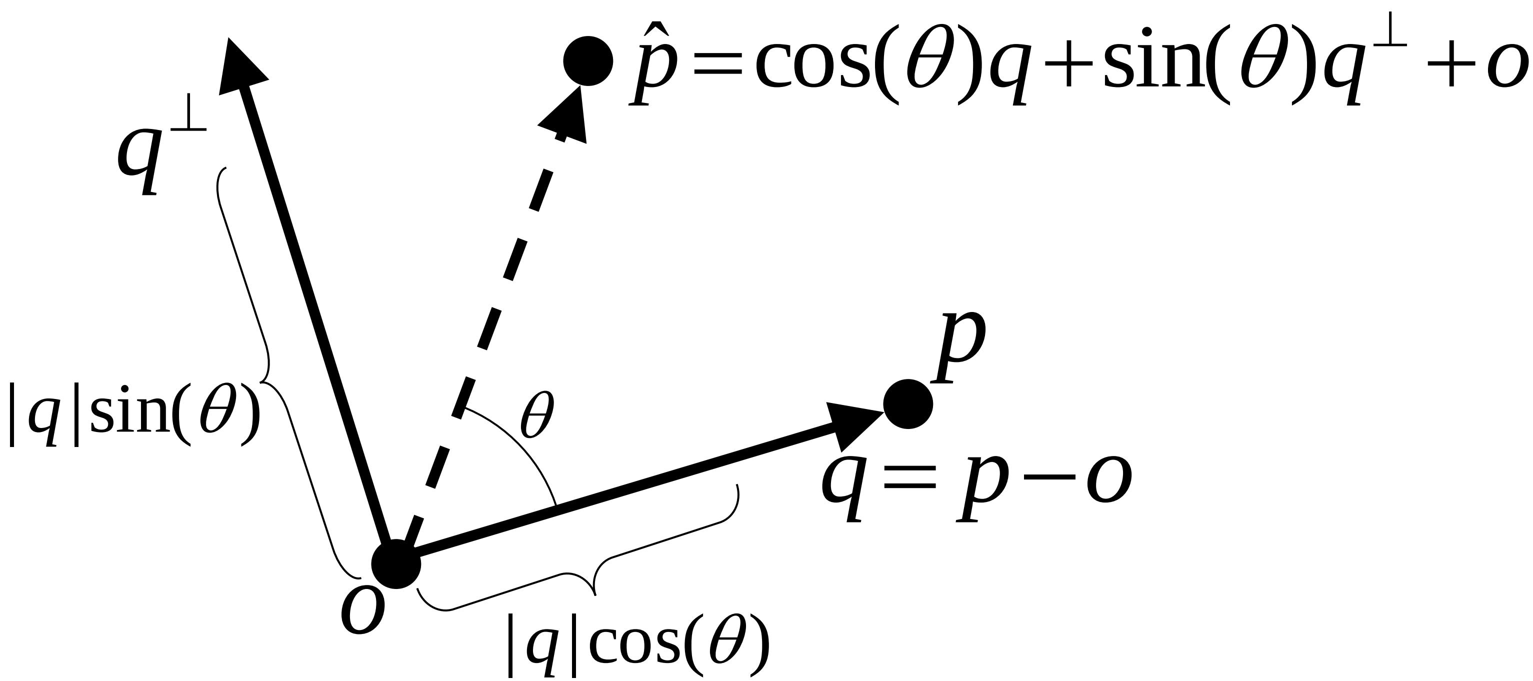
$$\begin{pmatrix} I & o \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} cI + s\_^{\perp} & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} I & -o \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ 1 \end{pmatrix} = \begin{pmatrix} \hat{p} \\ 1 \end{pmatrix}$$

Rotation



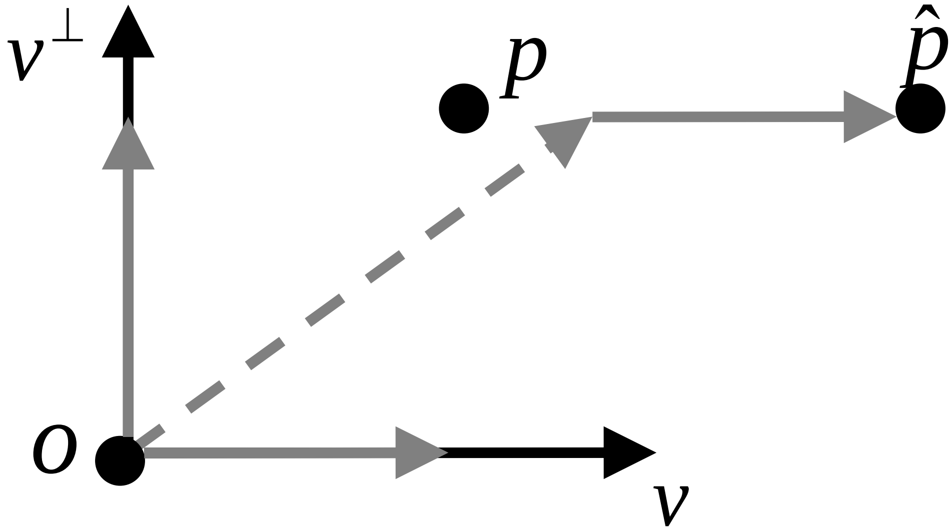
$$\begin{pmatrix} 1 & 0 & o_x \\ 0 & 1 & o_y \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \begin{pmatrix} 1 & 0 & -o_x \\ 0 & 1 & -o_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} = \begin{pmatrix} \hat{p}_x \\ \hat{p}_y \\ 1 \end{pmatrix}$$

