Smooth Curves

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TAMU Computer Science

Smooth Curves

- Interpolation
  - Interpolation through Linear Algebra
  - Lagrange interpolation
- Bezier curves
- B-spline curves

Smooth Curves

- How do we create smooth curves?

Smooth Curves

- How do we create smooth curves?

- Parametric curves with polynomials

$$p(t) = \left(x(t), y(t)\right)$$

Smooth Curves

- Controlling the shape of the curve

$$x(t) = a + bt + ct^2 + dt^3$$

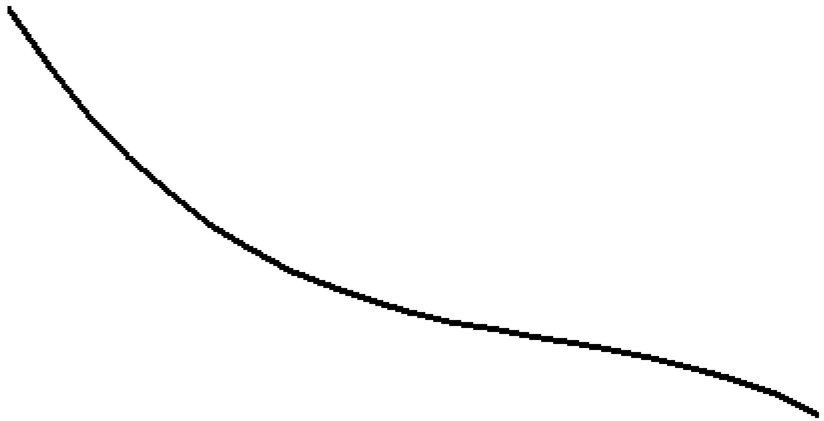
$$y(t) = e + ft + gt^2 + ht^3$$

Smooth Curves

- Controlling the shape of the curve

$$x(t) = t$$

$$y(t) = 1 - t + t^2 - t^3$$

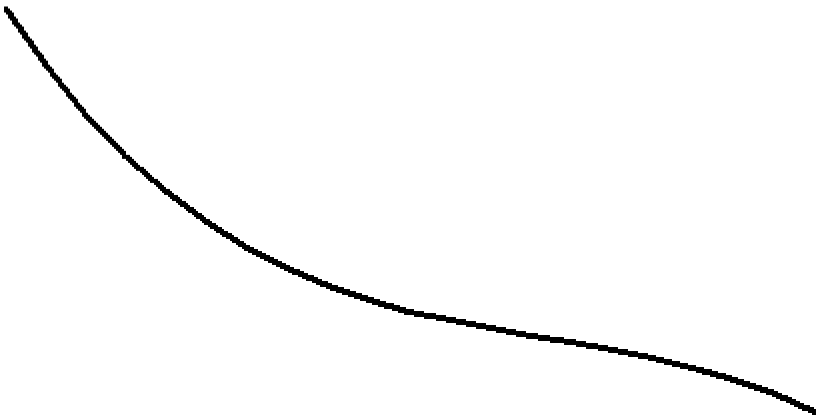


Smooth Curves

- Controlling the shape of the curve

$$x(t) = t$$

$$y(t) = 3 - t + t^2 - t^3$$



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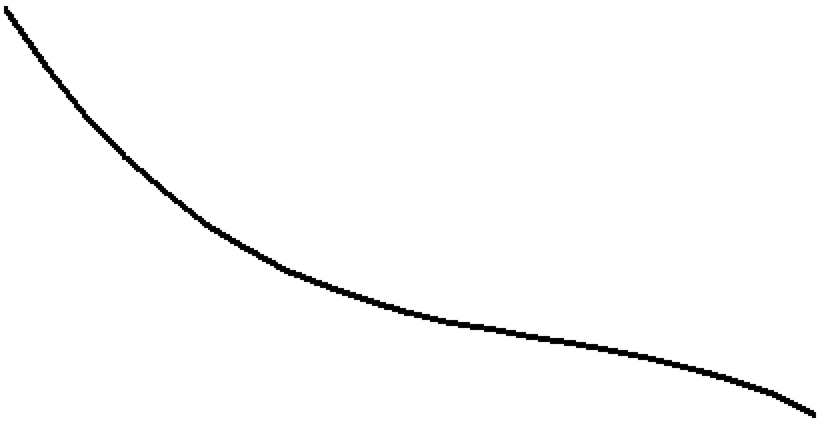

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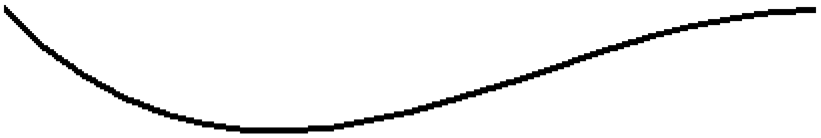

Smooth Curves

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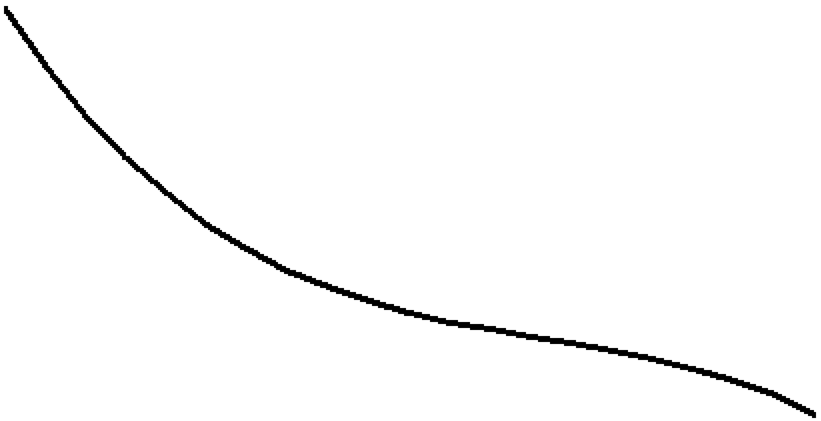
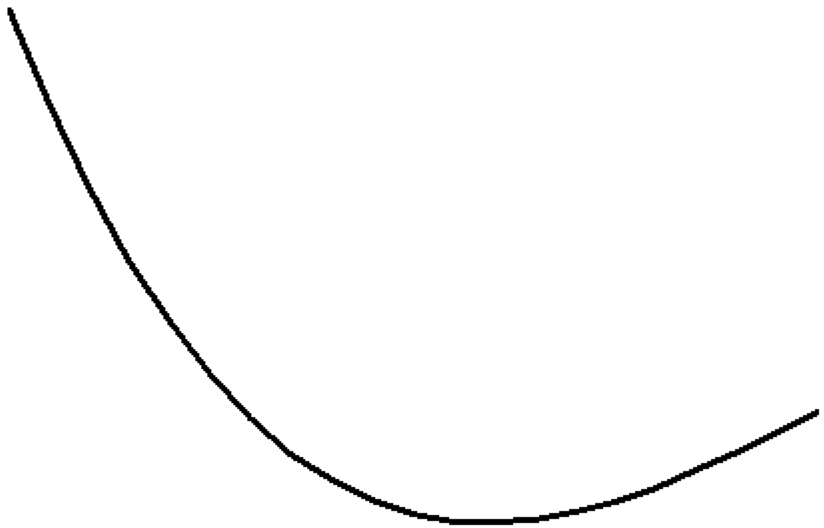


Smooth Curves

- Controlling the shape of the curve

$$x(t) = t$$

$$y(t) = 1 - t + 3t^2 - t^3$$



Smooth Curves

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$$x(t) = t$$

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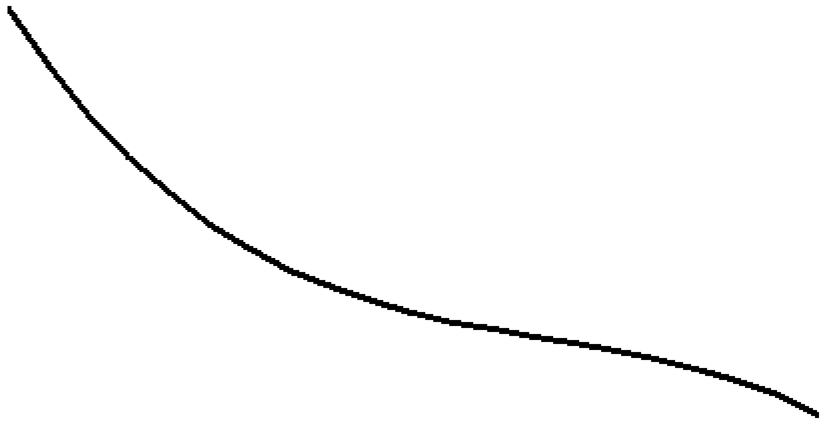

Smooth Curves

- Controlling the shape of the curve

$$x(t) = t$$

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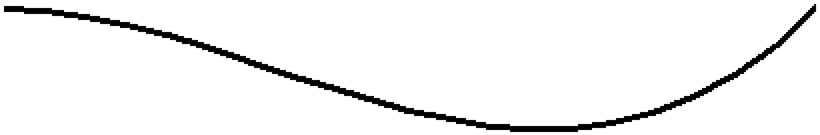


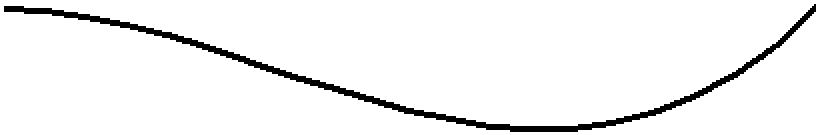
Smooth Curves

- Controlling the shape of the curve

$$x(t) = t$$

$$y(t) = 1 - t + t^2 + t^3$$

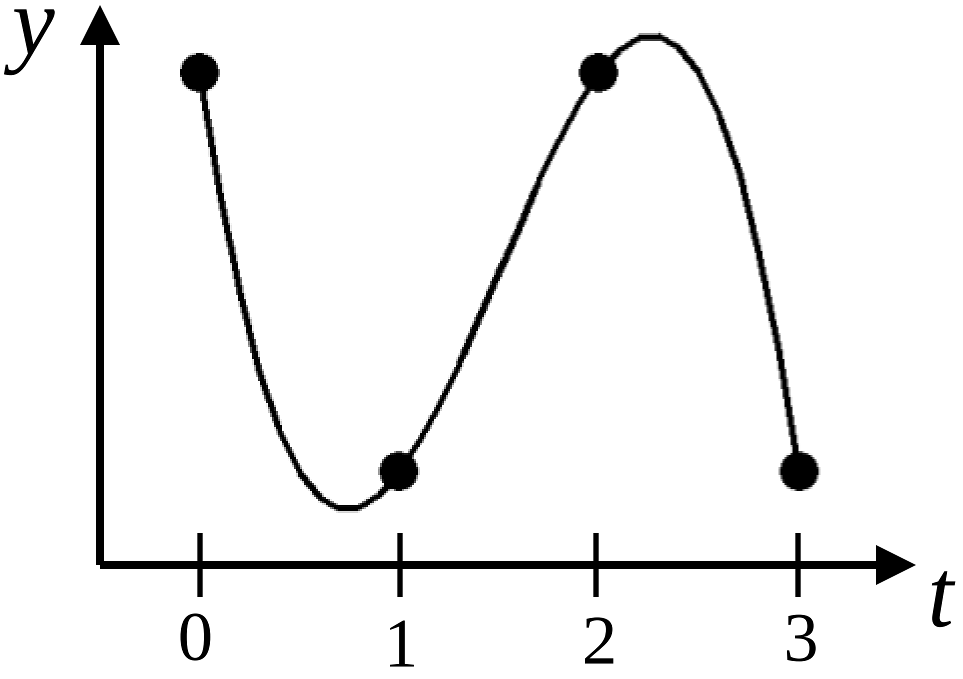
Power-basis coefficients not intuitive
for controlling shape of curve!!!



Interpolation

- Find a polynomial that passes through specified values

$$y(t) = a + bt + ct^2 + dt^3$$



Interpolation

- Find a polynomial that passes through specified values

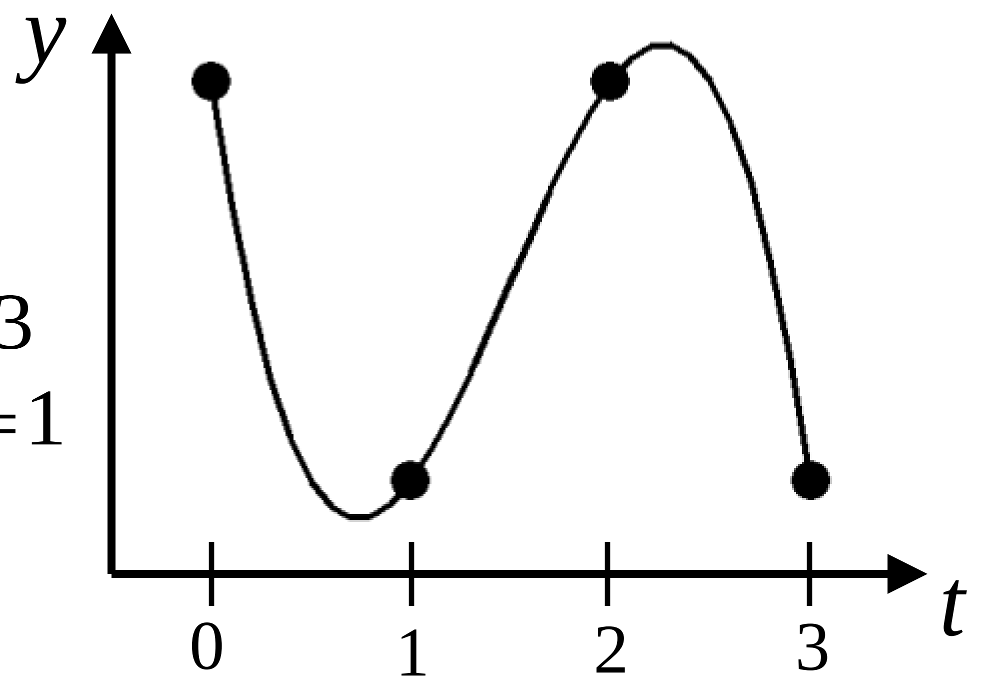
$$y(t) = a + bt + ct^2 + dt^3$$

$$y(0) = a = 3$$

$$y(1) = a + b + c + d = 1$$

$$y(2) = a + 2b + 4c + 8d = 3$$

$$y(3) = a + 3b + 9c + 27d = 1$$

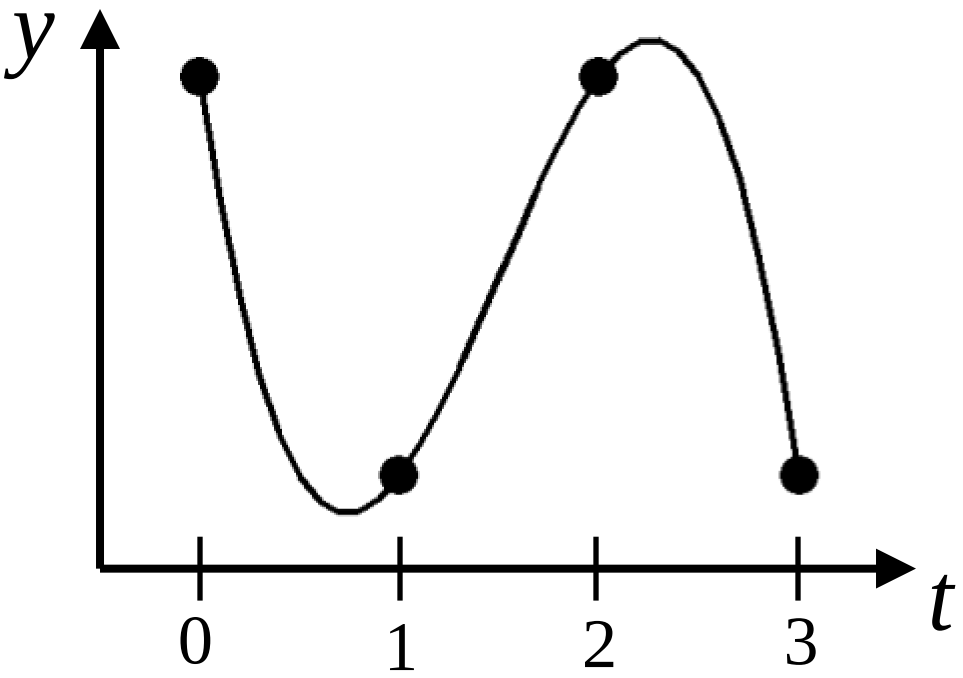


Interpolation

- Find a polynomial that passes through specified values

$$y(t) = a + bt + ct^2 + dt^3$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 3 \\ 1 \end{pmatrix}$$

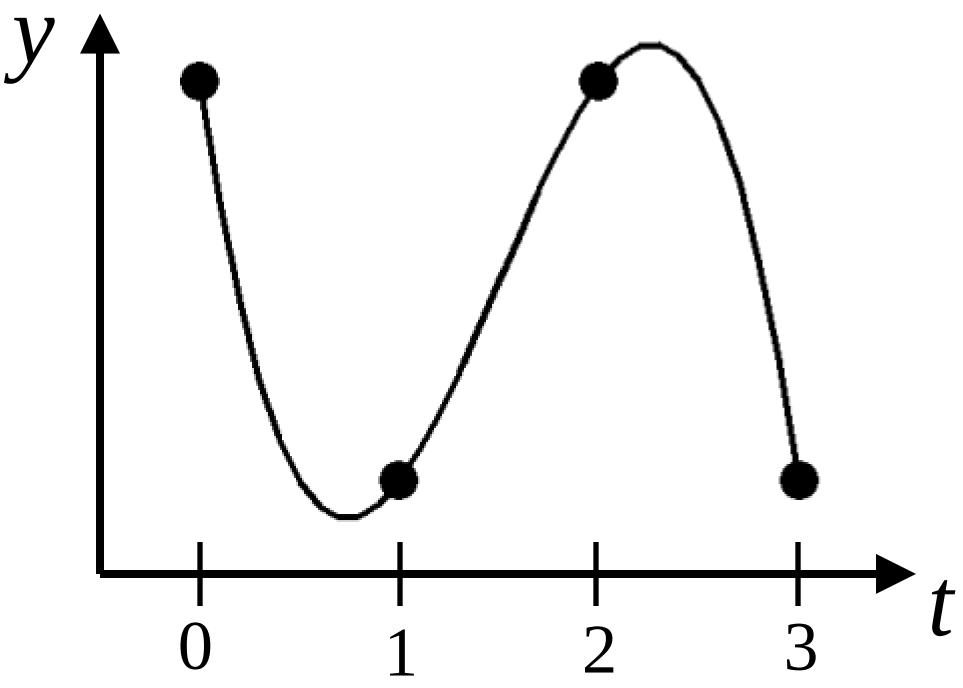


Interpolation

- Find a polynomial that passes through specified values

$$y(t) = a + bt + ct^2 + dt^3$$

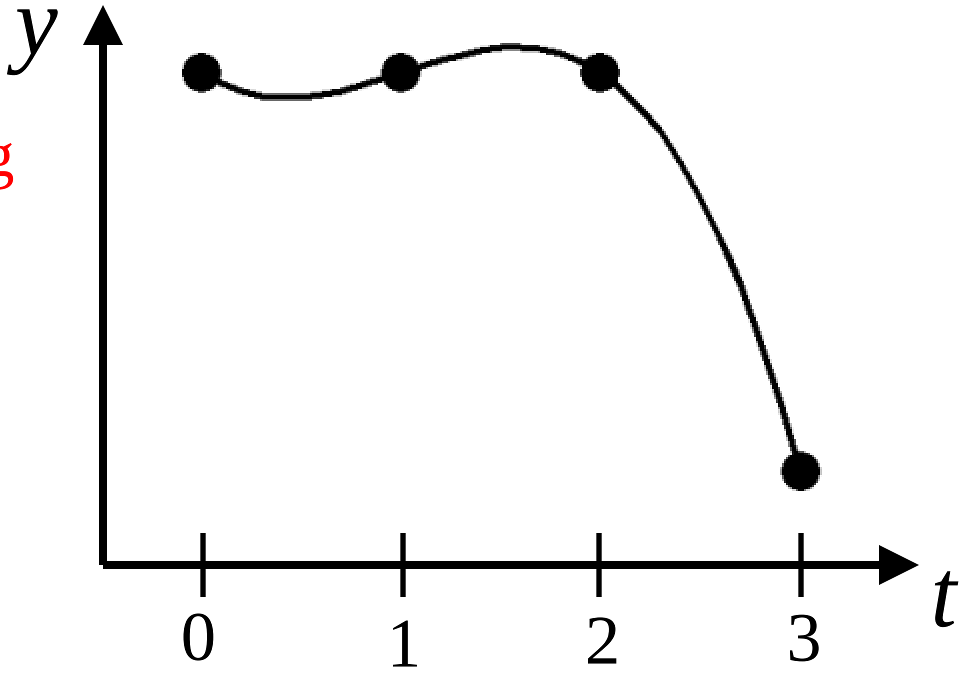
$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ -20/3 \\ 6 \\ -1/3 \end{pmatrix}$$