Rotation



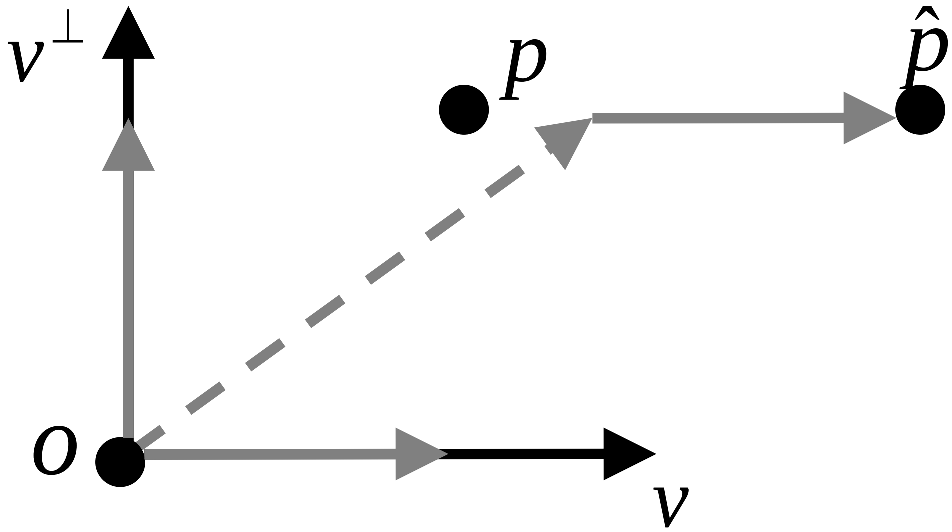
$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) & o_x + o_y \sin(\theta) - o_x \cos(\theta) \\ \sin(\theta) & \cos(\theta) & o_y - o_y \cos(\theta) - o_x \sin(\theta) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} = \begin{pmatrix} \hat{p}_x \\ \hat{p}_y \\ 1 \end{pmatrix}$$

Shear



$$\hat{p} = p + \alpha(v^{\perp} \cdot (p - o))v$$

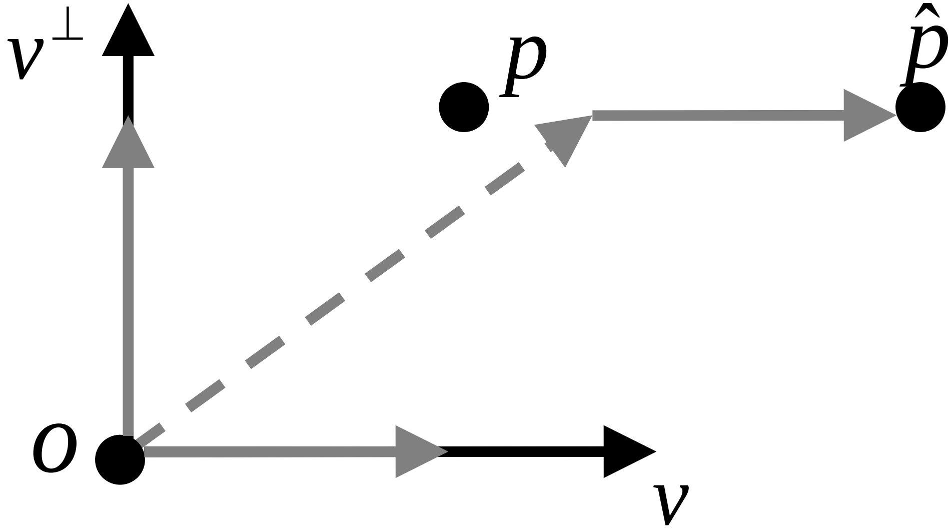
Shear



$$\hat{p} = p + \alpha(v^\perp \cdot (p - o))v$$

$$\begin{pmatrix} I + \alpha v (v^\perp)^T & -\alpha v (v^\perp)^T o \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ 1 \end{pmatrix} = \begin{pmatrix} \hat{p} \\ 1 \end{pmatrix}$$

Shear



$$\hat{p} = p + \alpha(v^\perp \cdot (p - o))v$$

$$\begin{pmatrix} 1-\alpha v_x v_y & \alpha v_x^2 & \alpha v_x(v_y o_x - v_x o_y) \\ -\alpha v_y^2 & 1+\alpha v_y v_x & \alpha v_y(v_y o_x - v_x o_y) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} = \begin{pmatrix} \hat{p}_x \\ \hat{p}_y \\ 1 \end{pmatrix}$$

Finding Affine Transformations

- Image of 3 points determines affine transformation

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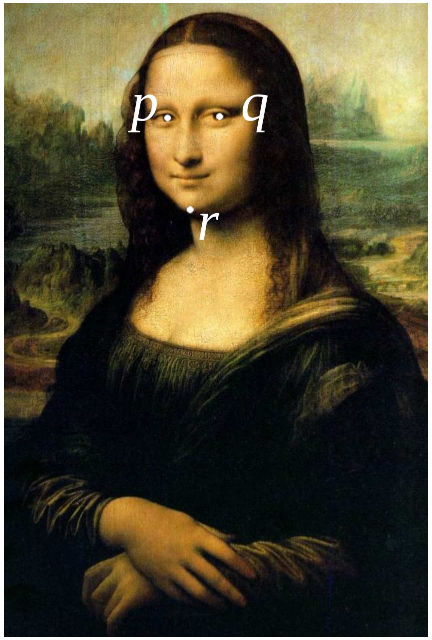
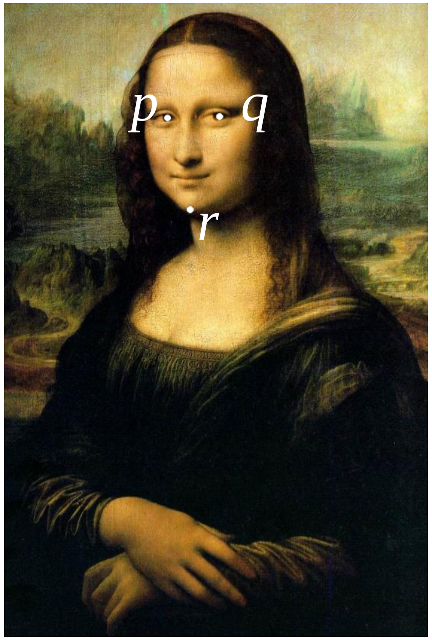
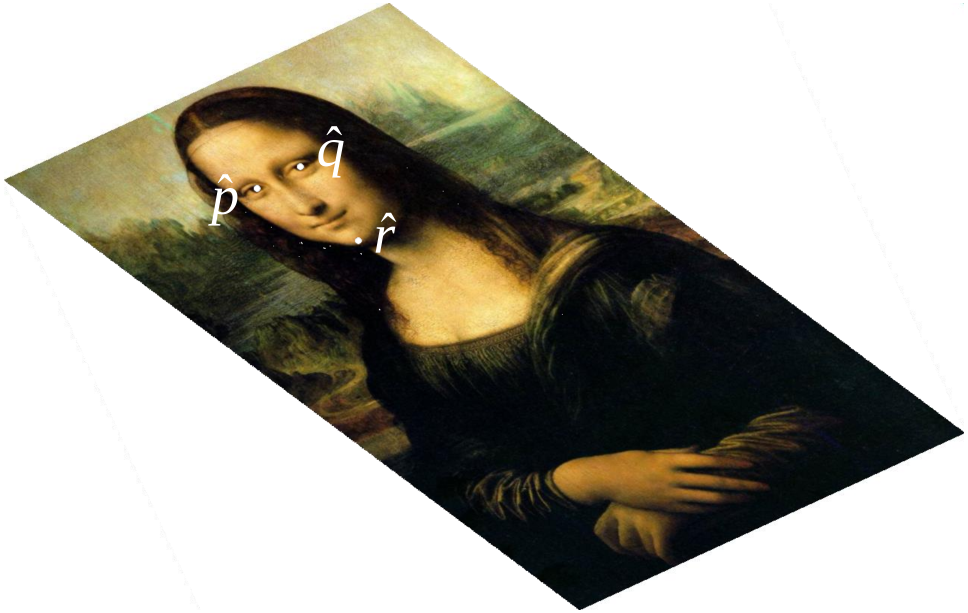


Finding Affine Transformations

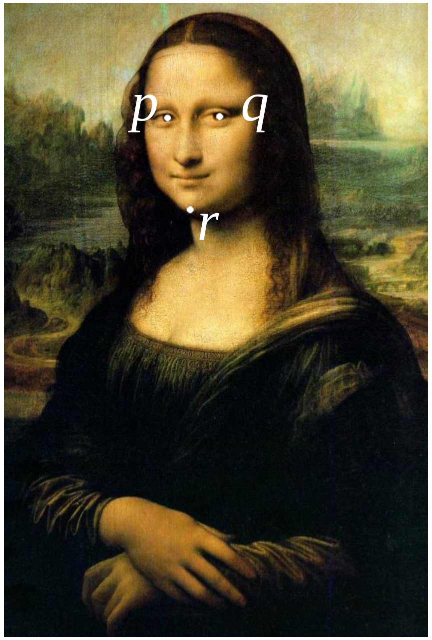
- Image of 3 points determines affine transformation