Interpolation

- Find a polynomial that passes through specified values

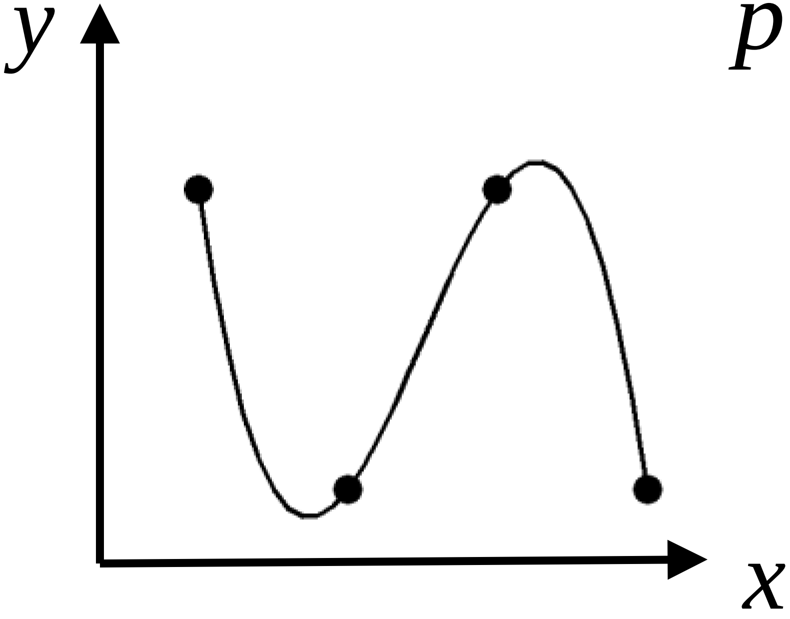
Intuitive control of curve using “control points”!!!



Interpolation

- Perform interpolation for each component separately
- Combine result to obtain parametric curve

$$p(t) = \big(x(t), y(t)\big)$$



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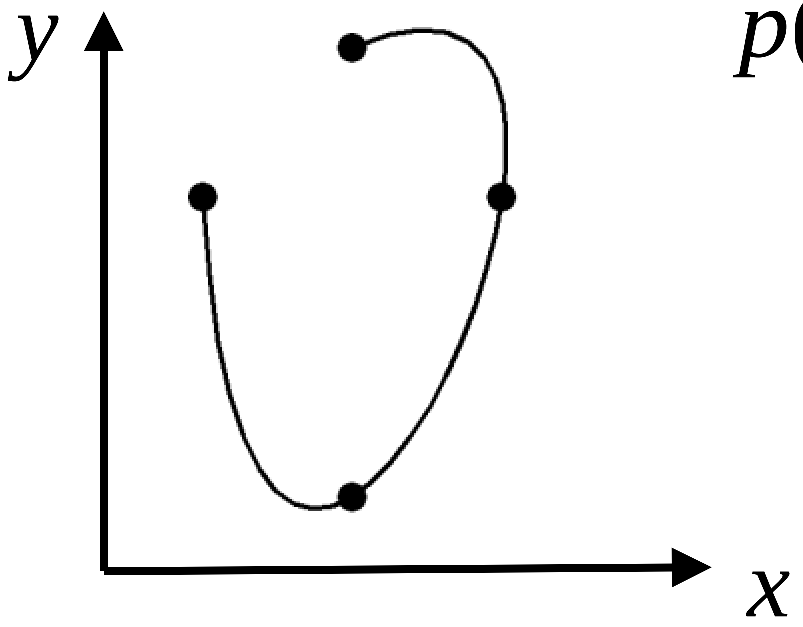

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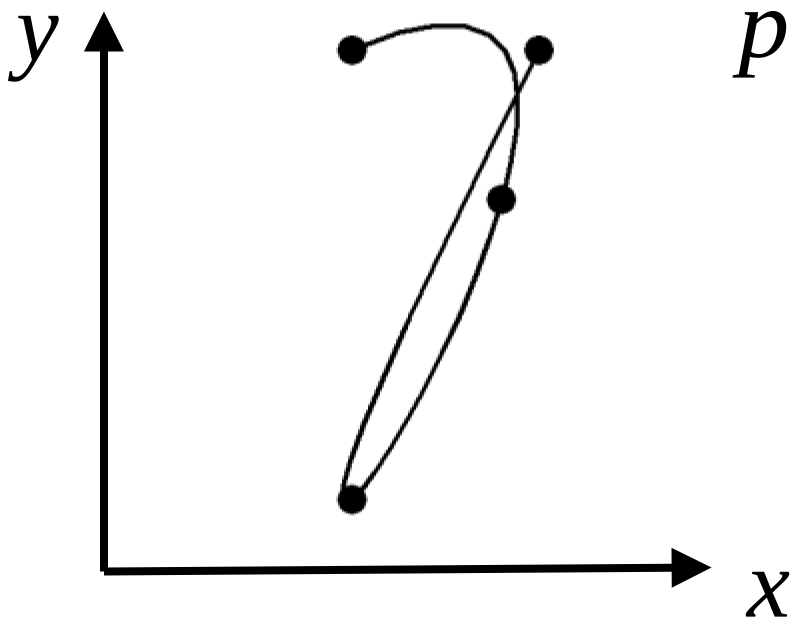
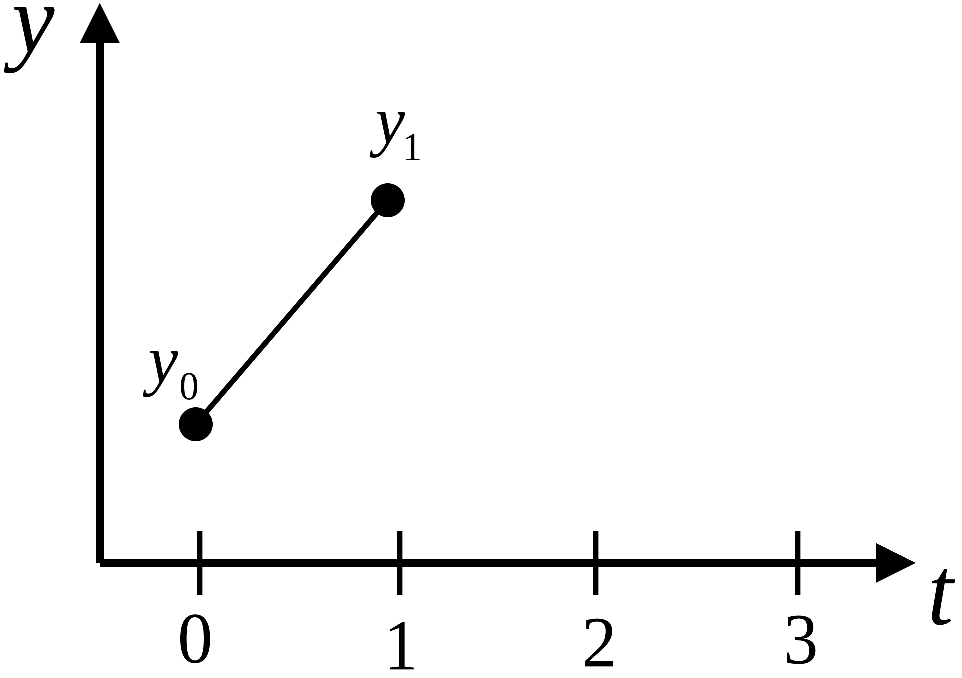


Lagrange Interpolation

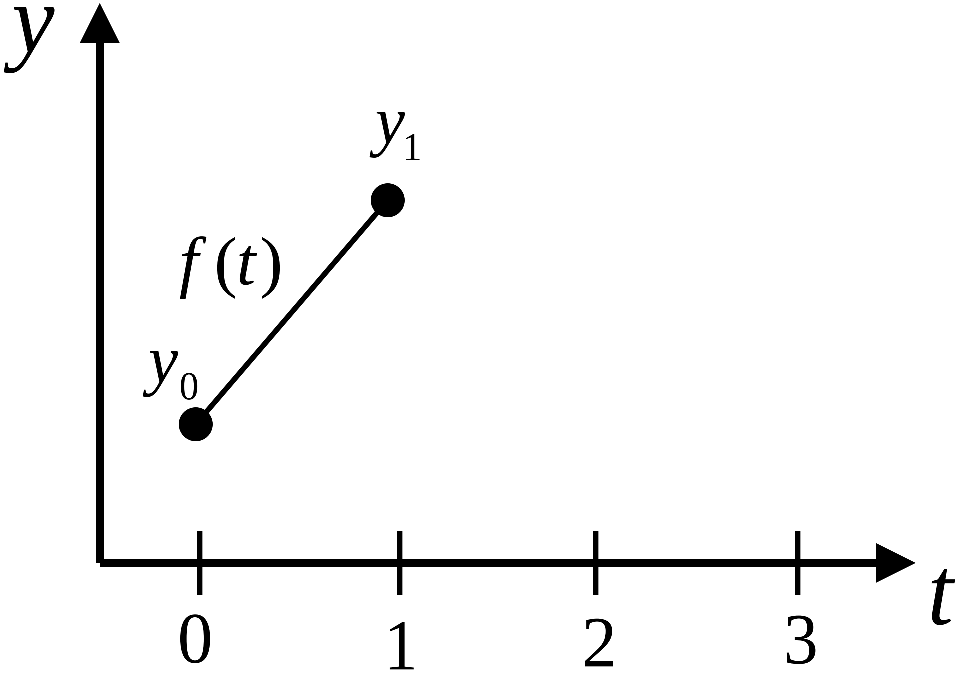
- Identical to matrix method but uses a geometric construction



Lagrange Interpolation

- Identical to matrix method but uses a geometric construction

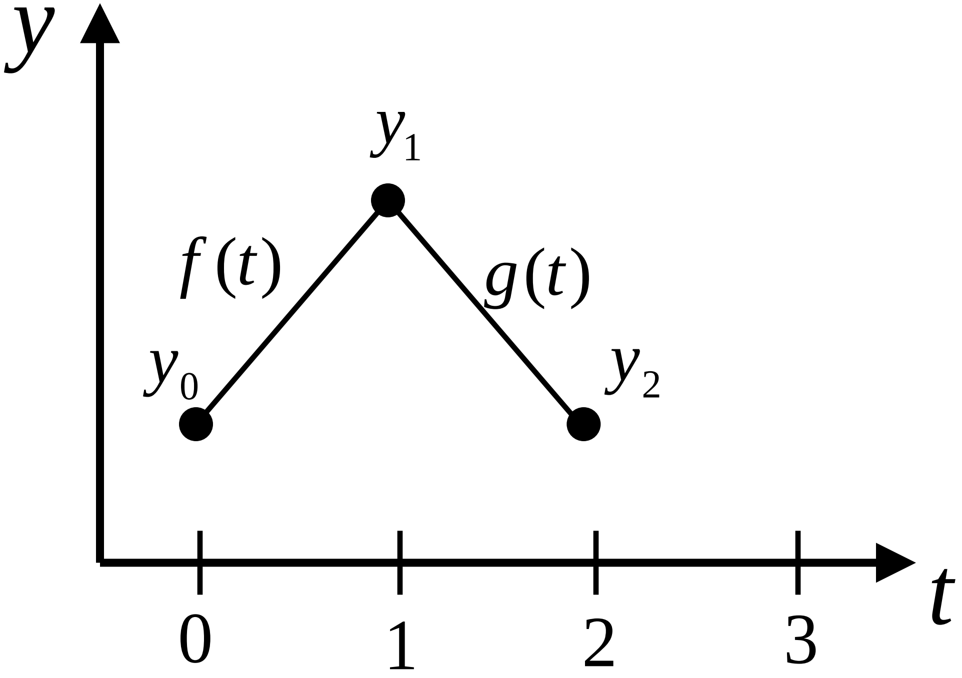
$$f(t) = (1-t)\,y_0 + t\,y_1$$



Lagrange Interpolation

- Identical to matrix method but uses a geometric construction

$$f(t) = (1-t)y_0 + t\,y_1$$
$$g(t) = (2-t)y_1 + (t-1)\,y_2$$

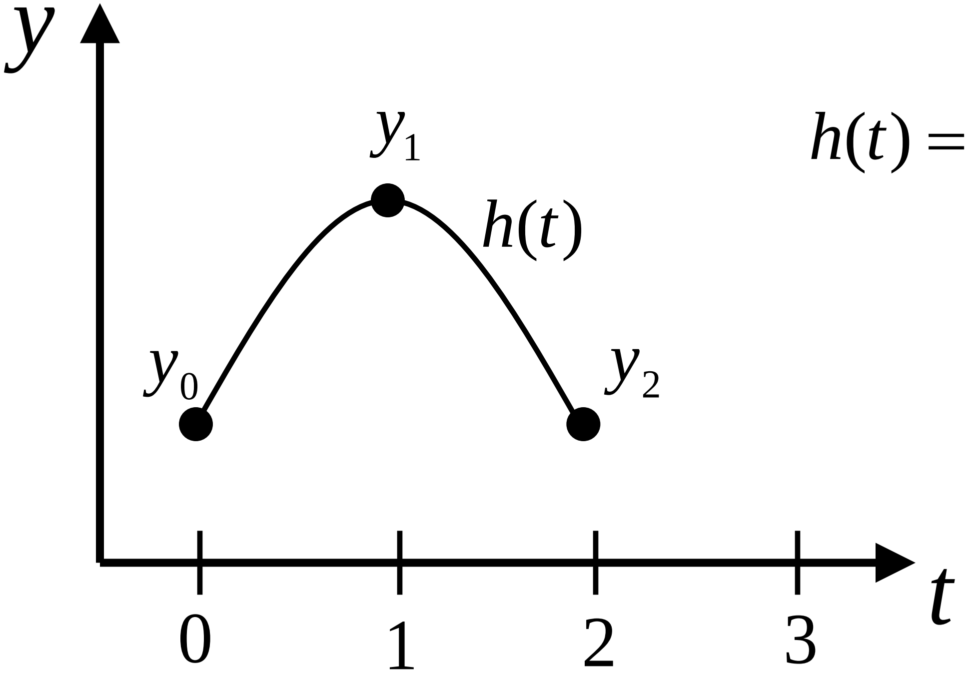


Lagrange Interpolation

- Identical to matrix method but uses a geometric construction

$$f(t) = (1-t)\,y_0 + t\,y_1$$
$$g(t) = (2-t)\,y_1 + (t-1)\,y_2$$

$$h(t) = \frac{(2-t)\,f(t) + t\,g(t)}{2}$$



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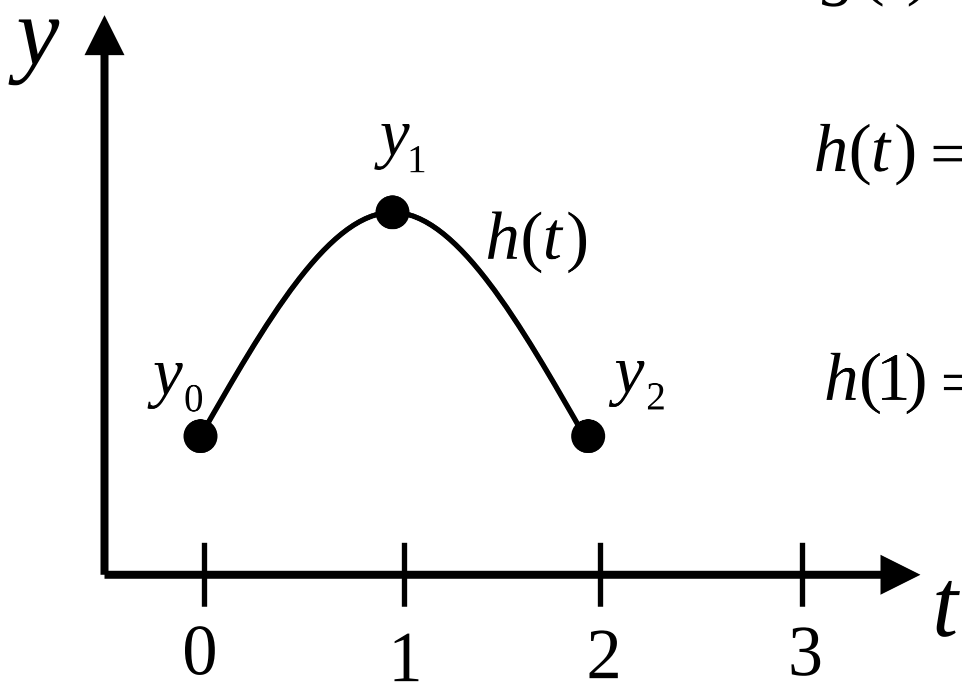