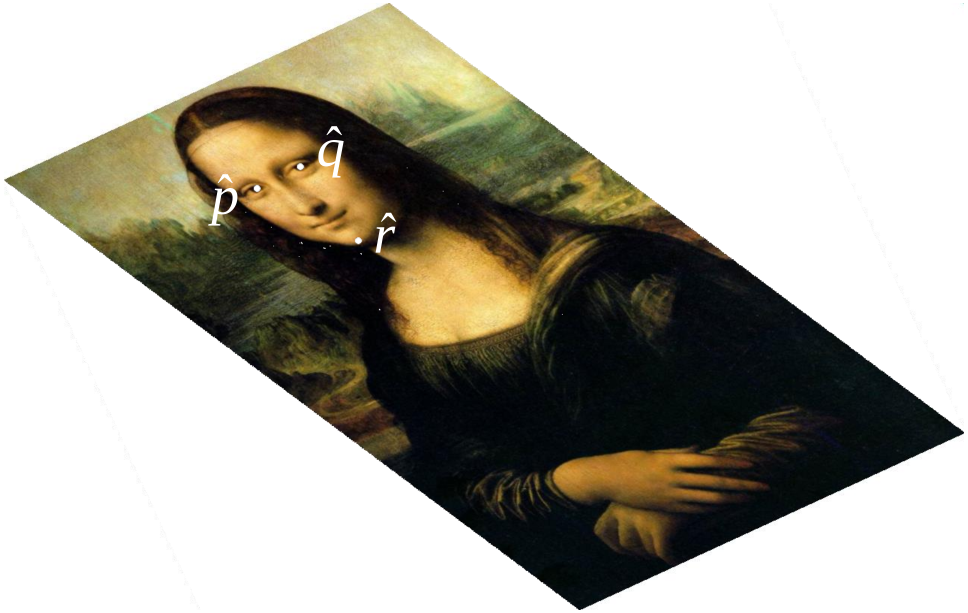




Finding Affine Transformations

- Image of 3 points determines affine transformation

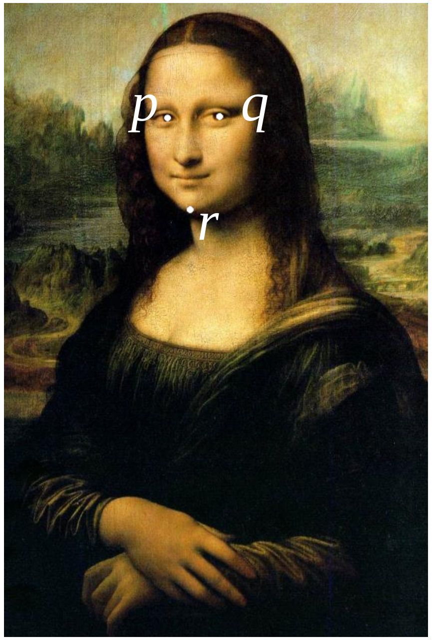


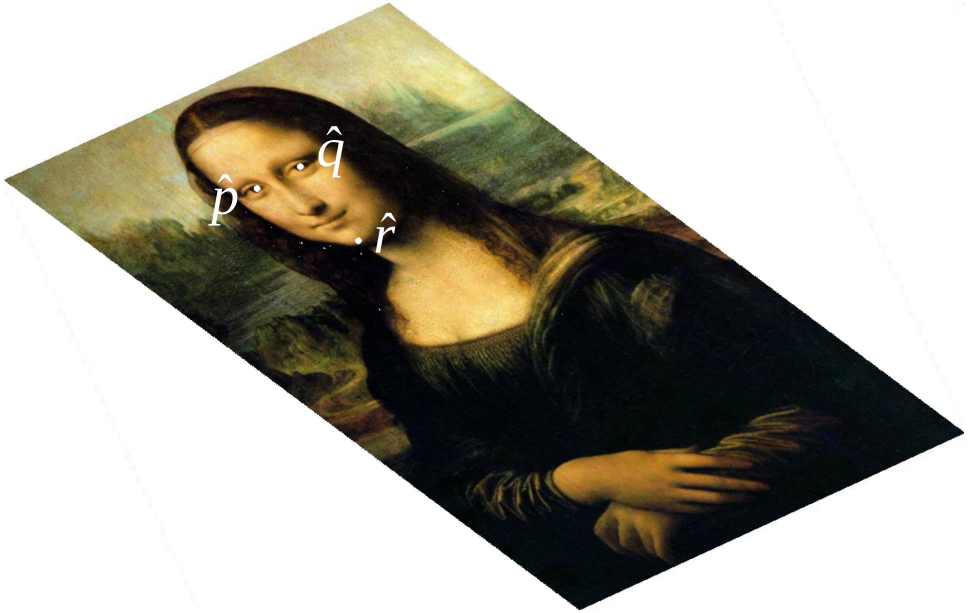


$$\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} p_x & q_x & r_x \\ p_y & q_y & r_y \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \hat{p}_x & \hat{q}_x & \hat{r}_x \\ \hat{p}_y & \hat{q}_y & \hat{r}_y \\ 1 & 1 & 1 \end{pmatrix}$$

Finding Affine Transformations

- Image of 3 points determines affine transformation

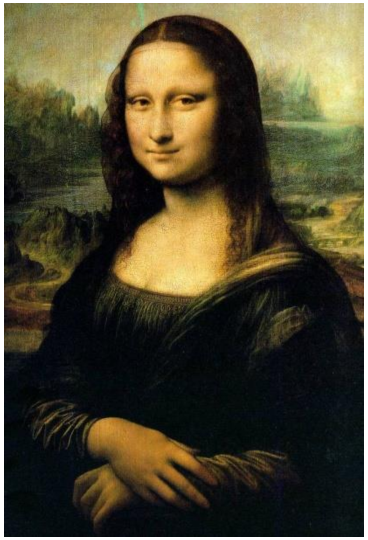




$$\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} = \begin{pmatrix} \hat{p}_x & \hat{q}_x & \hat{r}_x \\ \hat{p}_y & \hat{q}_y & \hat{r}_y \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} p_x & q_x & r_x \\ p_y & q_y & r_y \\ 1 & 1 & 1 \end{pmatrix}^{-1}$$

Composing Transformations

- Compose multiple transformations into a single matrix



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Composing Transformations

- Compose multiple transformations into a single matrix

$$\begin{pmatrix} \hat{p}_x \\ \hat{p}_y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix}$$

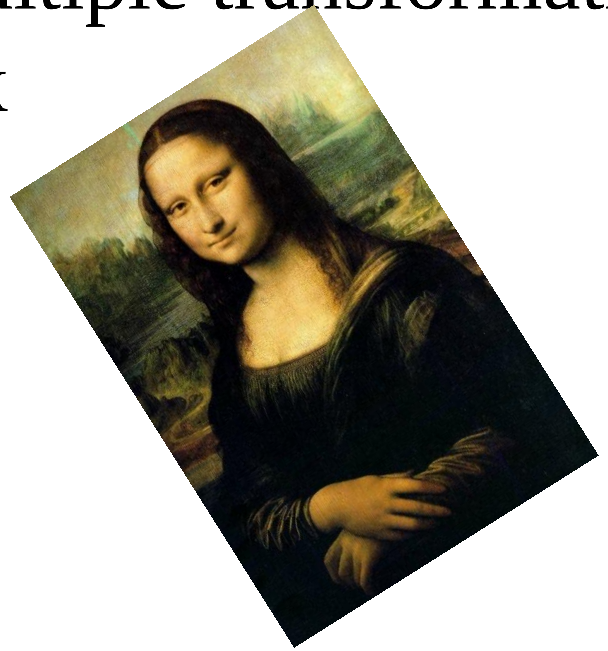
Composing Transformations

- Compose multiple transformations into a single matrix

$$\begin{pmatrix} \hat{p}_x \\ \hat{p}_y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix}$$

Composing Transformations

- Compose multiple transformations into a single matrix



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$$\begin{pmatrix}\hat{p}_x\\ \hat{p}_y\\ 1\end{pmatrix}=\begin{pmatrix}1 & 0 & -t_x\\ 0 & 1 & -t_y\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos(\theta) & -\sin(\theta) & 0\\ \sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & t_x\\ 0 & 1 & t_y\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}p_x\\ p_y\\ 1\end{pmatrix}$$

Composing T… nsformations

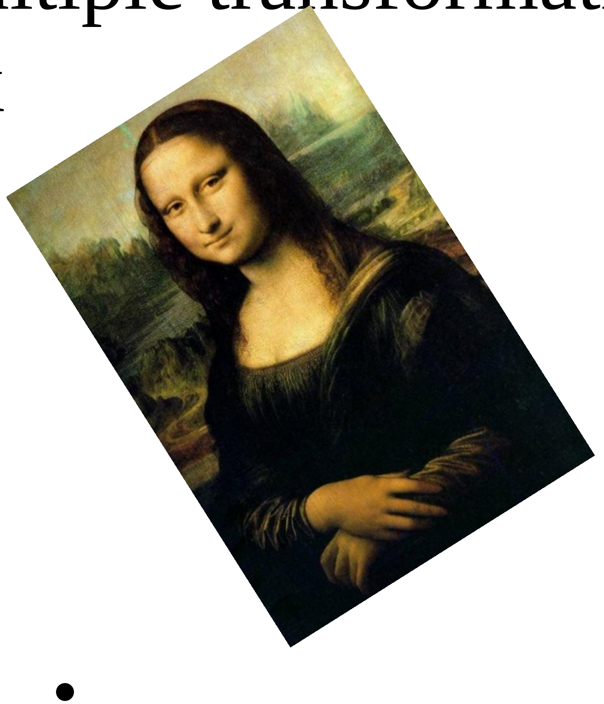
- Compose m… sformations into a single matrix

Order matters!!!

$$\begin{pmatrix}\hat{p}_x\\ \hat{p}_y\\ 1\end{pmatrix}=\begin{pmatrix}1 & 0 & t_x\\ 0 & 1 & t_y\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos(\theta) & -\sin(\theta) & 0\\ \sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -t_x\\ 0 & 1 & -t_y\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}p_x\\ p_y\\ 1\end{pmatrix}$$

Composing Transformations

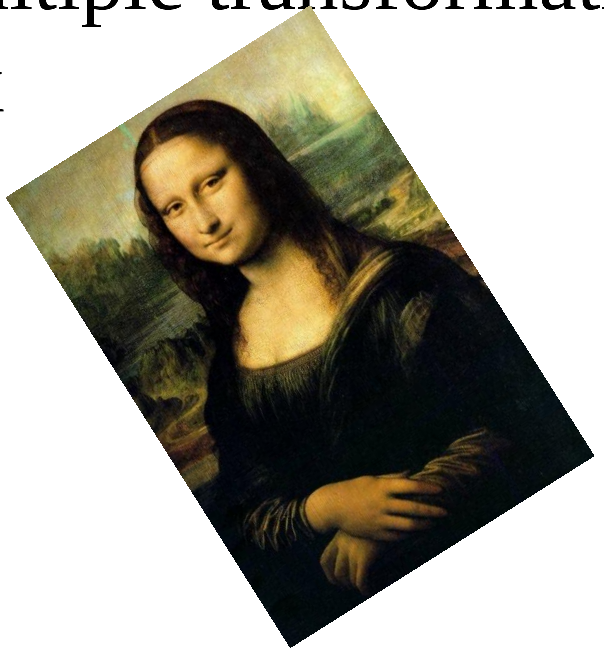
- Compose multiple transformations into a single matrix



$$\begin{pmatrix} \hat{p}_x \\ \hat{p}_y \\ 1 \end{pmatrix} = \left(\begin{pmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \right) \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix}$$