$$h(t) = \frac{(2-t)\,f(t) + t\,g(t)}{2}$$

$$h(1) = \frac{(2-1)\,f(1) + 1\,g(1)}{2}$$

Lagrange Interpolation

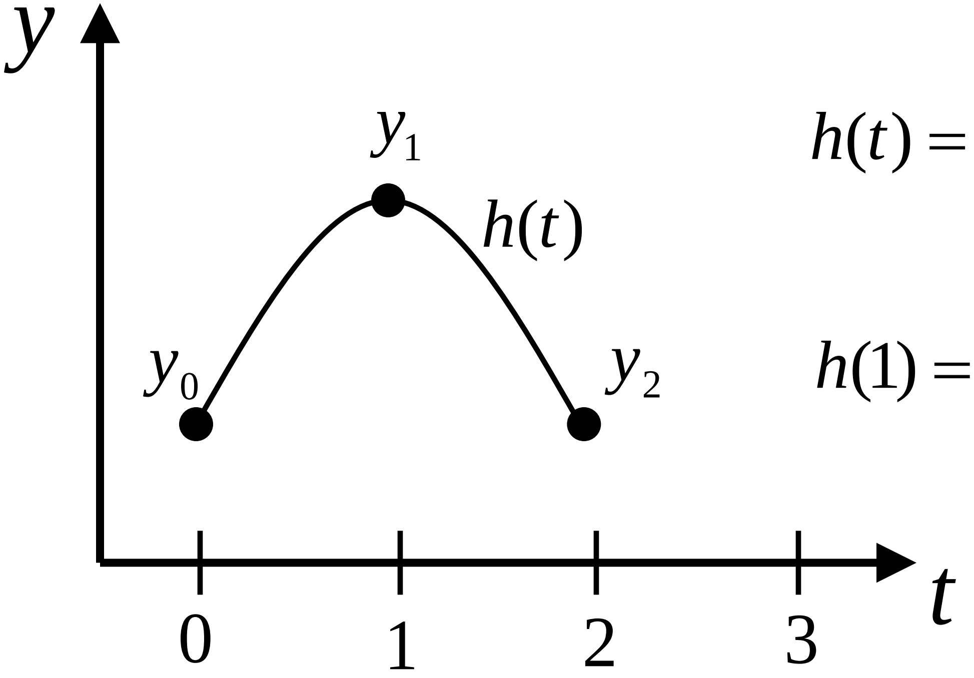
- Identical to matrix method but uses a geometric construction

$$f(t) = (1-t)\,y_0 + t\,y_1$$

$$g(t) = (2-t)\,y_1 + (t-1)\,y_2$$

$$h(t) = \frac{(2-t)\,f(t) + t\,g(t)}{2}$$

$$h(1) = \frac{y_1 + y_1}{2} = y_1$$



Lagrange Interpolation

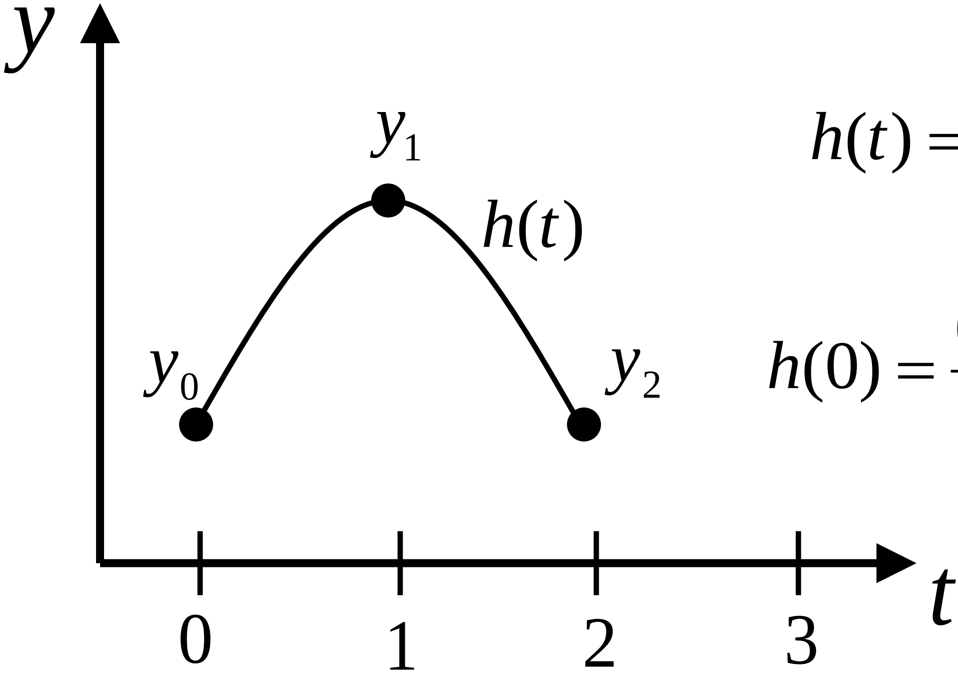
- Identical to matrix method but uses a geometric construction

$$f(t) = (1-t)\,y_0 + t\,y_1$$

$$g(t) = (2-t)\,y_1 + (t-1)\,y_2$$

$$h(t) = \frac{(2-t)\,f(t) + t\,g(t)}{2}$$

$$h(0) = \frac{(2-0)\,f(0) + 0\,g(0)}{2}$$



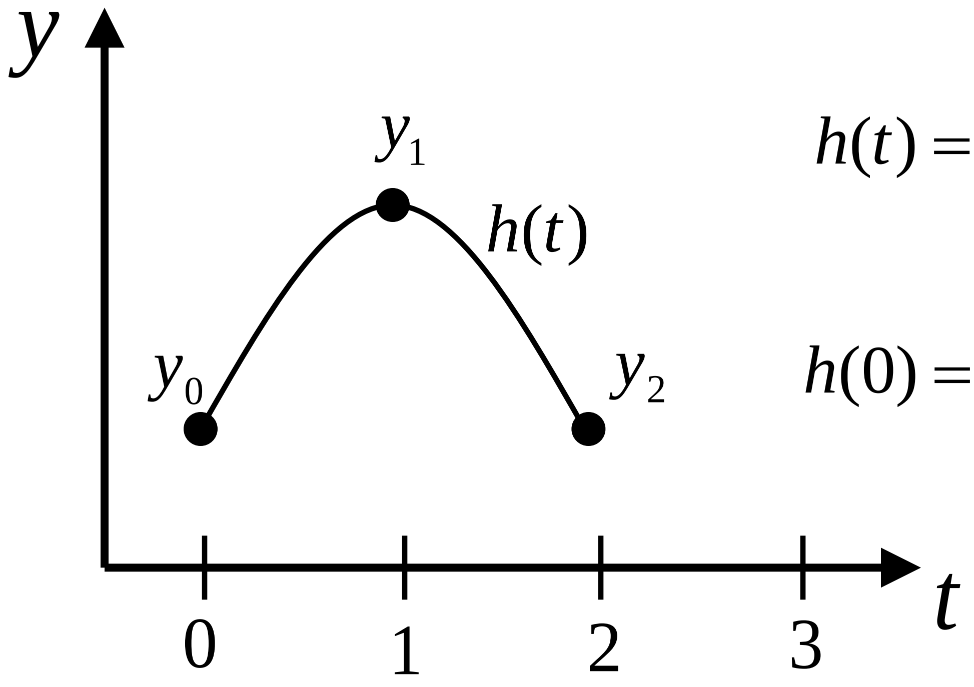
Lagrange Interpolation

- Identical to matrix method but uses a geometric construction

$$f(t) = (1-t)\,y_0 + t\,y_1$$
$$g(t) = (2-t)\,y_1 + (t-1)\,y_2$$

$$h(t) = \frac{(2-t)\,f(t) + t\,g(t)}{2}$$

$$h(0) = \frac{2y_0}{2} = y_0$$



Lagrange Interpolation

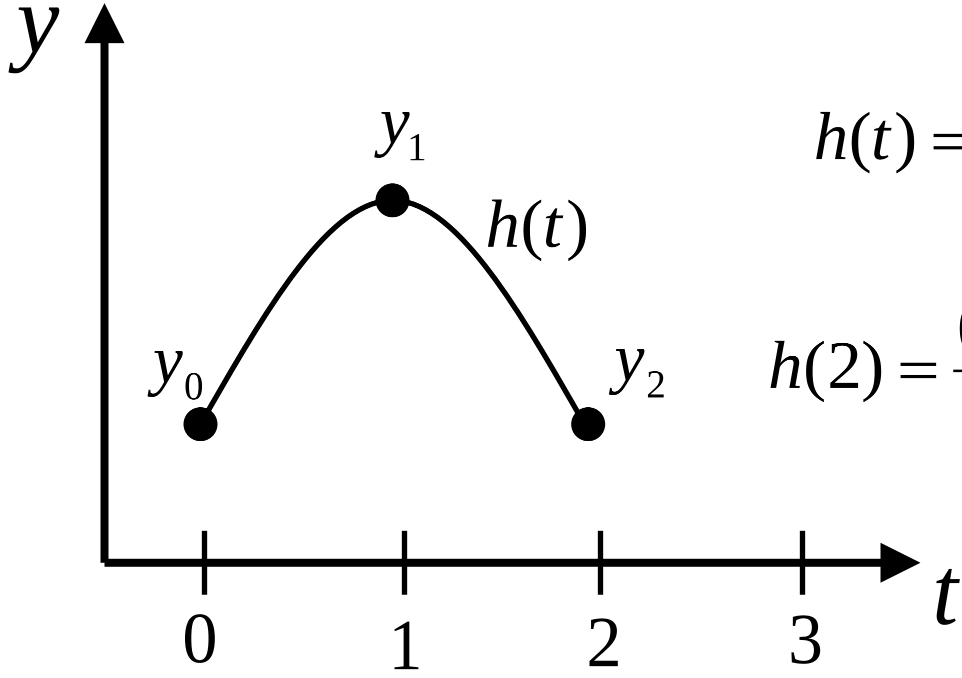
- Identical to matrix method but uses a geometric construction

$$f(t) = (1-t)y_0 + t\,y_1$$

$$g(t) = (2-t)y_1 + (t-1)\,y_2$$

$$h(t) = \frac{(2-t)\,f(t) + t\,g(t)}{2}$$

$$h(2) = \frac{(2-2)\,f(2) + 2\,g(2)}{2}$$



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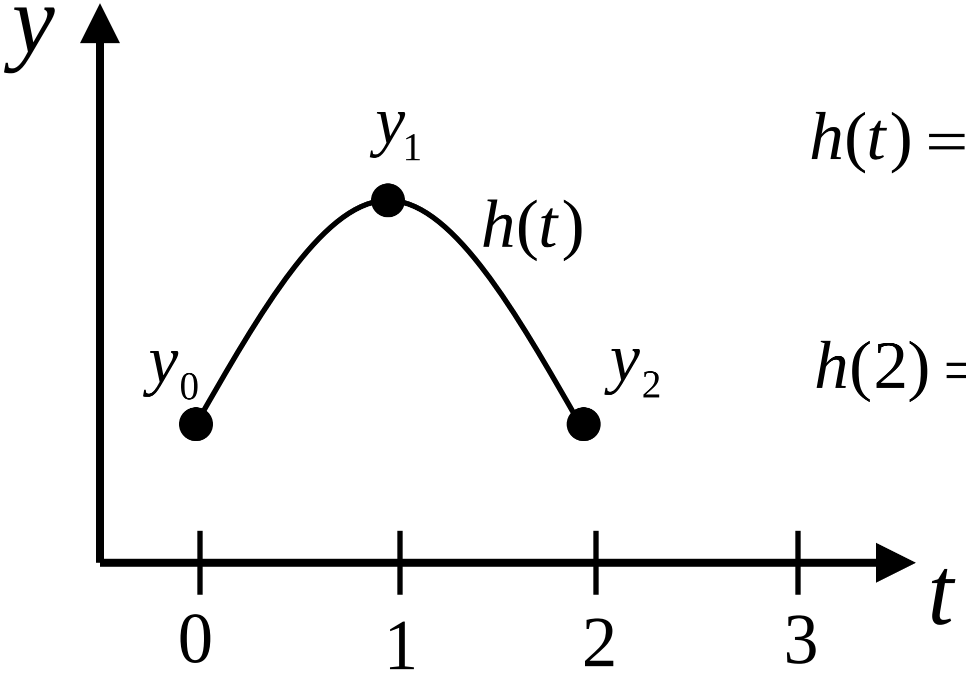
- Identical to matrix method but uses a geometric construction

$$f(t) = (1-t)y_0 + t\,y_1$$

$$g(t) = (2-t)y_1 + (t-1)\,y_2$$

$$h(t) = \frac{(2-t)\,f(t) + t\,g(t)}{2}$$

$$h(2) = \frac{2\,y_2}{2} = y_2$$



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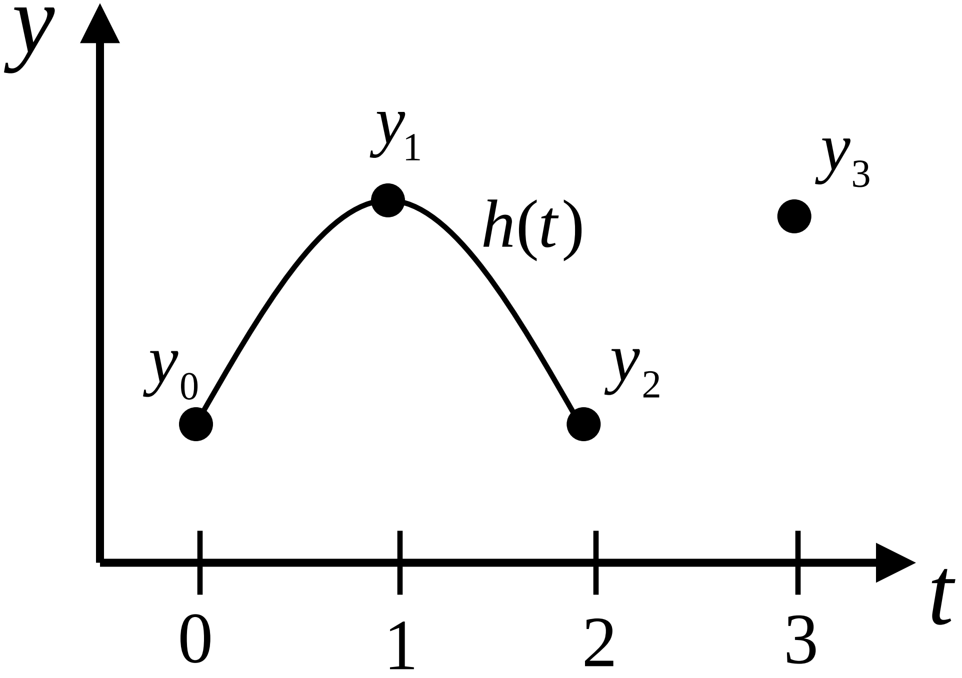

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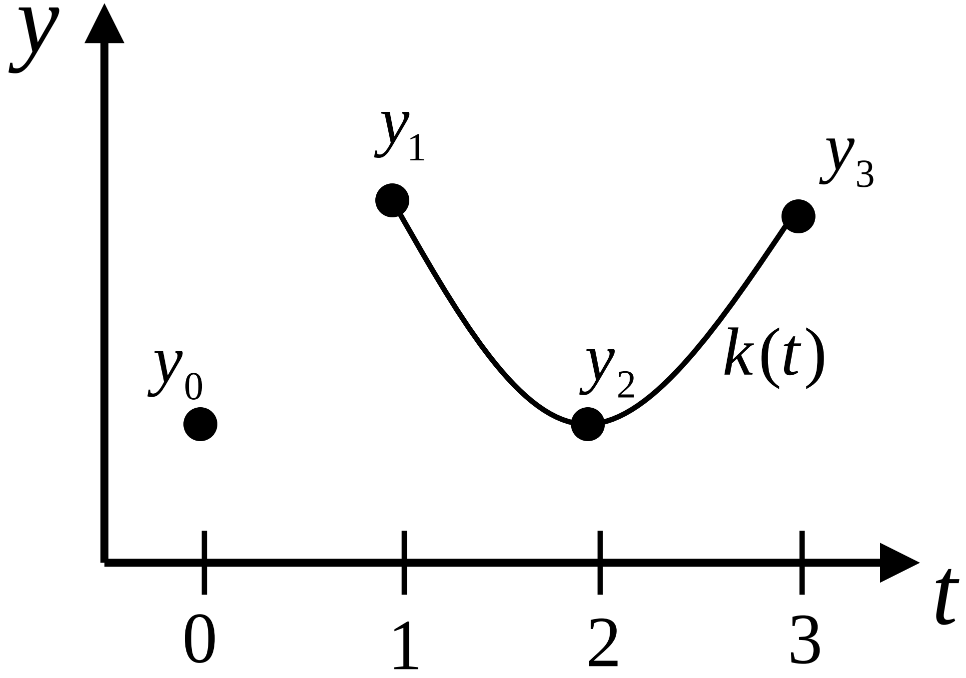
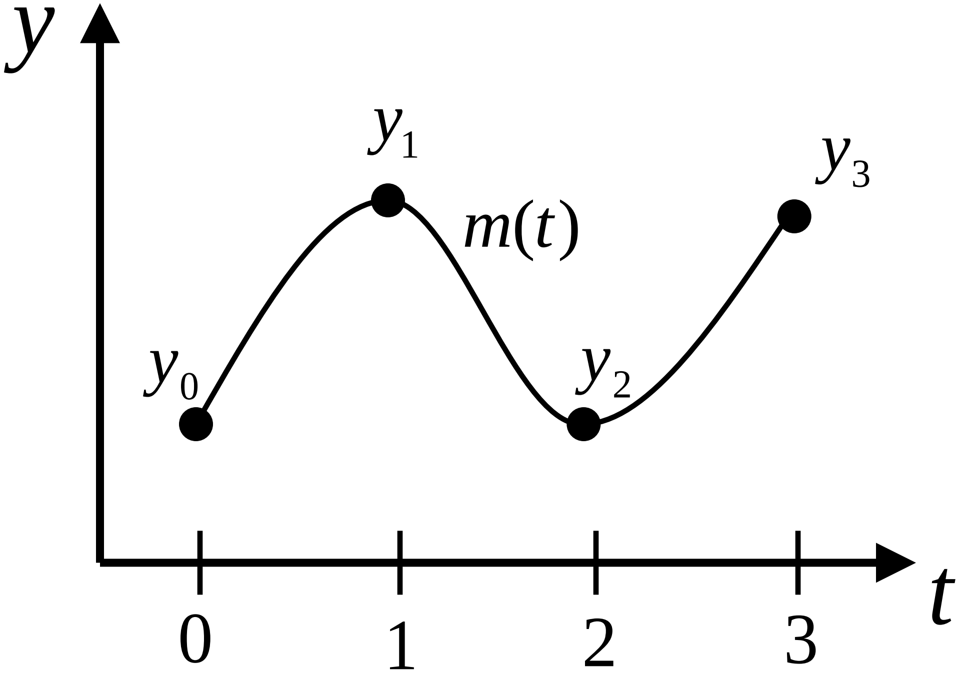


Lagrange Interpolation

- Identical to matrix method but uses a geometric construction

$$m(t) = \frac{(3-t)h(t) + t\,k(t)}{3}$$



Uniform Lagrange Interpolation

- Base Case
  - Linear interpolation between two points
- Inductive Step
  - Assume we have points $y_{\mathrm{i}}, \ldots, y_{n+1+\mathrm{i}}$
  - Build interpolating polynomials $f(t)$, $g(t)$ of degree n for $y_{\mathrm{i}}, \ldots, y_{n+i}$ and $y_{\mathrm{i}+1}, \ldots, y_{n+1+\mathrm{i}}$
  - $h(t) = \dfrac{(n+1+i-t)\,f(t)+(t-i)\,g(t)}{n+1}$
  - $h(t)$ interpolates all points $y_{\mathrm{i}}, \ldots, y_{n+1+\mathrm{i}}$ and is of degree $n+1$
  - Moreover, $h(t)$ is unique

Lagrange Evaluation Pseudocode

Lagrange(*pts*, *i*, *t*)

    n = length(*pts*)-2

    if *n* is 0

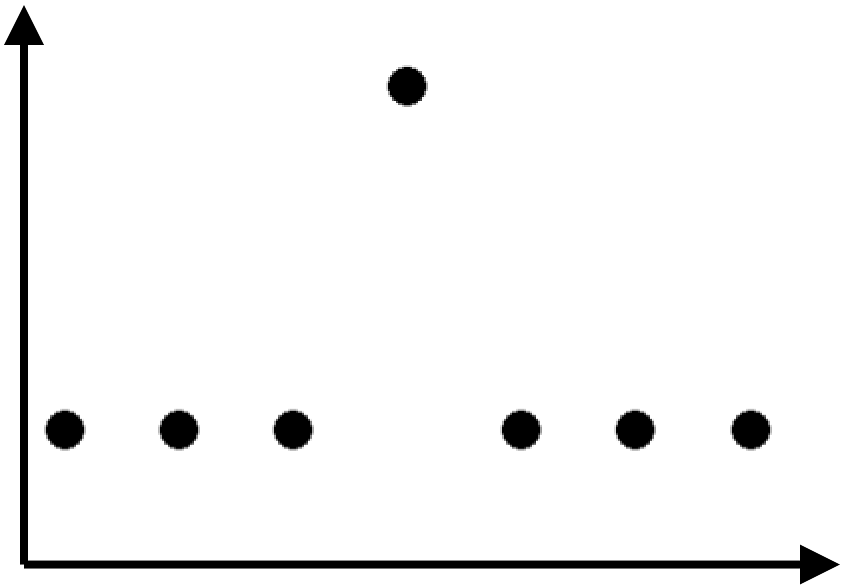
        return *pts*[0](*i*+1-*t*)+*pts*[1](*t*-*i*)

    f=Lagrange(*pts* without last element, *i*, *t*)

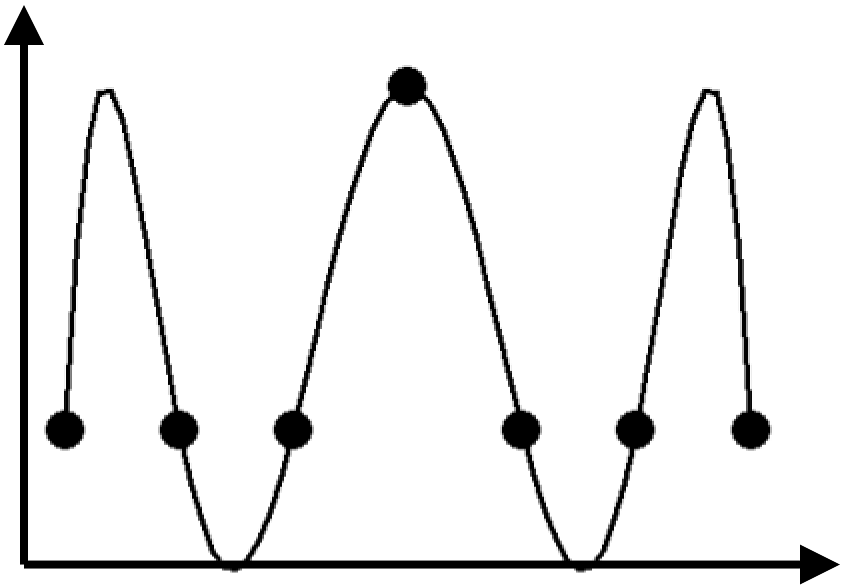
    g=Lagrange(*pts* without first element, *i*+1, *t*)

    return ((*n*+1+*i*-*t*)f+(*t*-*i*)*g*)/(*n*+1)

Problems with Interpolation

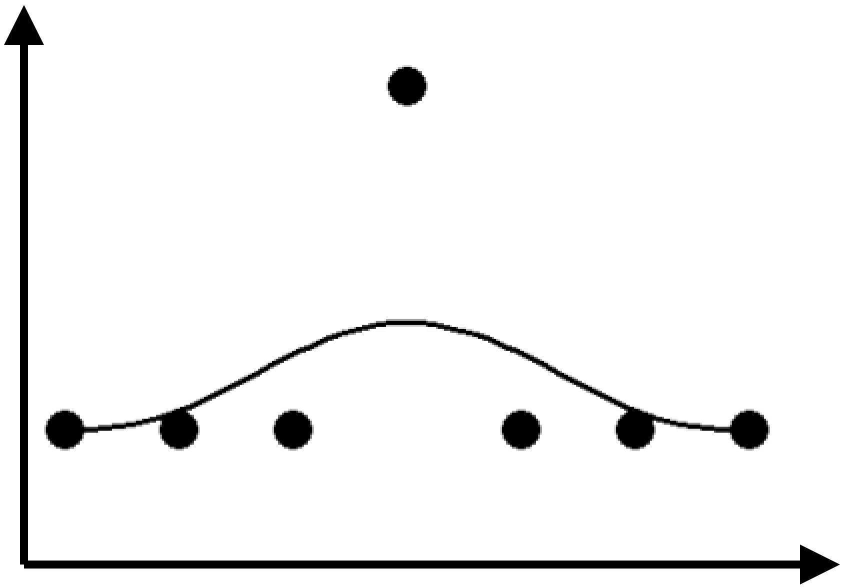


Problems with Interpolation



Bezier Curves

- Polynomial curves that seek to approximate rather than to interpolate



Bernstein Polynomials

- Degree 1: $(1-t)$, t
- Degree 2: $(1-t)^2$, $2(1-t)t$, t^2
- Degree 3: $(1-t)^3$, $3(1-t)^2t$, $3(1-t)t^2$, t^3

Bernstein Polynomials

- Degree 1: $(1\text{-}t)$, t
- Degree 2: $(1\text{-}t)^2$, $2(1\text{-}t)t$, t^2
- Degree 3: $(1\text{-}t)^3$, $3(1\text{-}t)^2t$, $3(1\text{-}t)t^2$, t^3
- Degree 4: $(1\text{-}t)^4$, $4(1\text{-}t)^3t$, $6(1\text{-}t)^2t^2$, $4(1\text{-}t)t^3$,t^4

Bernstein Polynomials

- Degree 1: $(1\text{-}t)$, t
- Degree 2: $(1\text{-}t)^2$, $2(1\text{-}t)t$, t^2
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- Degree 4: $(1\text{-}t)^4$, $4(1\text{-}t)^3t$, $6(1\text{-}t)^2t^2$, $4(1\text{-}t)t^3$,t^4
- Degree 5: $(1\text{-}t)^5$, $5(1\text{-}t)^4t$, $10(1\text{-}t)^3t^2$, $10(1\text{-}t)^2t^3$,$5(1\text{-}t)t^4$,t^5
- …
- Degree n: $\binom{n}{i}(1-t)^{n-i}t^i$ for $0 \le i \le n$

Bezier Curves

$$p(t) = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i p_i$$

Bezier Curves

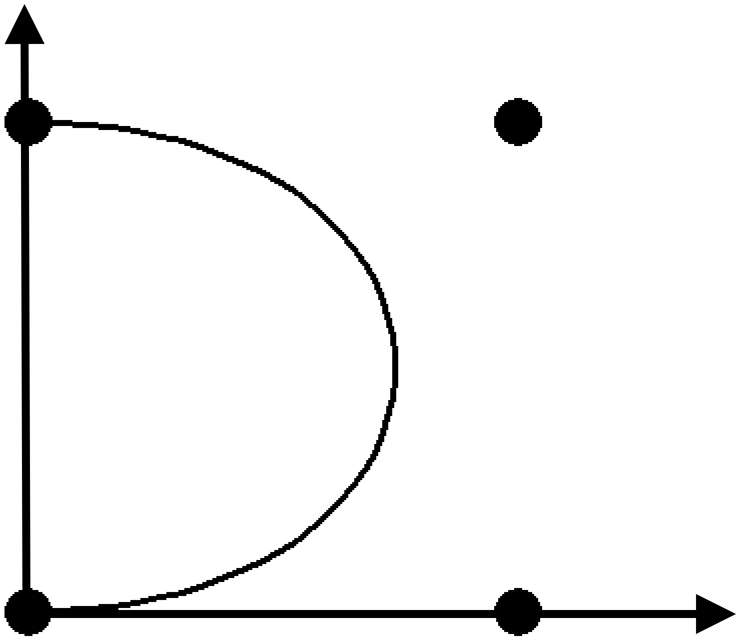
$$p(t) = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i p_i$$

$$p(t) = (1-t)^3(0,0) + 3(1-t)^2 t(1,0) + 3(1-t)t^2(1,1) + t^3(0,1)$$

Bezier Curves

$$p(t) = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i p_i$$

$$p(t) = (3(1-t)t, (3-2t)t^2)$$



Bezier Curve Properties

- Interpolate end-points

$$p(t) = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i p_i$$

Bezier Curve Properties

- Interpolate end-points

$$p(0) = \sum_{i=0}^{n} \binom{n}{i} (1-0)^{n-i} 0^i p_i = p_0$$

Bezier Curve Properties

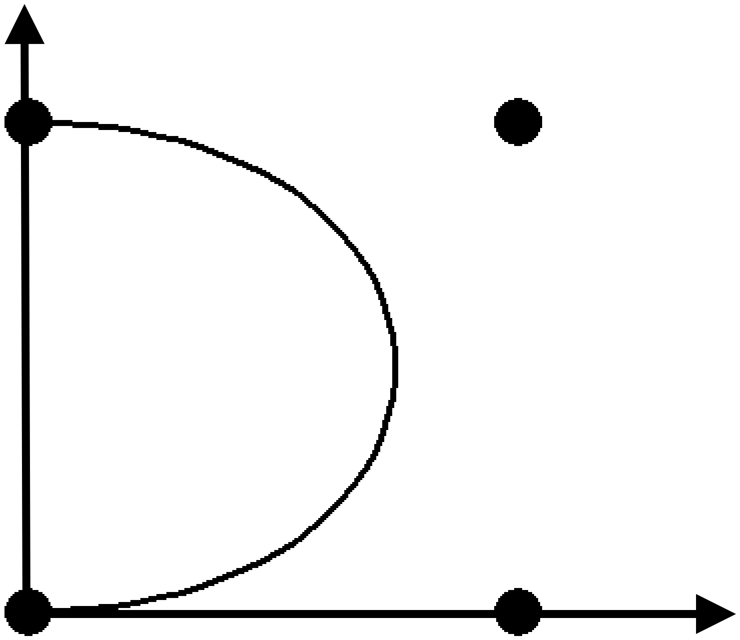
- Interpolate end-points

$$p(1)=\sum_{i=0}^{n}\binom{n}{i}(1-1)^{n-i}1^{i}p_i=p_n$$

Bezier Curve Properties

- Interpolate end-points

$$p(1) = \sum_{i=0}^{n} \binom{n}{i} (1-1)^{n-i} 1^i \, p_i = p_n$$



Bezier Curve Properties

- Interpolate end-points
- Tangent at end-points in direction of first/last edge

$$p(t) = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i p_i$$

Bezier Curve Properties

- Interpolate end-points
- Tangent at end-points in direction of first/last edge

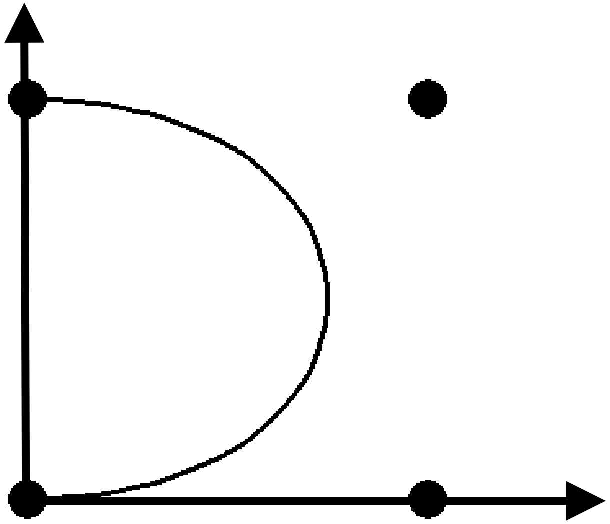
$$\frac{\partial p(t)}{\partial t} = \frac{\partial}{\partial t} \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^{i} p_i = \sum_{i=0}^{n} \binom{n}{i} p_i \left(i(1-t)^{n-i} t^{i-1} - (n-i)(1-t)^{n-i-1} t^{i} \right)$$

Bezier Curve Properties

- Interpolate end-points
- Tangent at end-points in direction of first/last edge

$$\frac{\partial p(0)}{\partial t} = n(p_1 - p_0)$$

$$\frac{\partial p(1)}{\partial t} = n(p_n - p_{n-1})$$

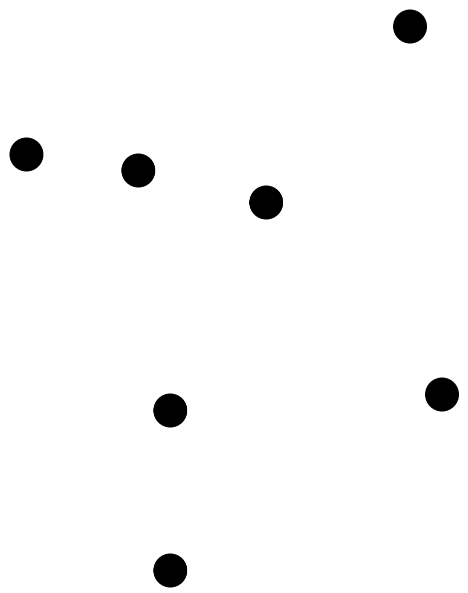


Bezier Curve Properties

- Interpolate end-points
- Tangent at end-points in direction of first/last edge
- Curve lies within the convex hull of the control points

Convex Hull

- The smallest convex set containing all points p_i



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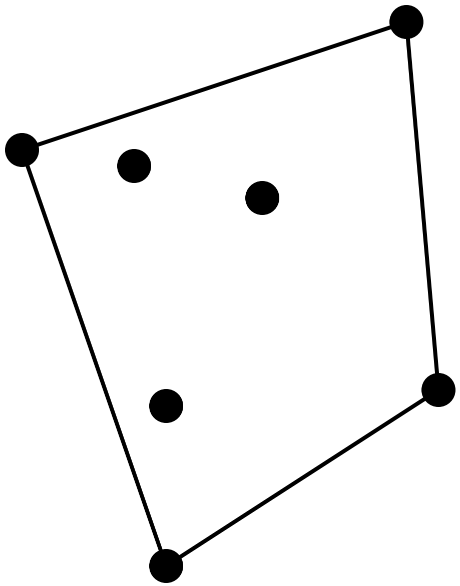

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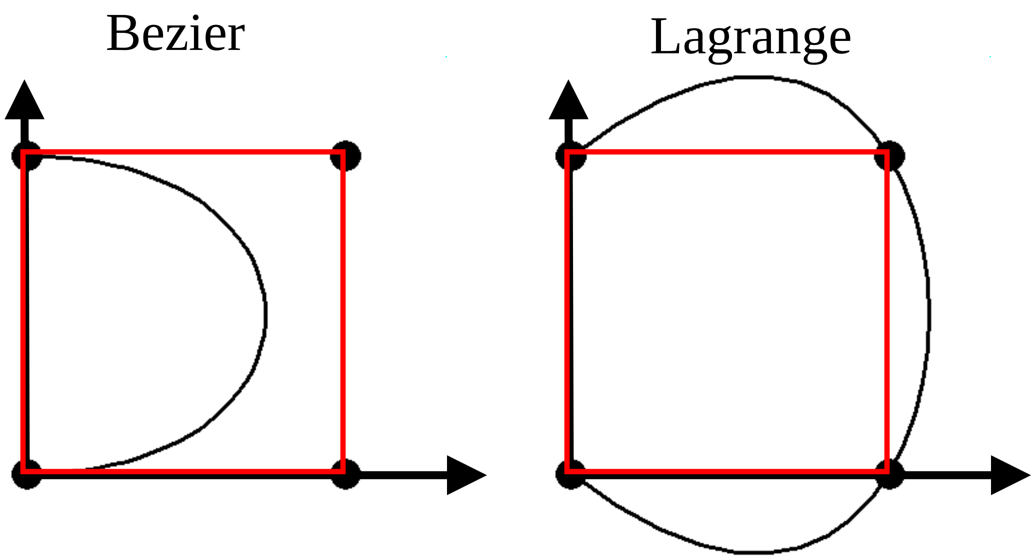
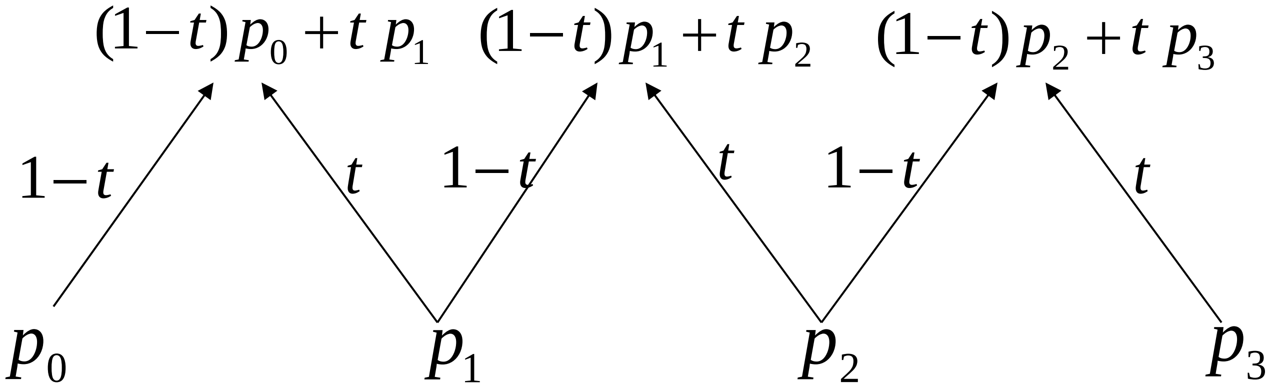


Pyramid Algorithms for Bezier Curves

- Polynomials aren't pretty
- Is there an easier way to evaluate the equation of a Bezier curve?