Composing Transformations

- Compose multiple transformations into a single matrix



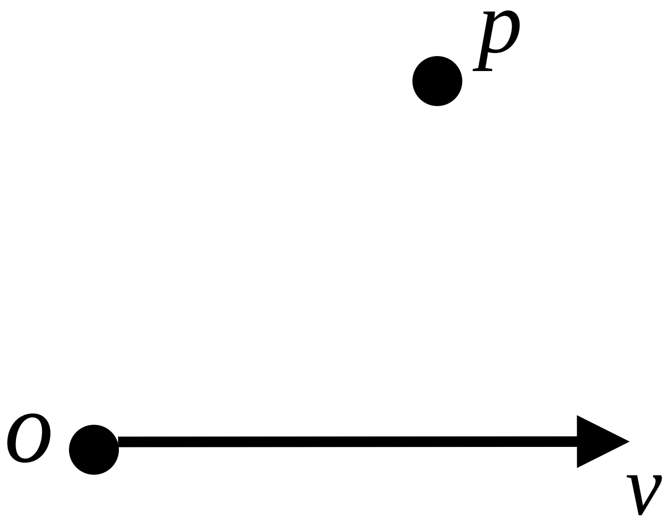
$$\begin{pmatrix} \hat{p}_x \\ \hat{p}_y \\ 1 \end{pmatrix} = M \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix}$$