$$p(t) = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i p_i$$

Pyramid Algorithms for Bezier Curves



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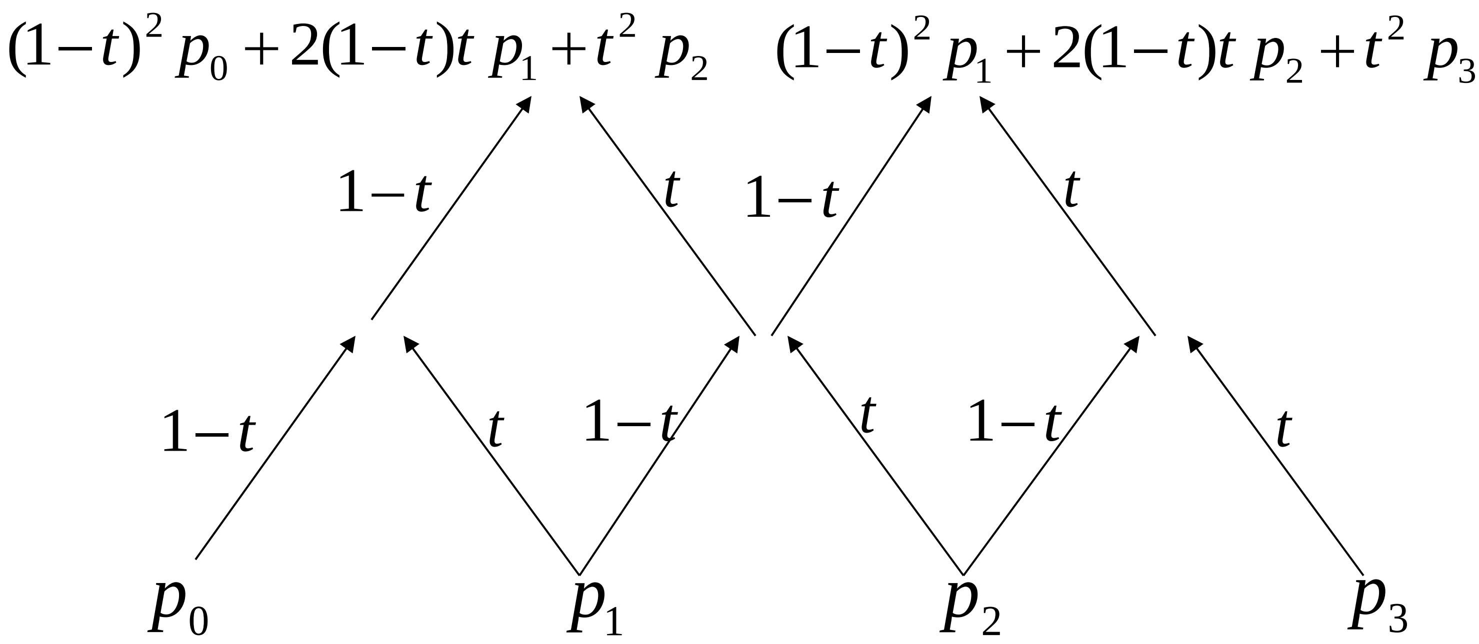
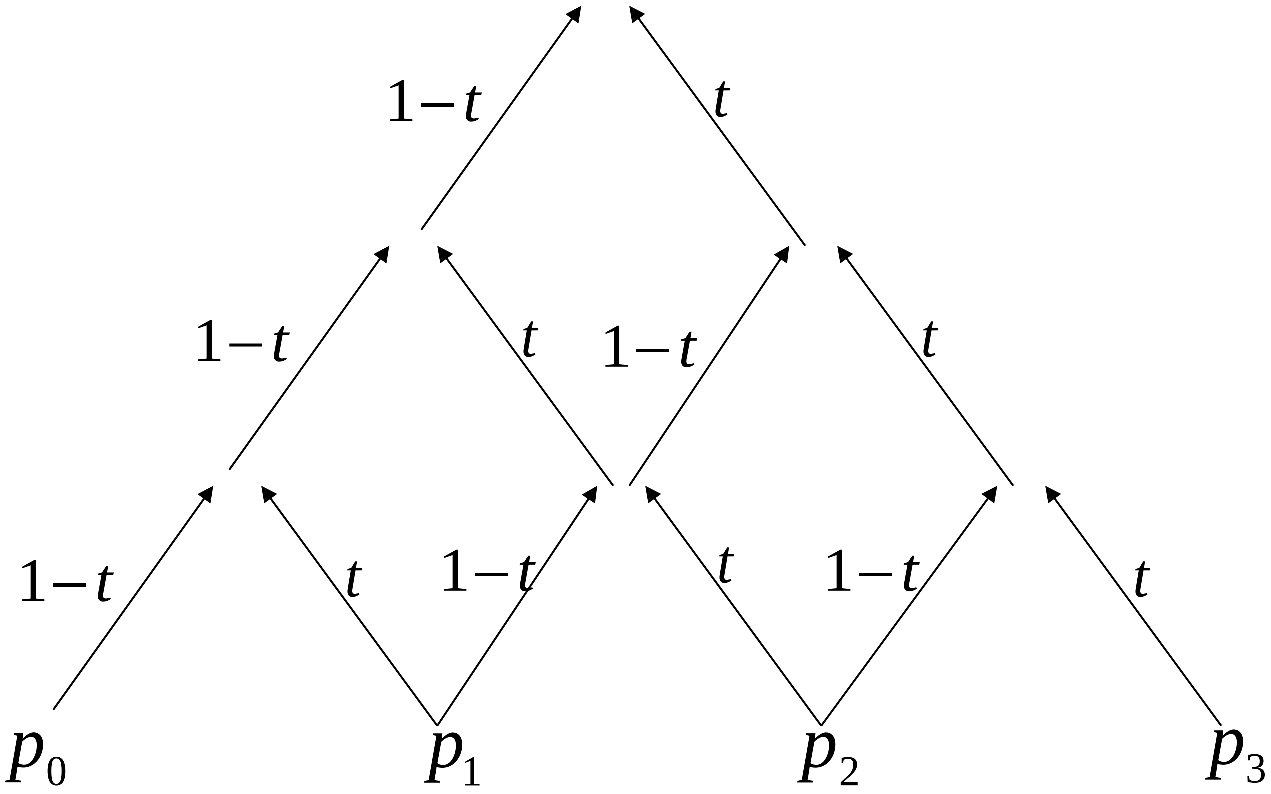


Pyramid Algorithms for Bezier Curves

$$(1-t)^3 p_0 + 3(1-t)^2 t\, p_1 + 3(1-t)t^2 p_2 + t^3\, p_3$$

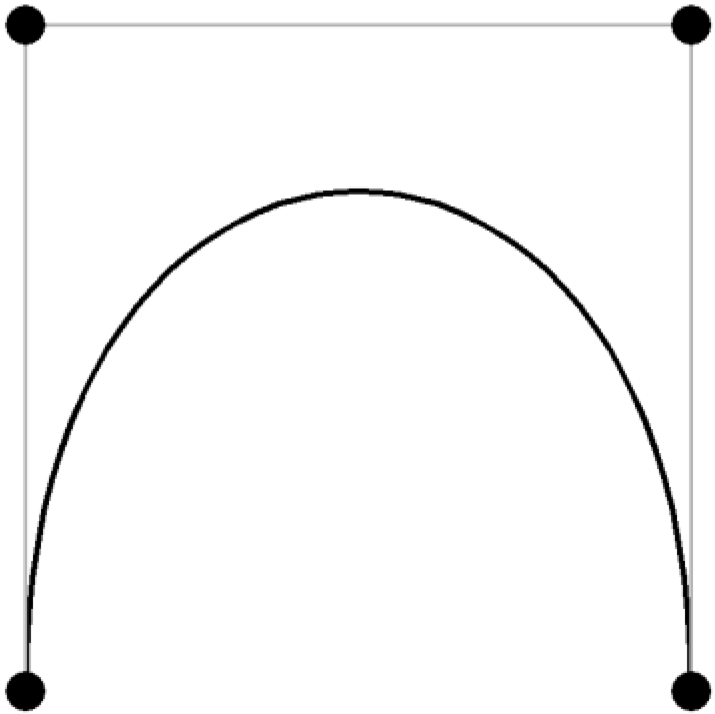


Bezier Evaluation Pseudocode

```
Bezier(pts, t)
    if ( length(pts) is 1)
        return pts[0]
    newPts={}
    For ( i = 0, i < length(pts) – 1, i++ )
        Add to newPts (1-t)pts[i] + t pts[i+1]
    return Bezier ( newPts, t )
```

Subdividing Bezier Curves

- Given a single Bezier curve, construct two smaller Bezier curves whose union is exactly the original curve



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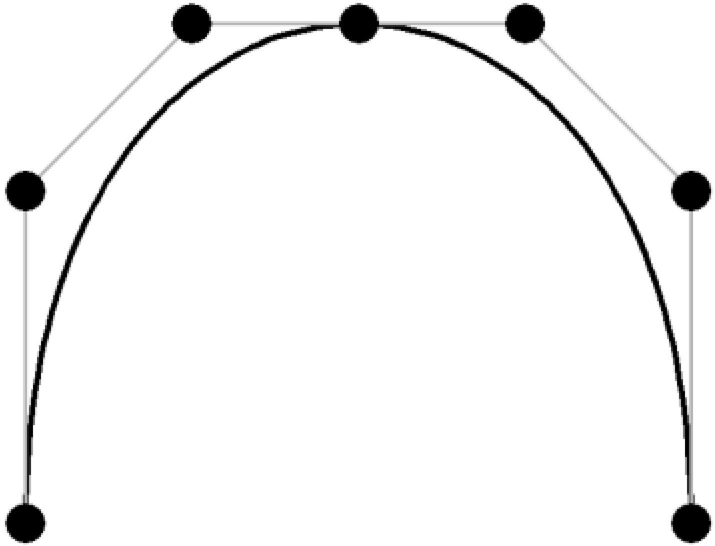

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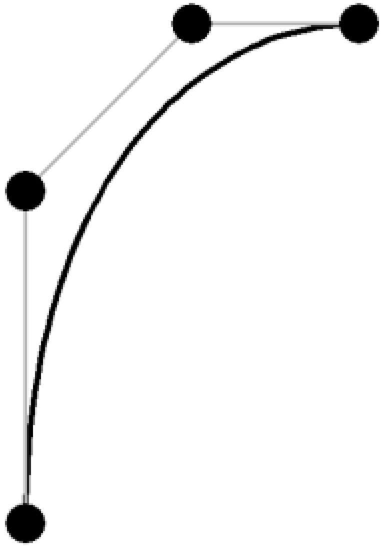

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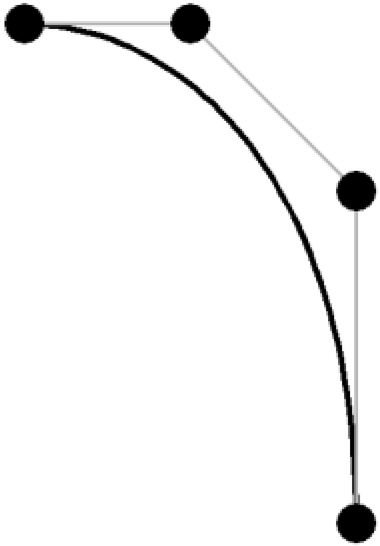

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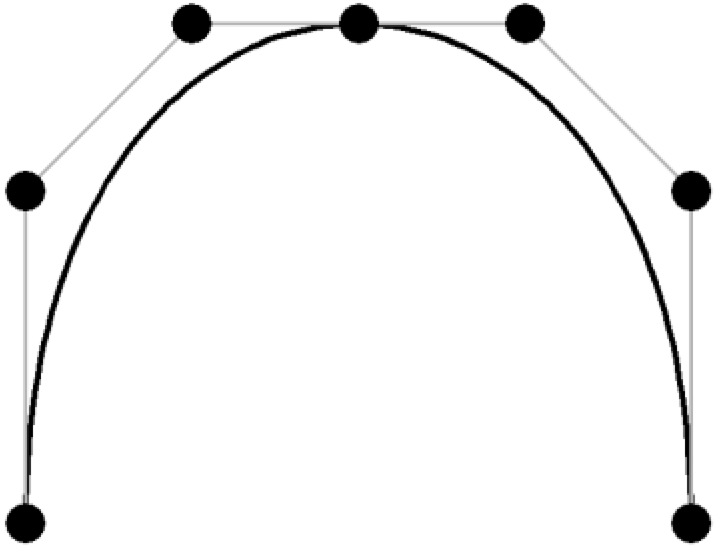
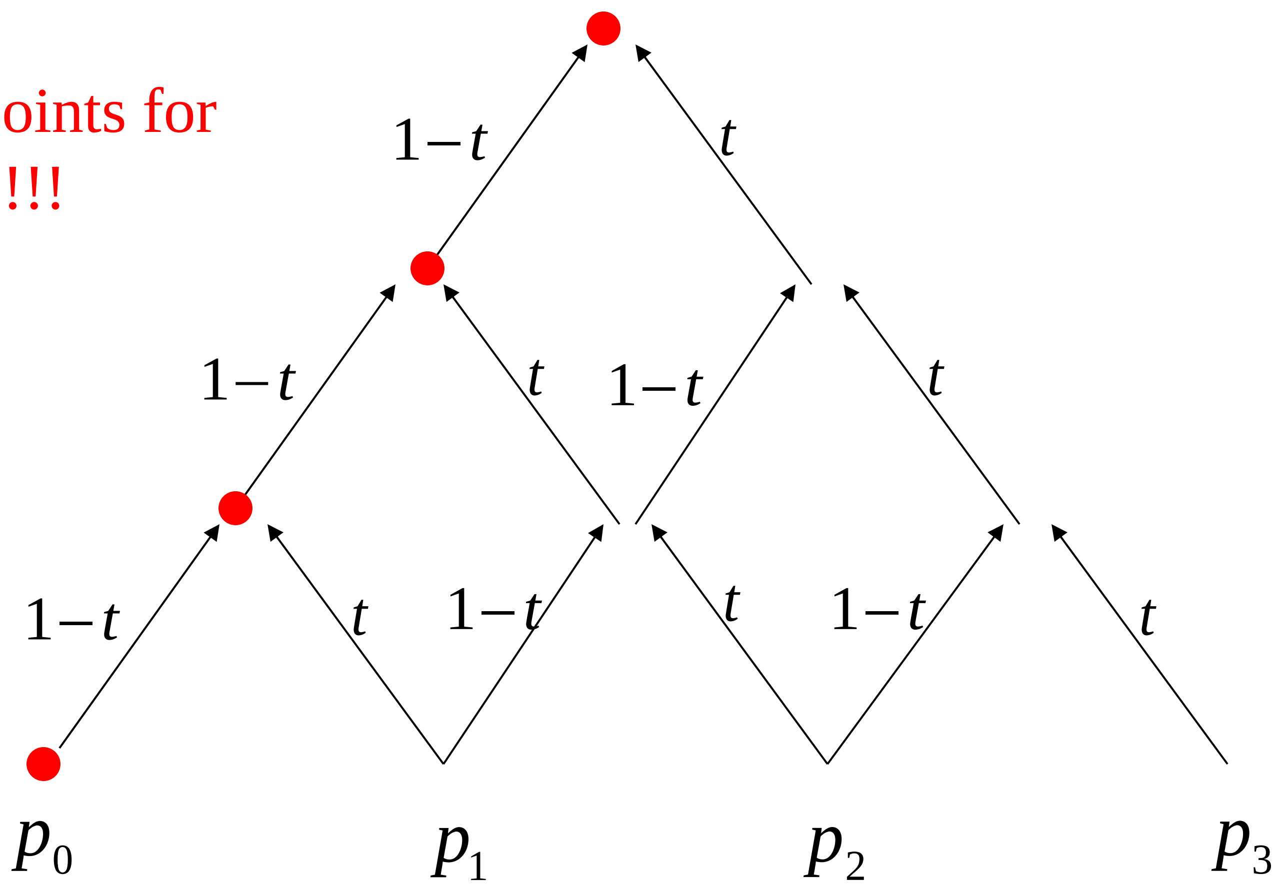


Subdividing Bezier Curves

Control points for
left curve!!!



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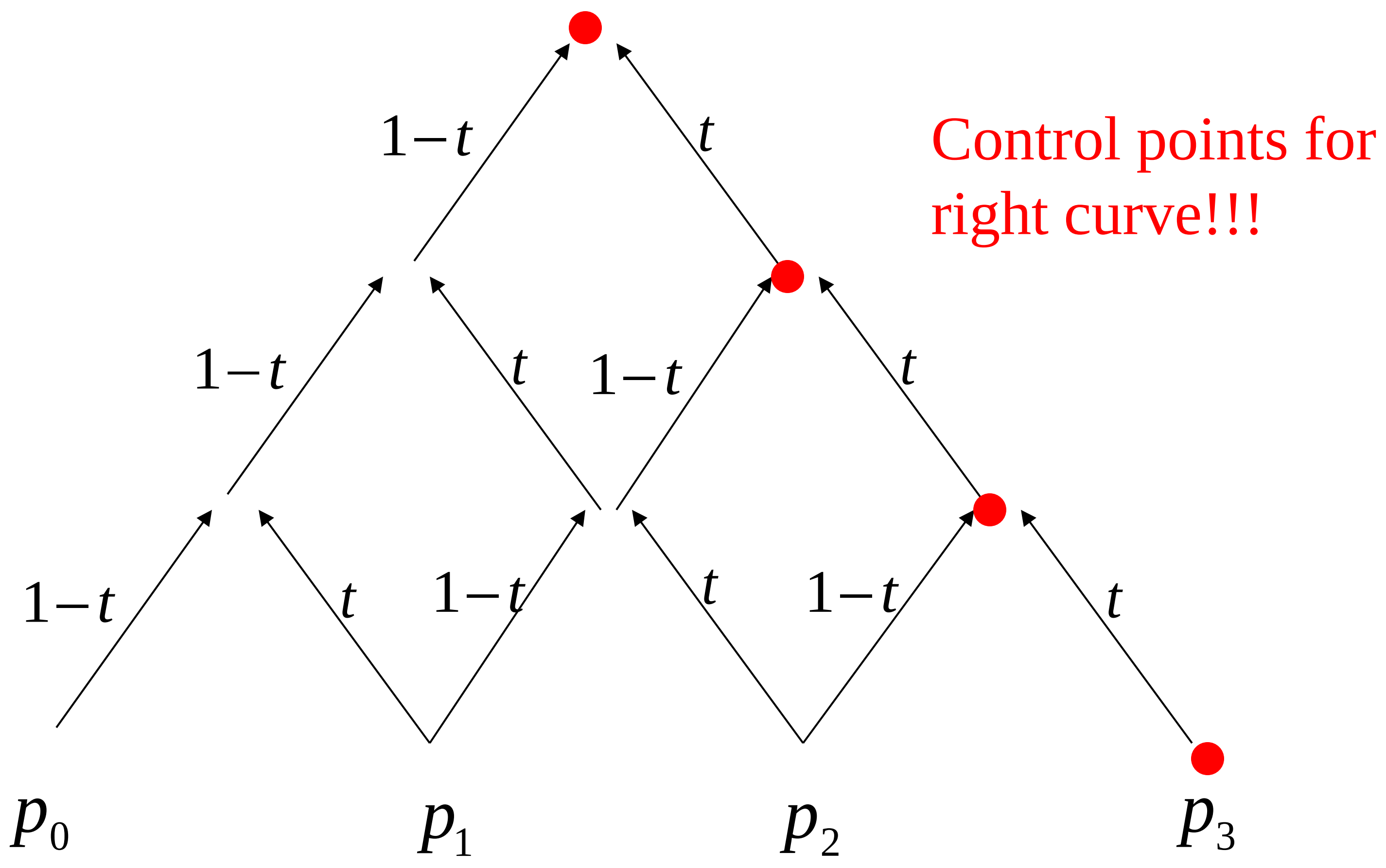