Reflection

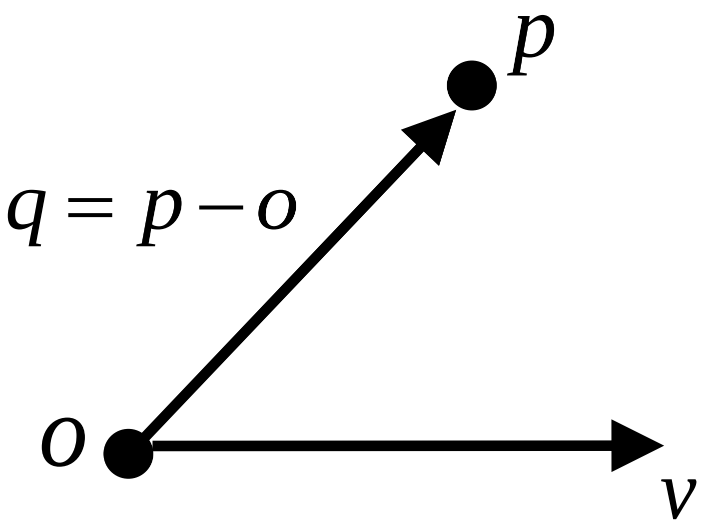
Reflection

Reflection

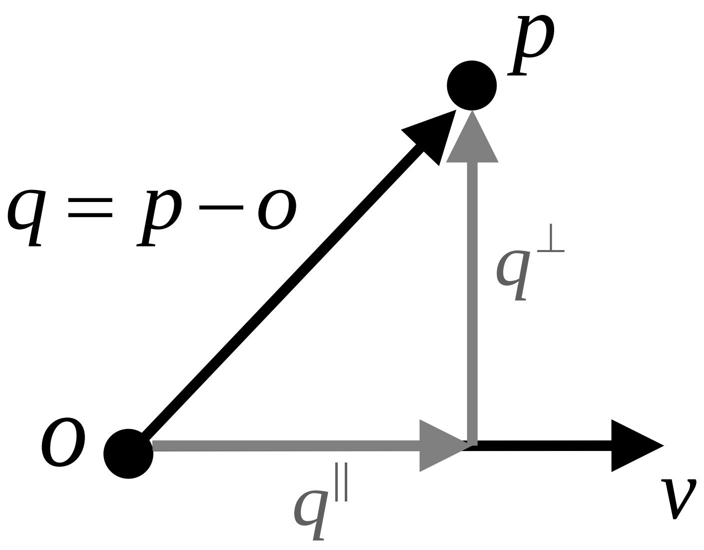
Reflection



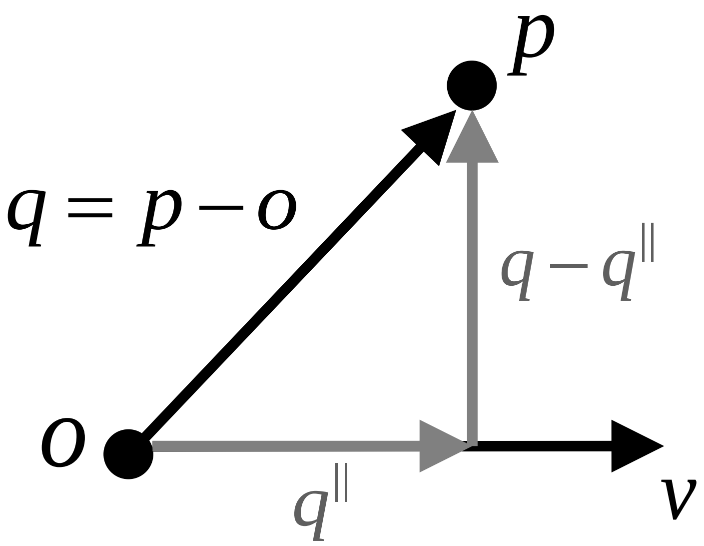
Reflection



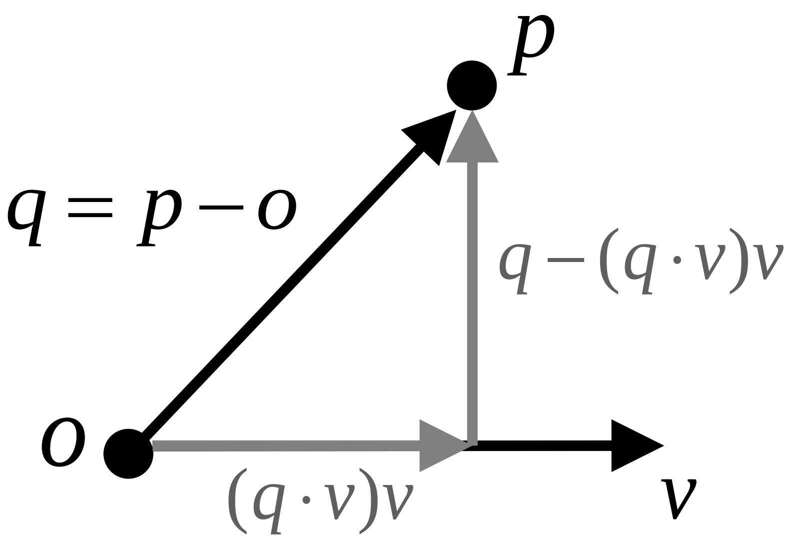
Reflection



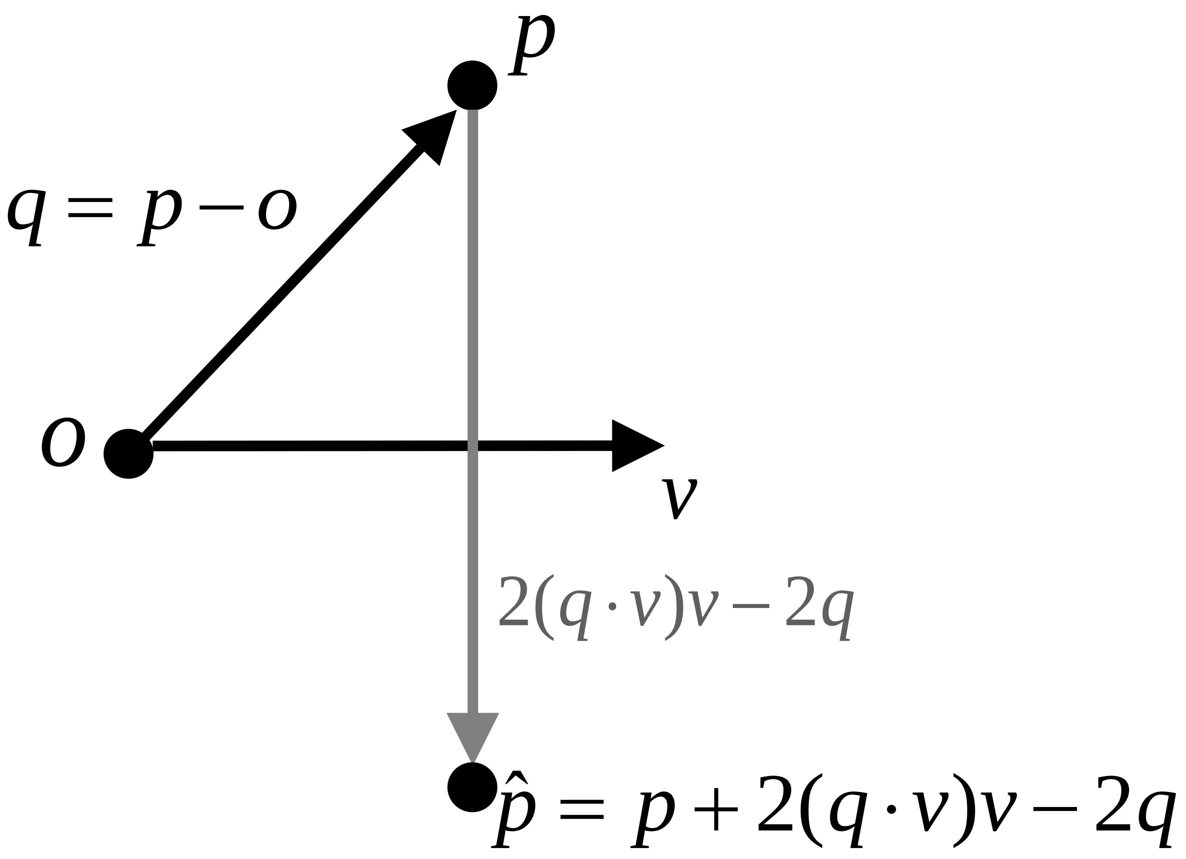
Reflection



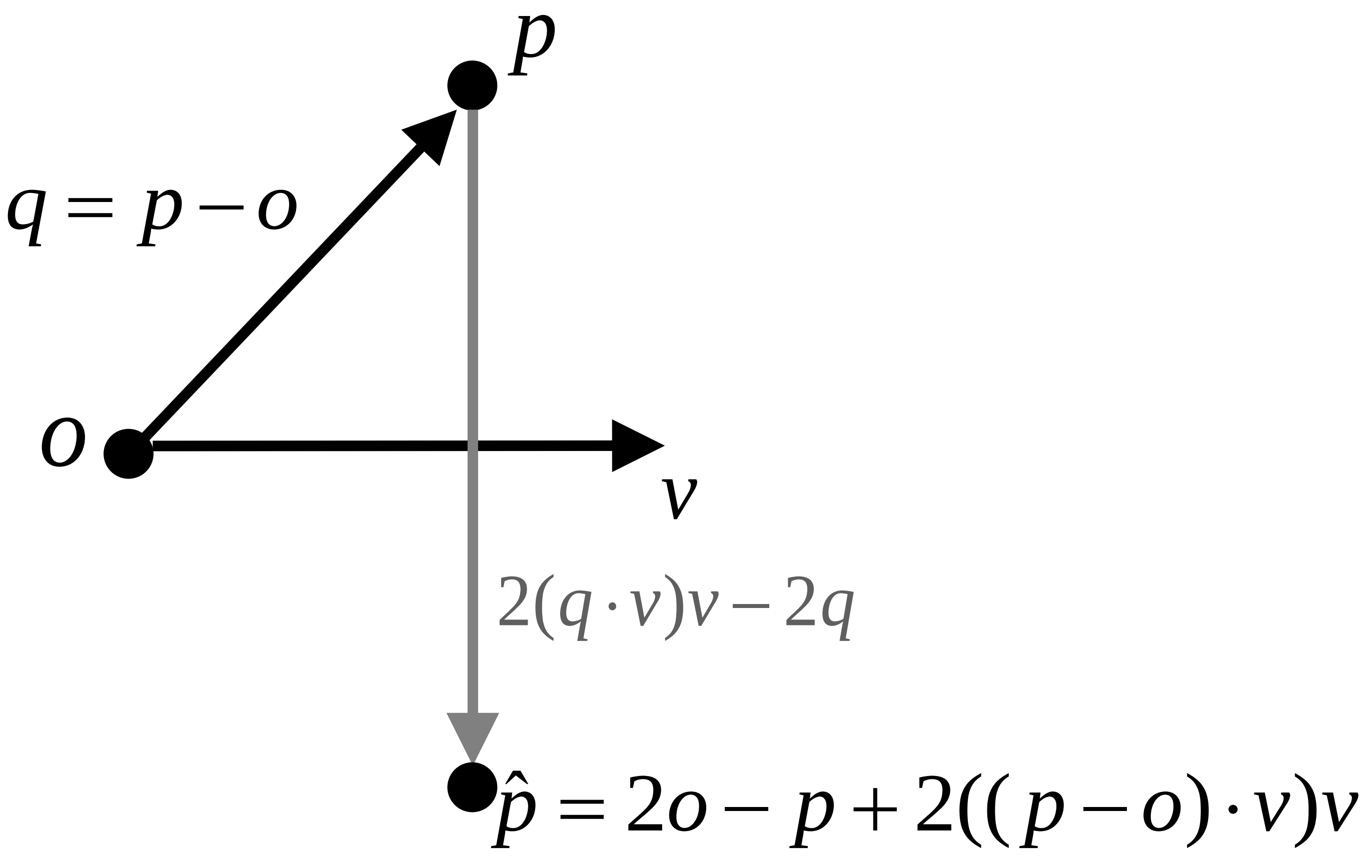
Reflection



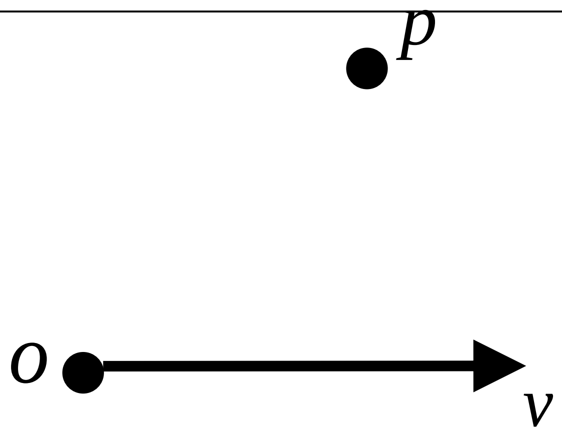