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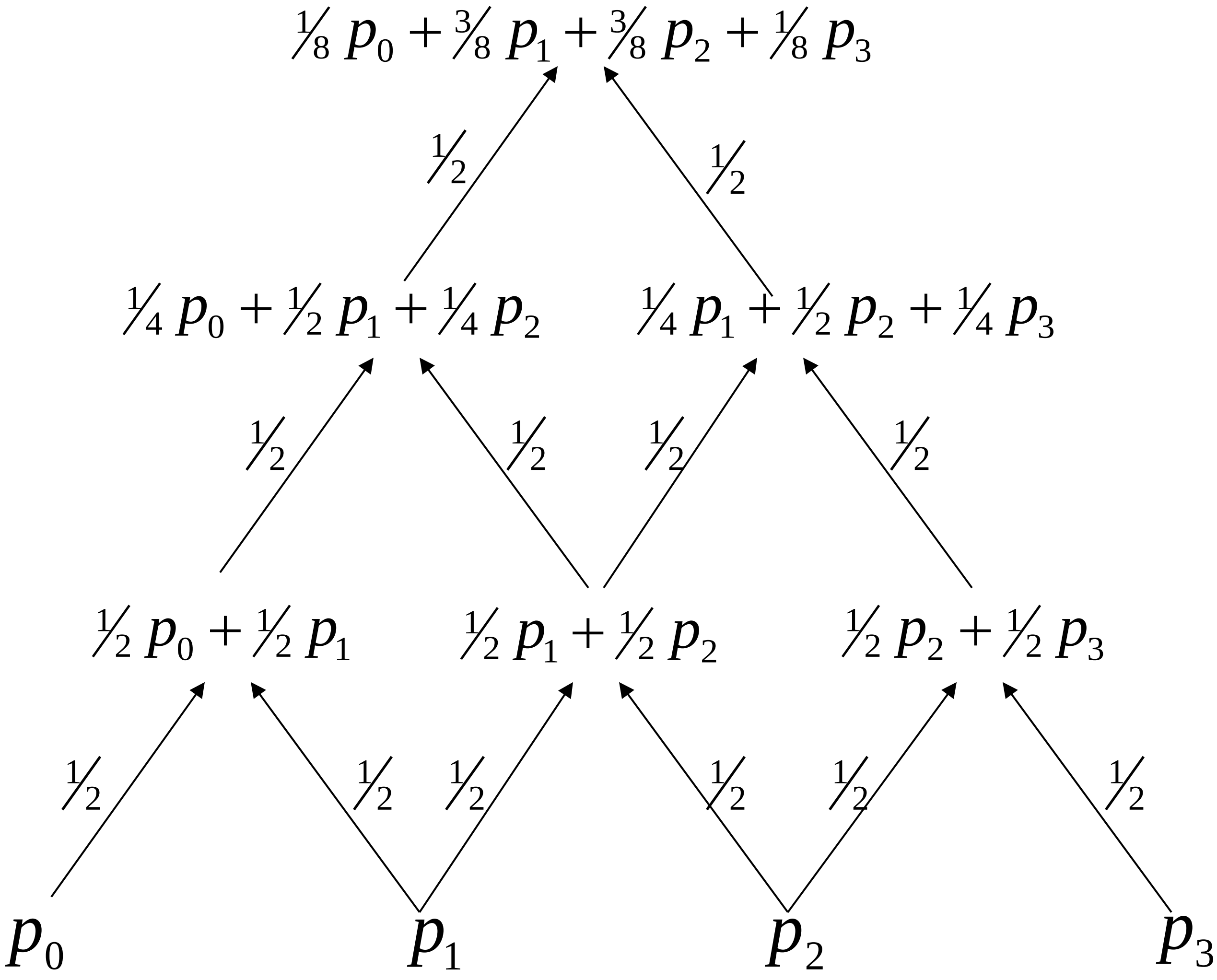

Adaptive Rendering of Bezier Curves

- If control polygon is close to a line, draw the control polygon
- If not, subdivide and recur on subdivided pieces

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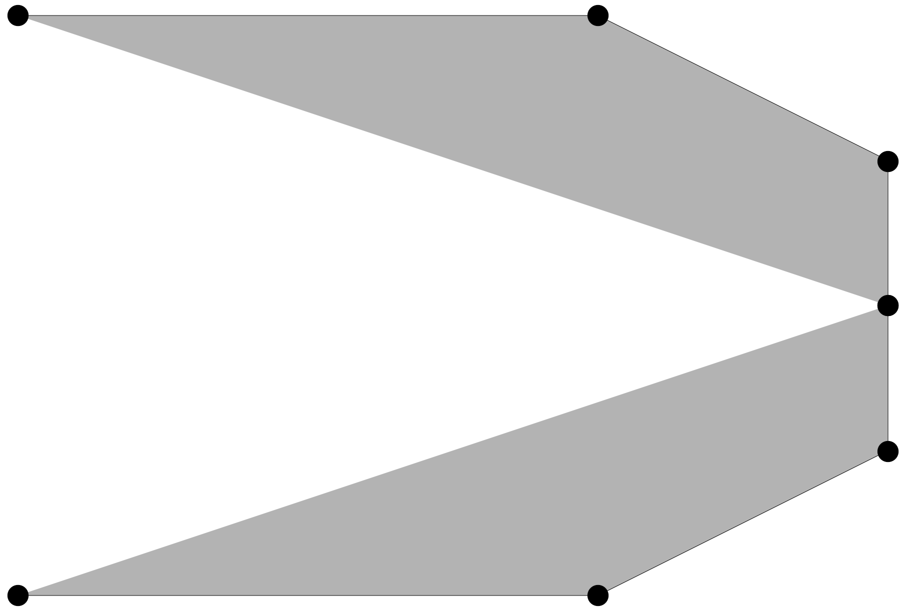

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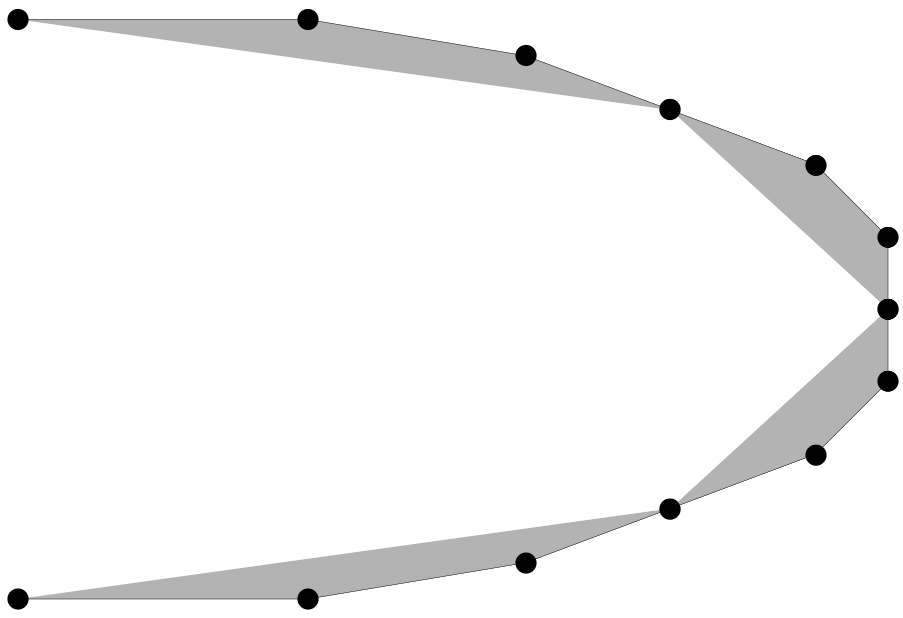

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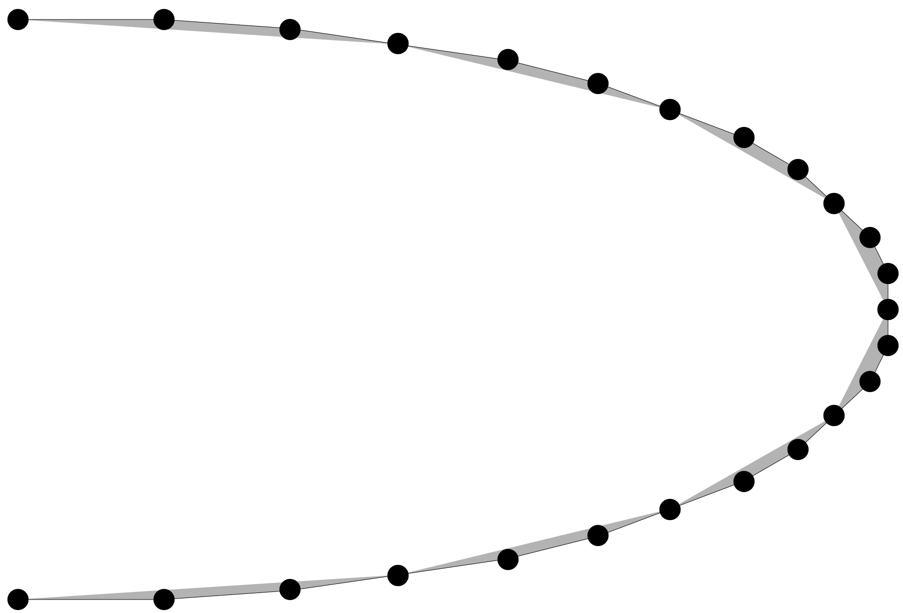

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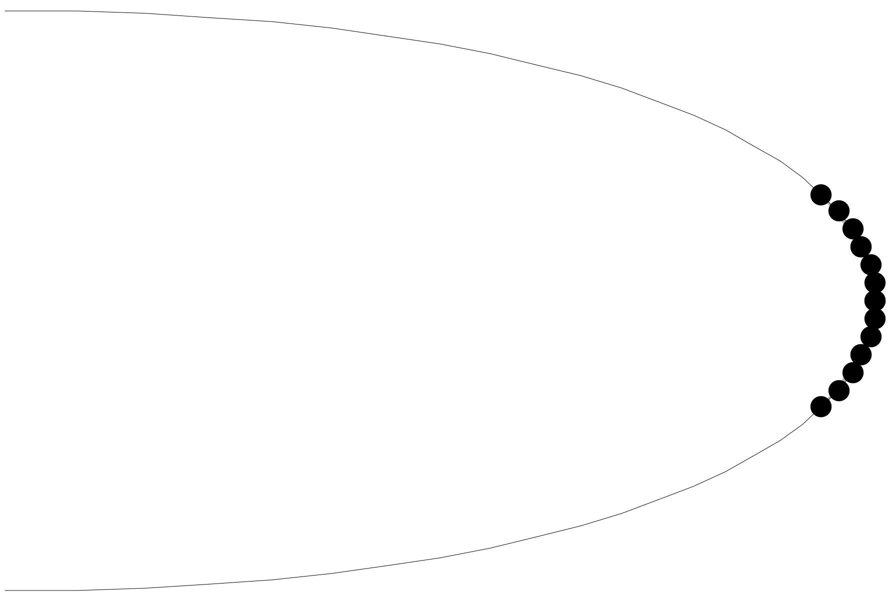

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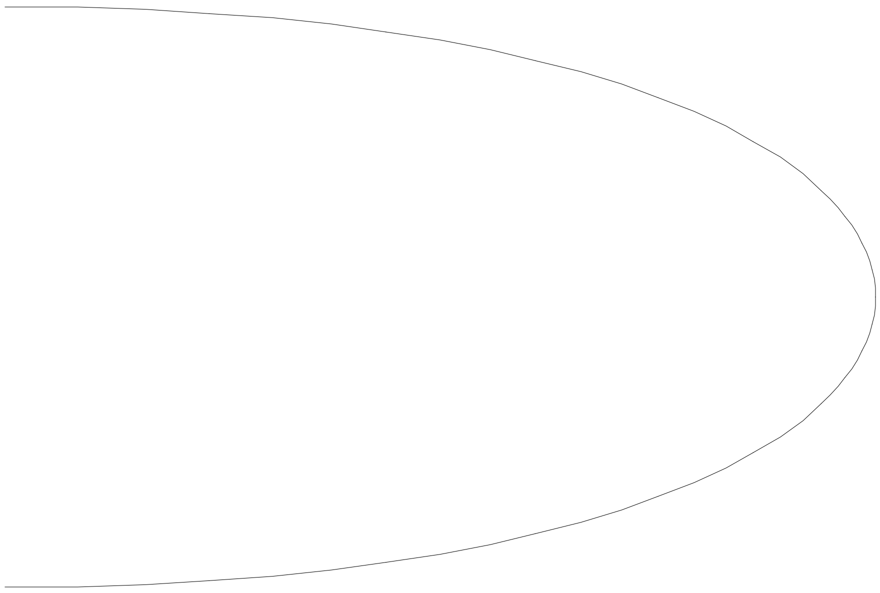


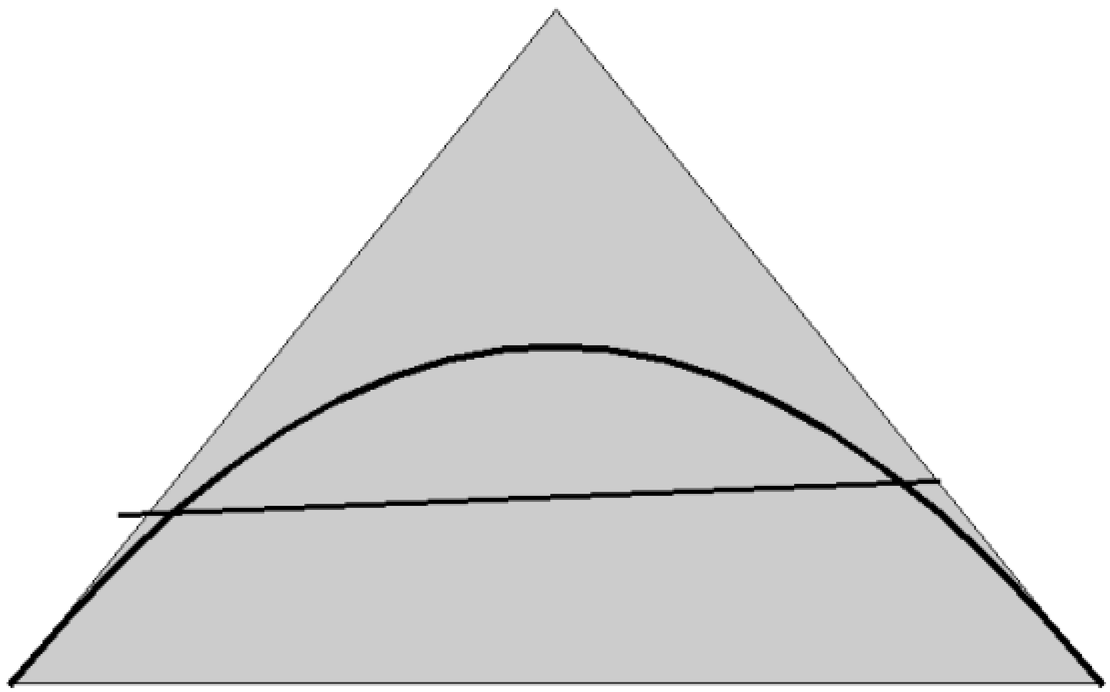
Applications: Intersection

- Given two Bezier curves, determine if and where they intersect

Applications: Intersection

- Check if convex hulls intersect
- If not, return no intersection
- If both convex hulls can be approximated with a straight line, intersect lines and return intersection
- Otherwise subdivide and recur on subdivided pieces

Applications: Intersection



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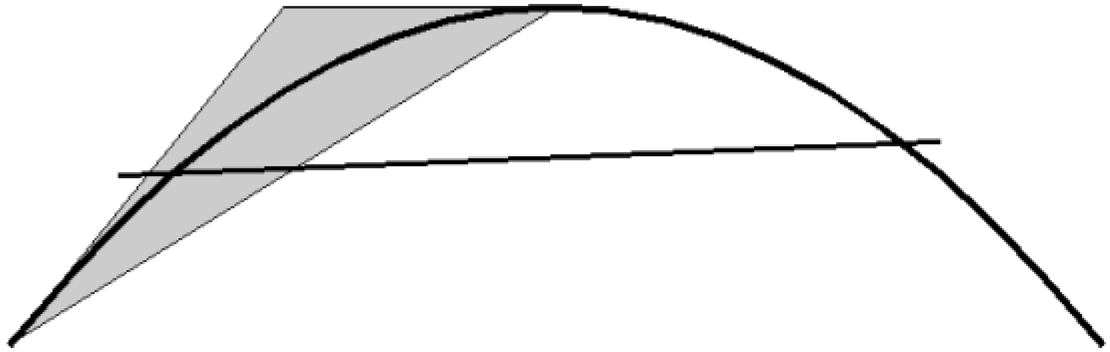

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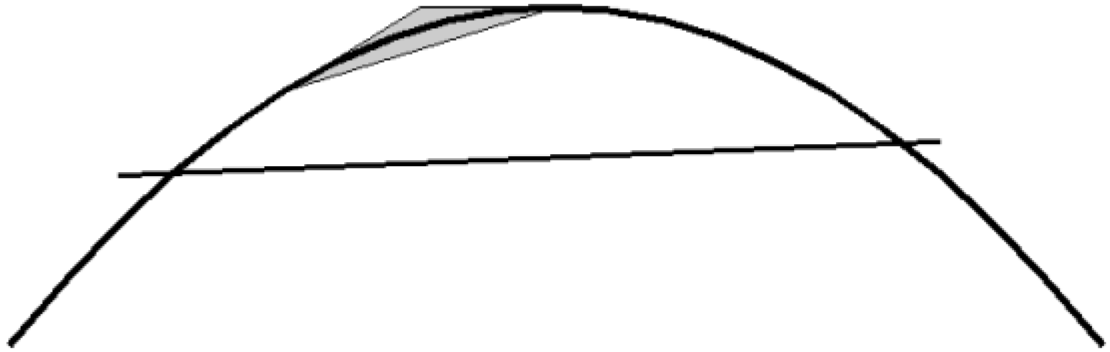

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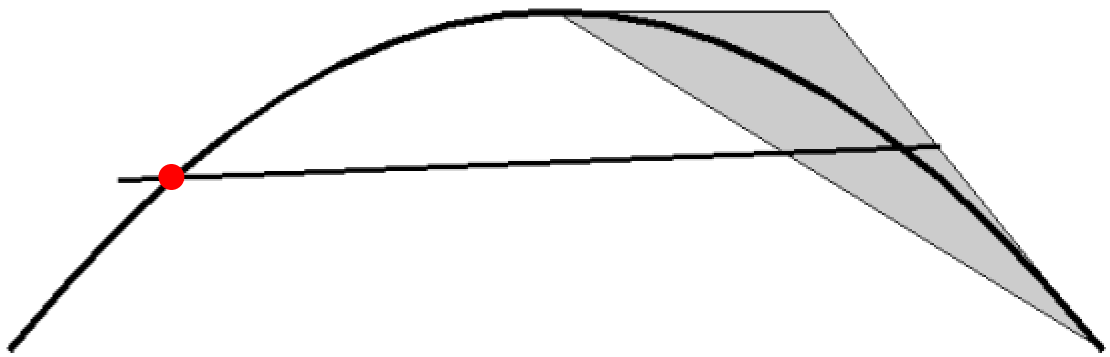

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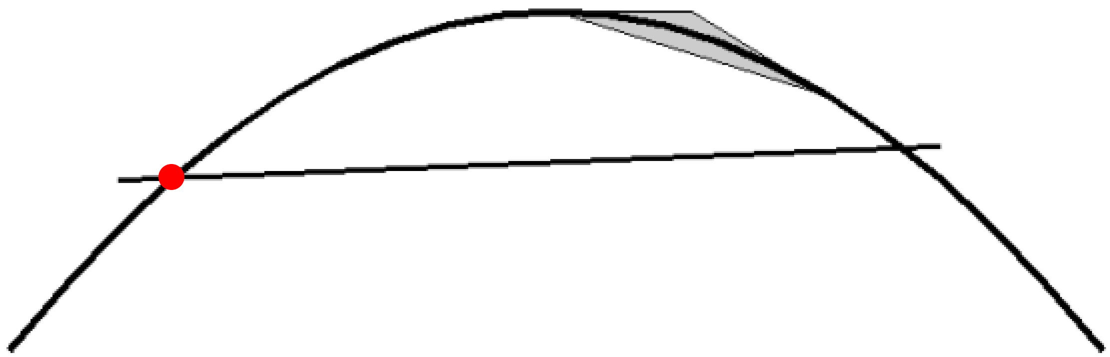

Applications: Intersection

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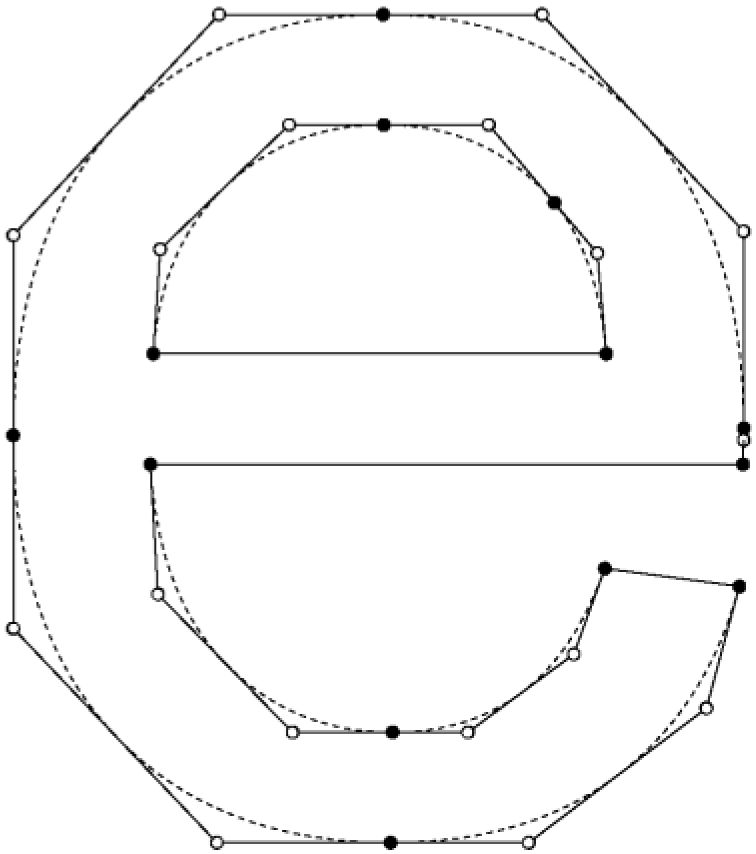
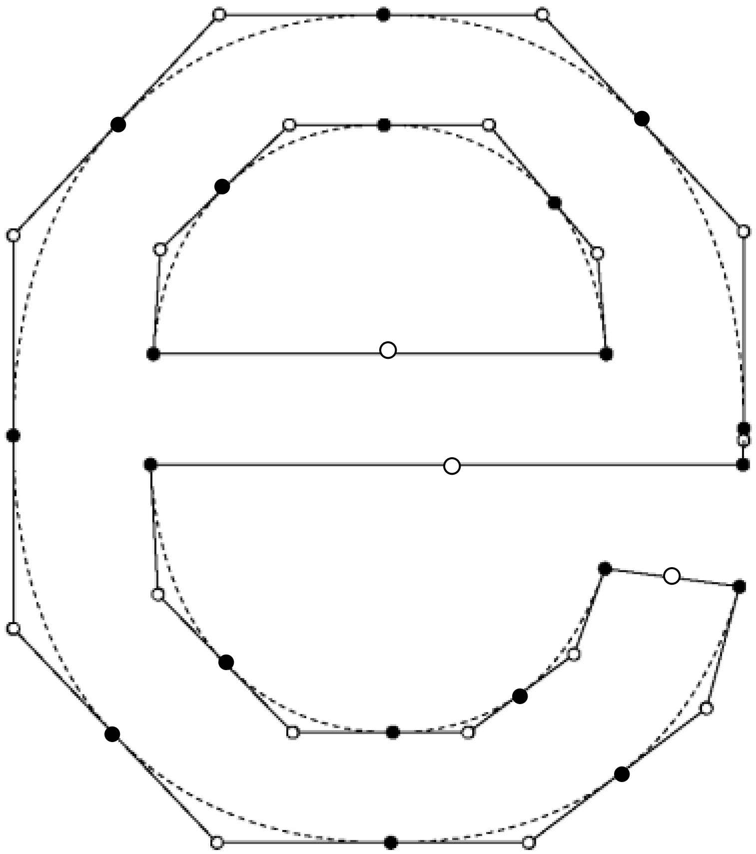
Applications: Intersection

Application: Font Rendering

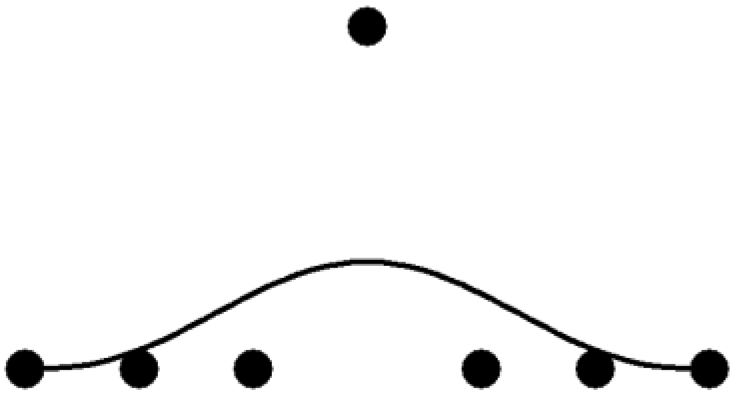


Application: Font Rendering



Problems with Bezier Curves

- More control points means higher degree
- Moving one control point affects the entire curve



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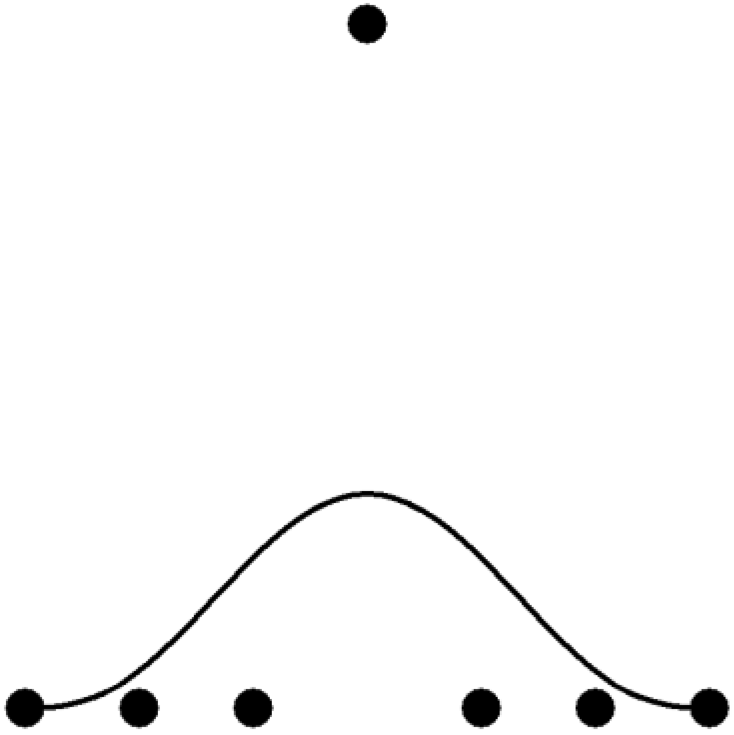
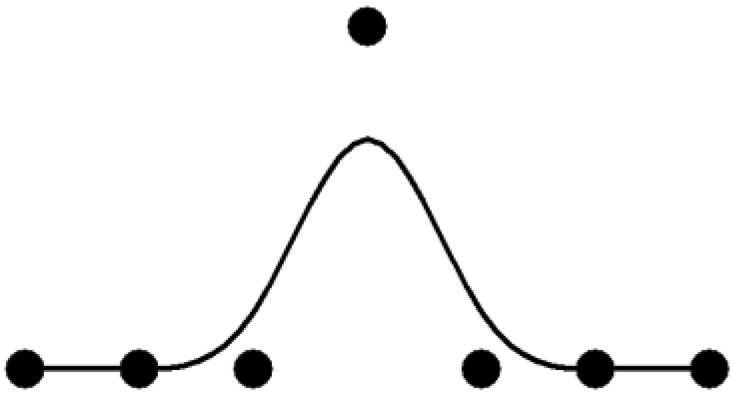


B-spline Curves

- Not a single polynomial, but lots of polynomials that meet together smoothly
- Local control



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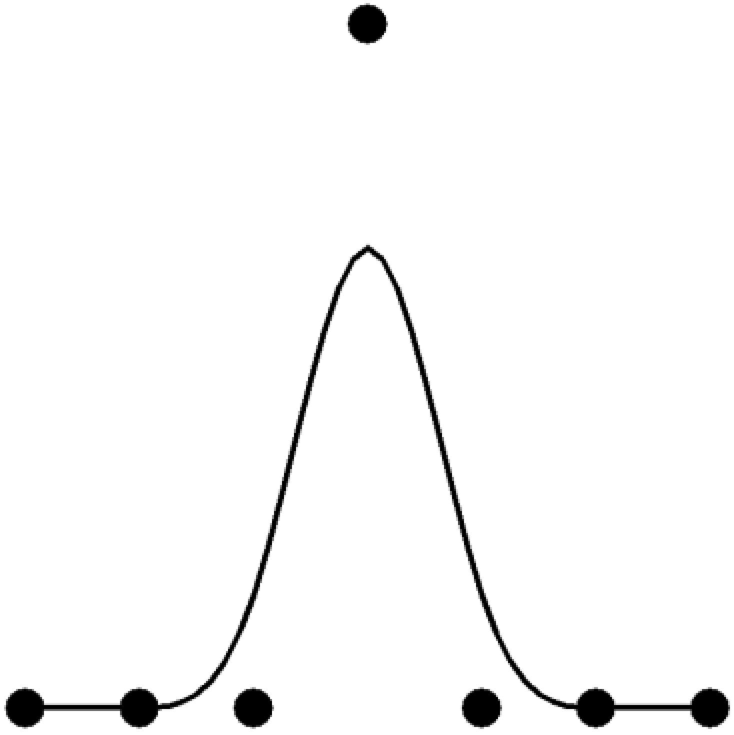


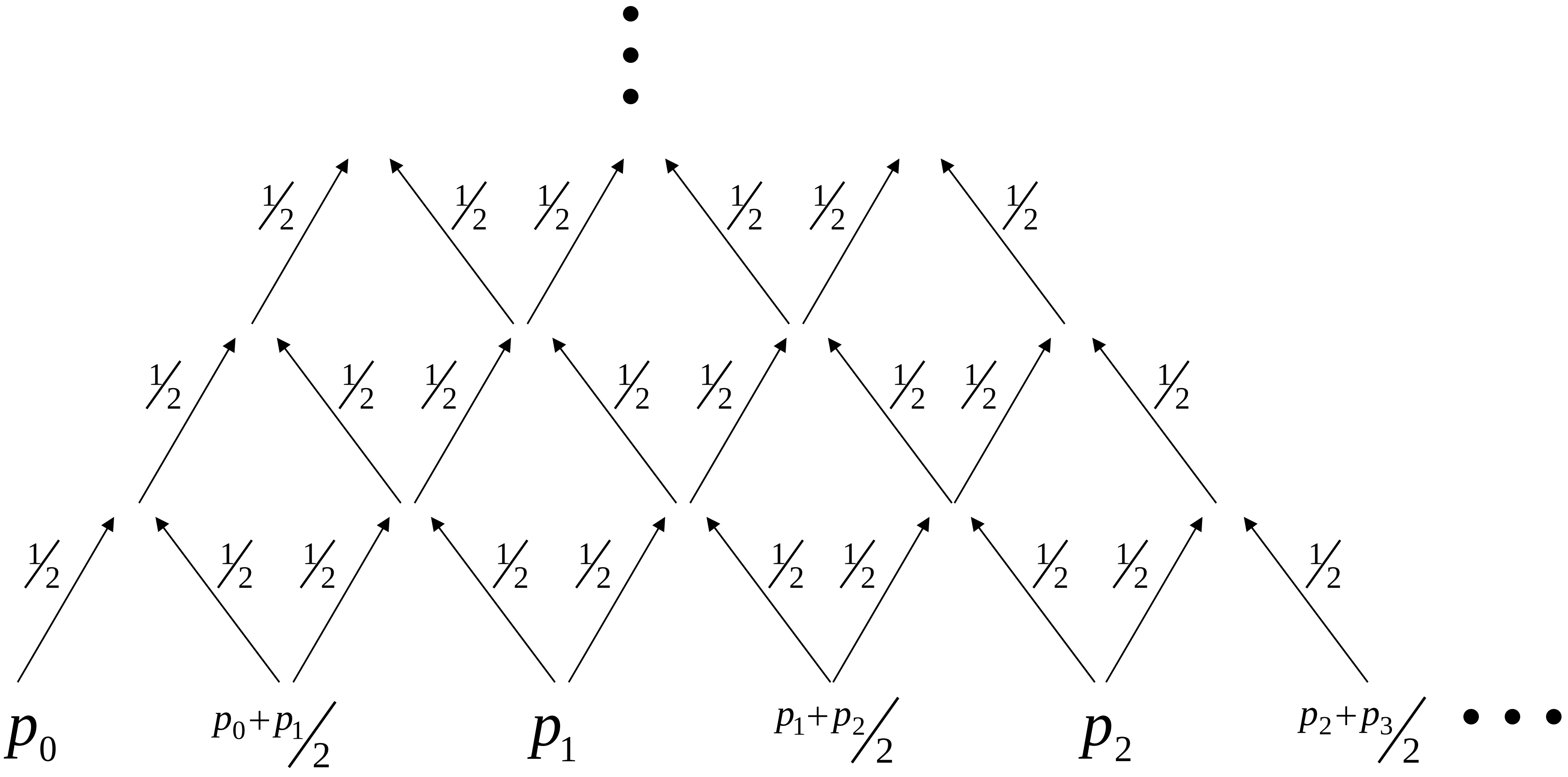
Rendering B-spline Curves

Lane-Reisenfeld subdivision algorithm

- Linearly subdivide the curve by inserting the midpoint on each edge
- Perform averaging by replacing each edge by its midpoint d times

Subdivide the curve repeatedly

Rendering B-spline Curves



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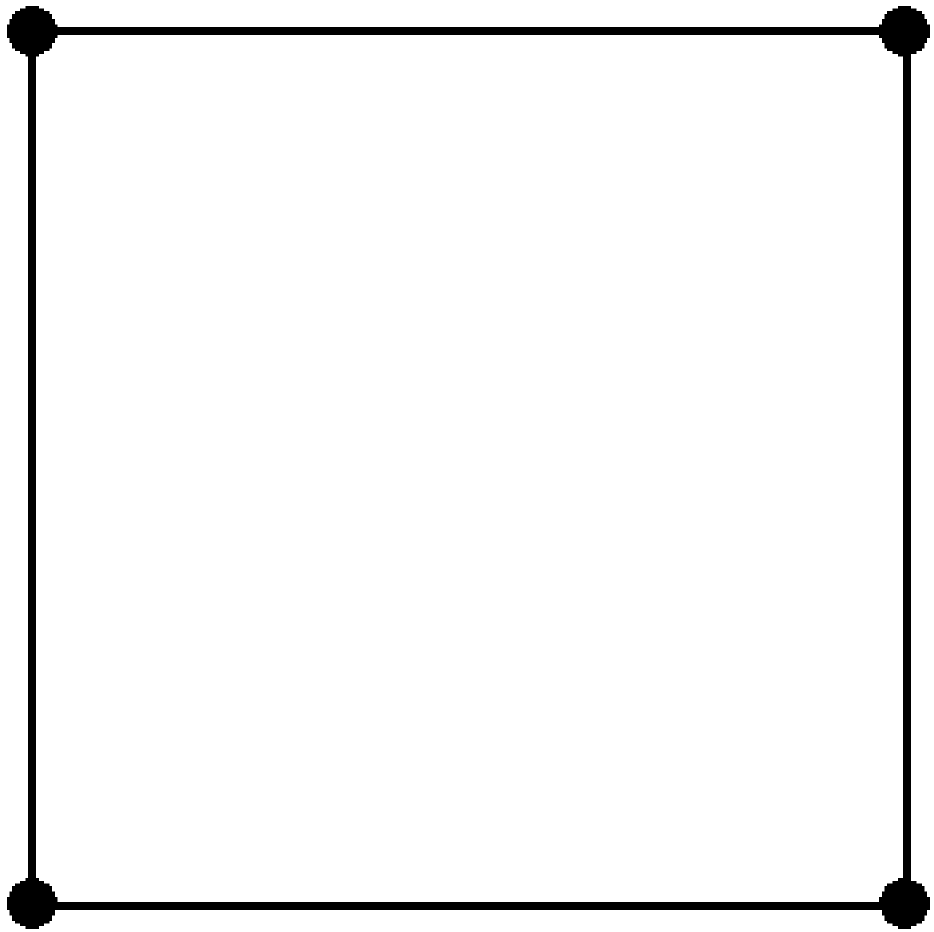

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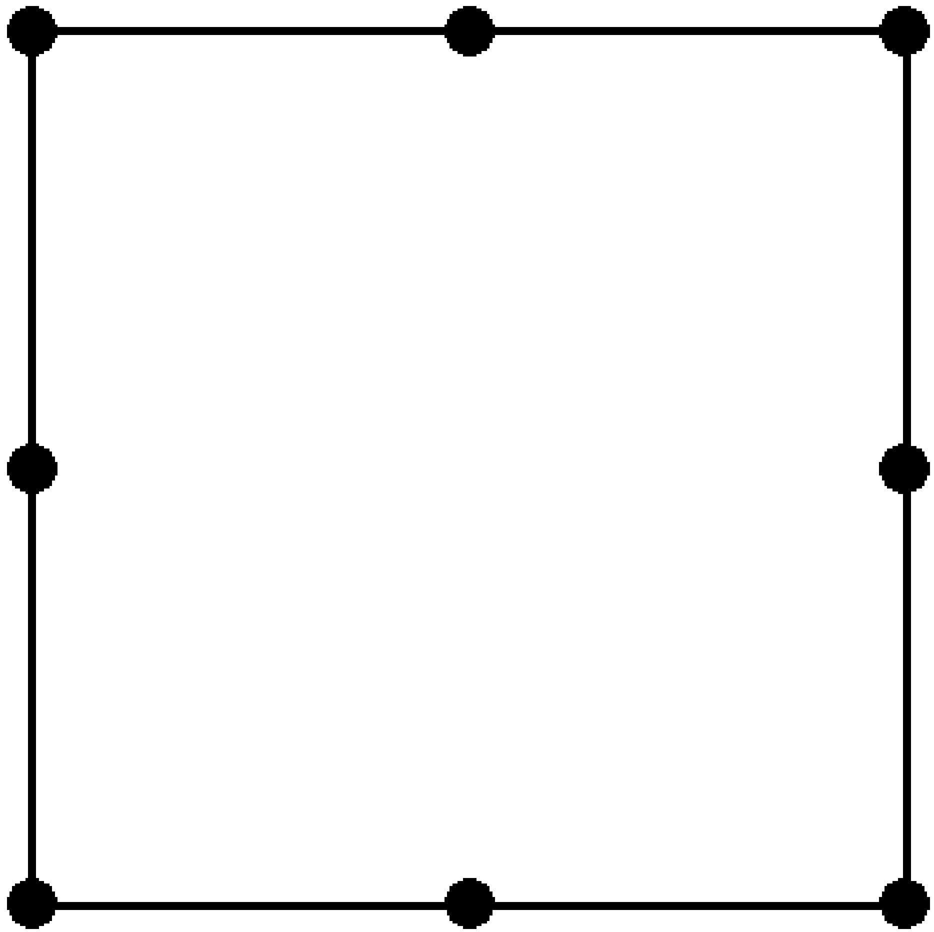