Reflection



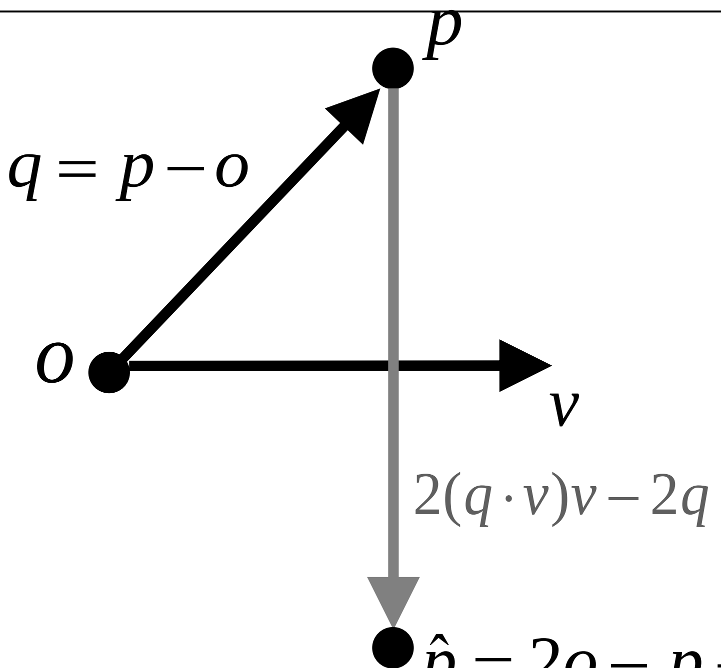
Reflection



Reflection



Reflection



$$\hat{p} = 2o - p + 2((p - o) \cdot v)v$$

$$\begin{pmatrix} 2{v_x}^2 - 1 & 2v_x v_y & 2(o_x - v_x(o_x v_x + o_y v_y)) \\ 2v_y v_x & 2{v_y}^2 - 1 & 2(o_y - v_y(o_x v_x + o_y v_y)) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} = \begin{pmatrix} \hat{p}_x \\ \hat{p}_y \\ 1 \end{pmatrix}$$