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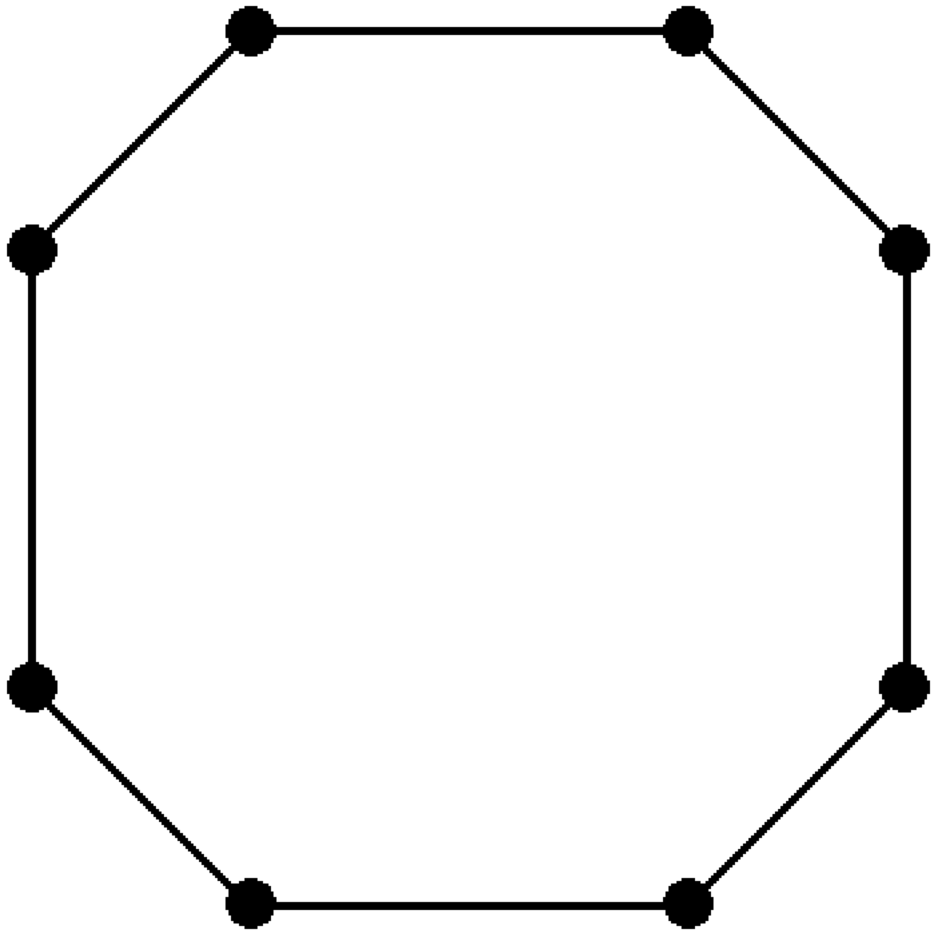

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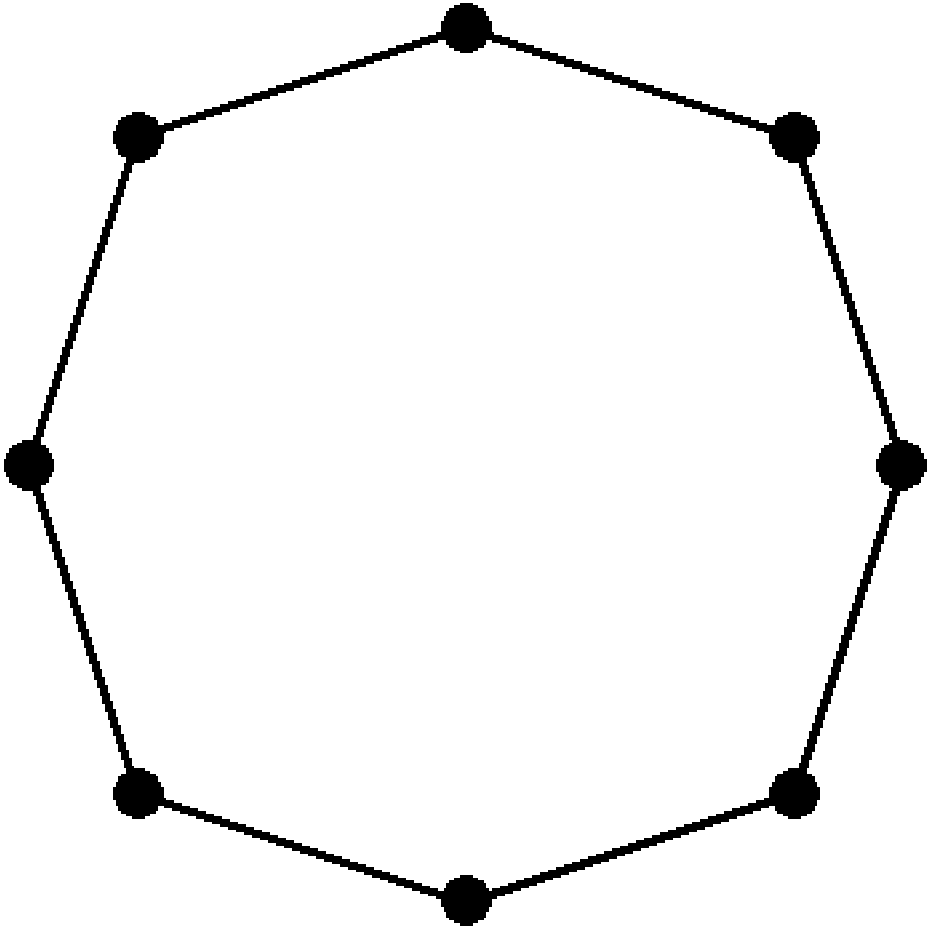

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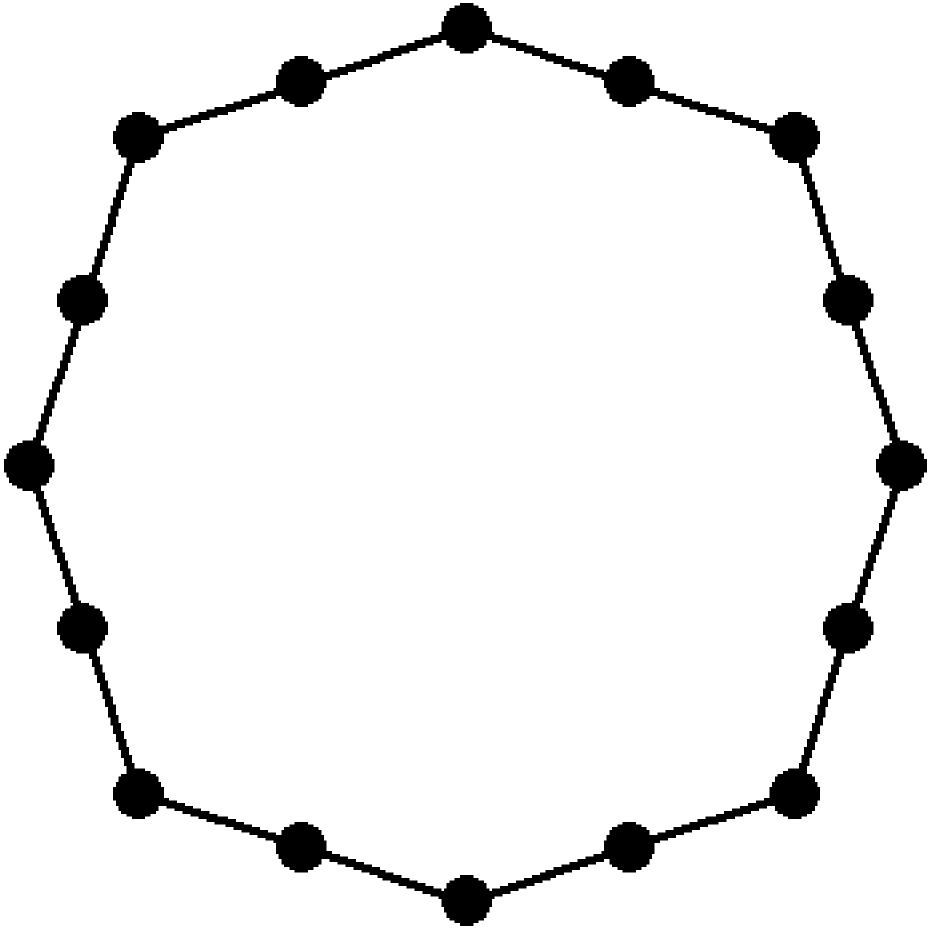

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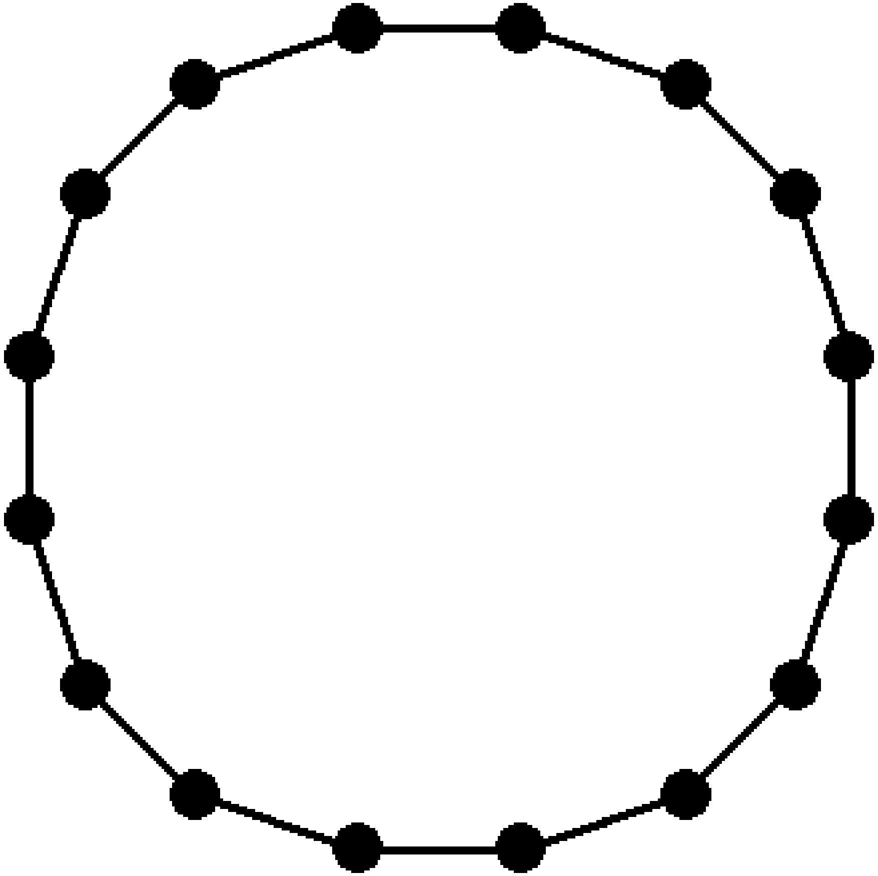

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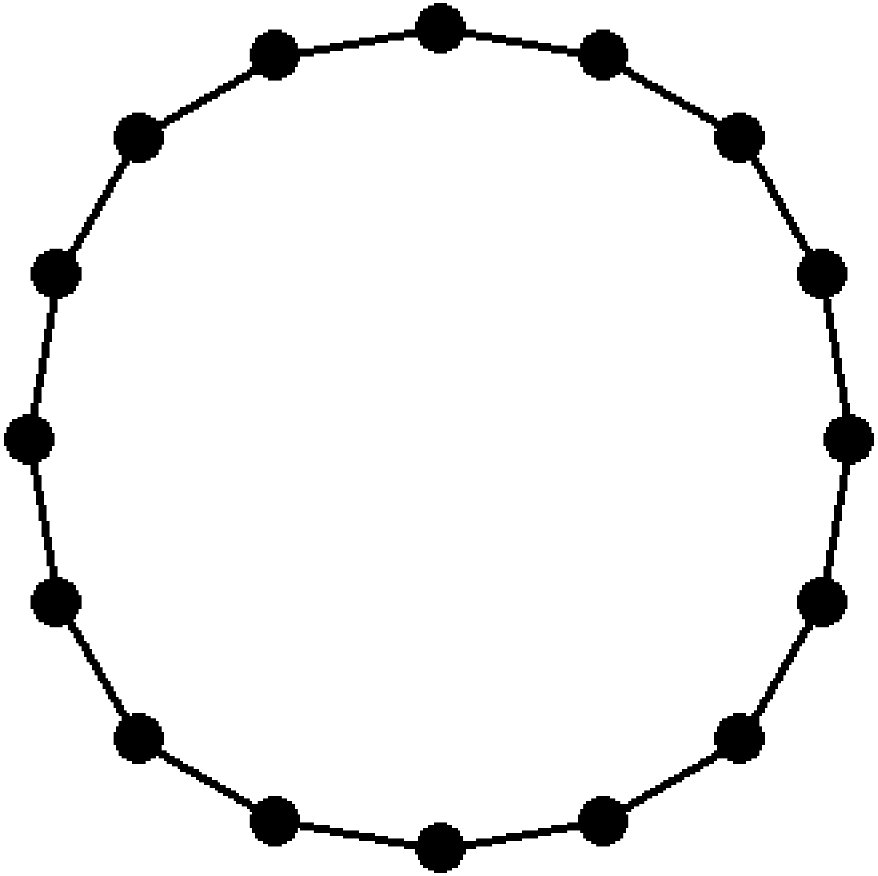

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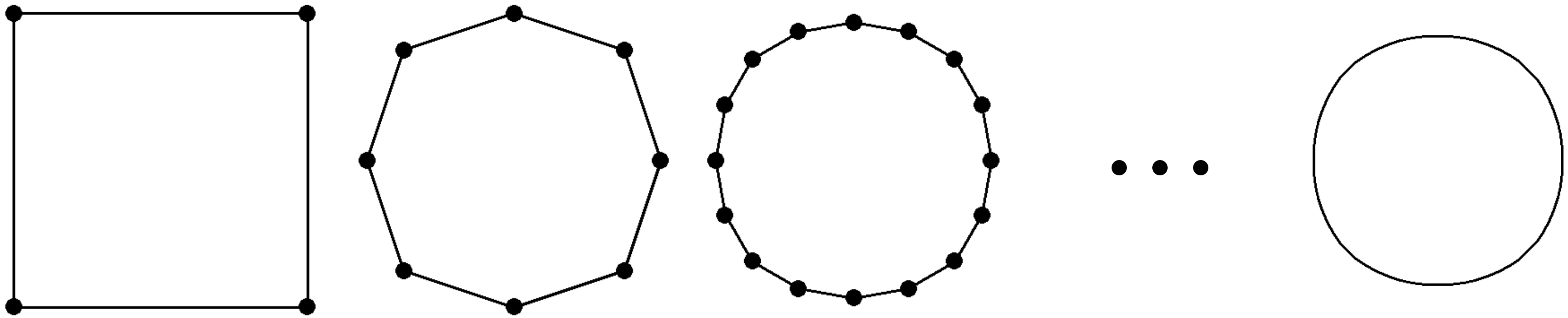
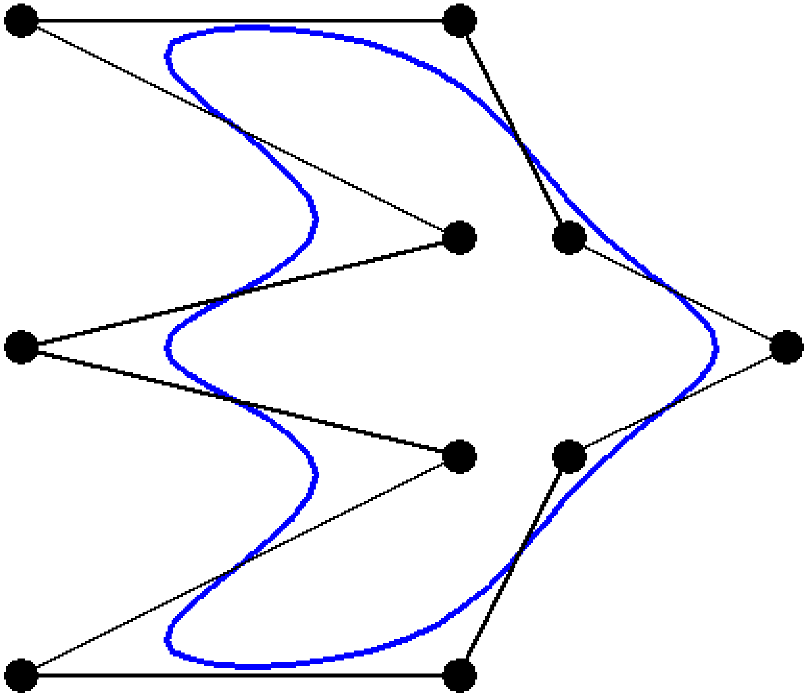


B-spline Properties

- Curve lies within convex hull of control points
- Variation Diminishing: Curve wiggles no more than control polygon
- Influence of one control point is bounded
- Degree of curve increases by one with each averaging step
- Smoothness increases by one with each averaging step

B-spline Curve Example



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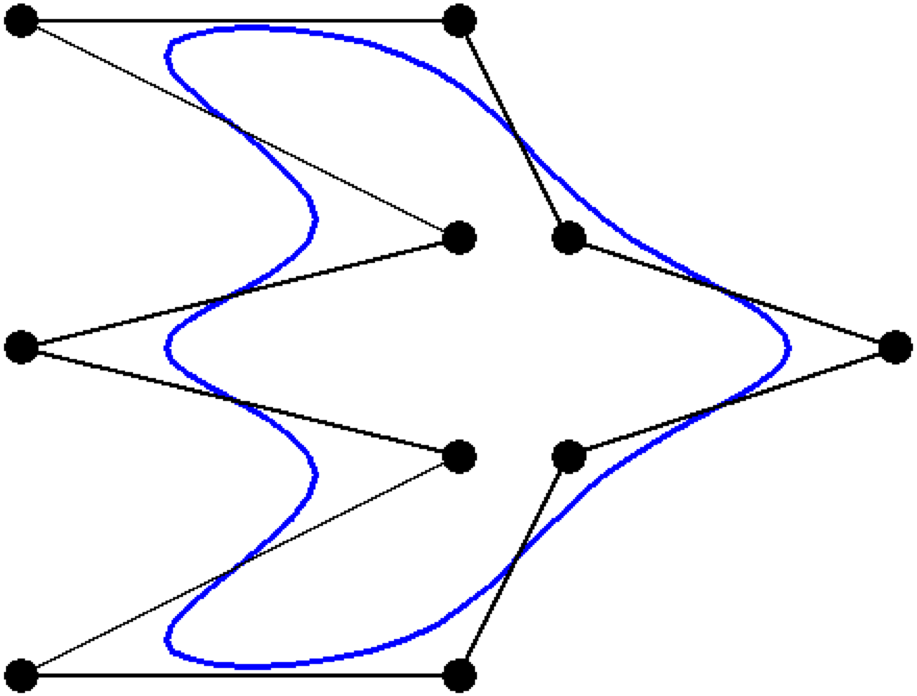

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