Fractals and
Iterated Affine Transformations

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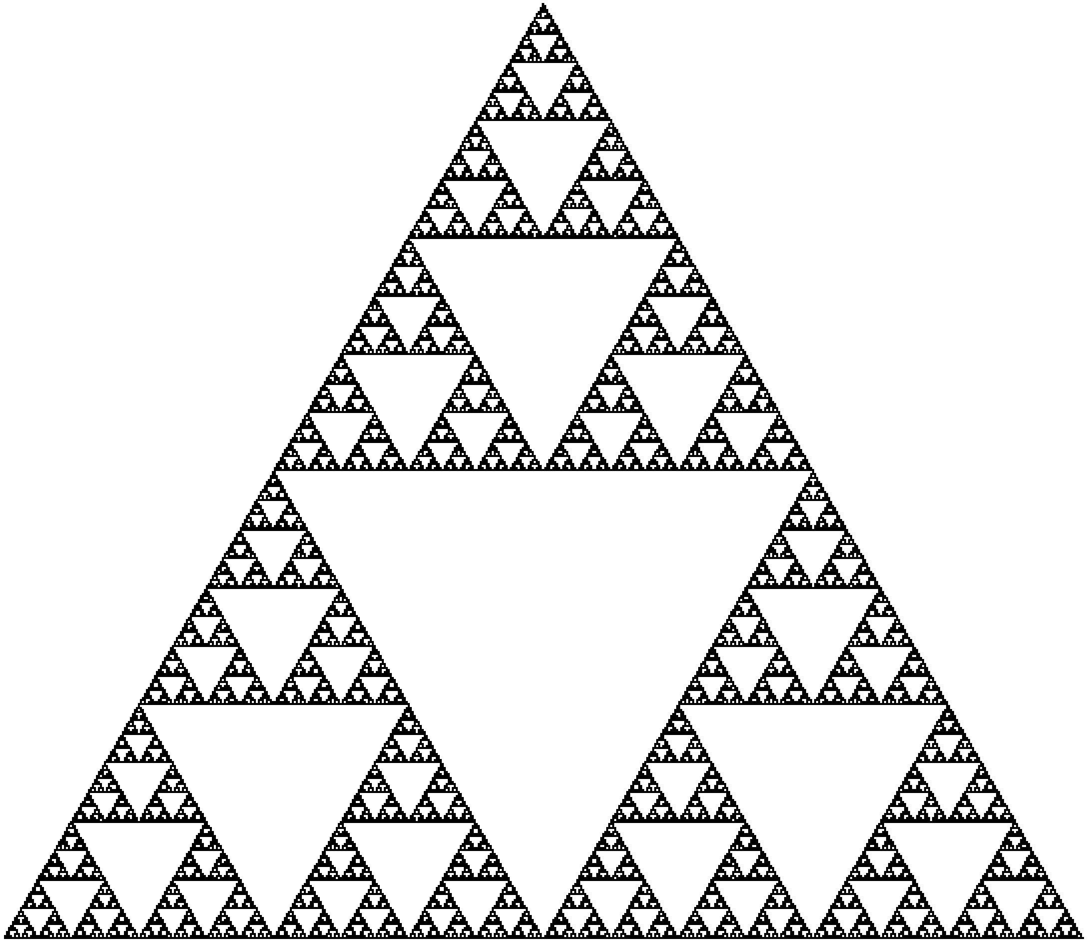
TAMU Computer Science

What are Fractals?



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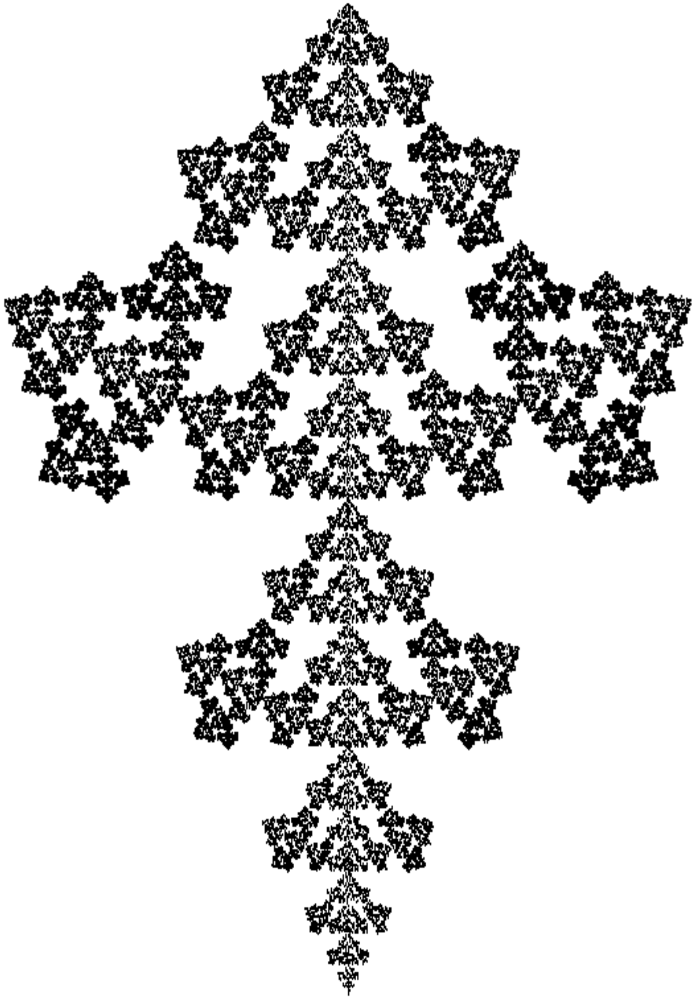

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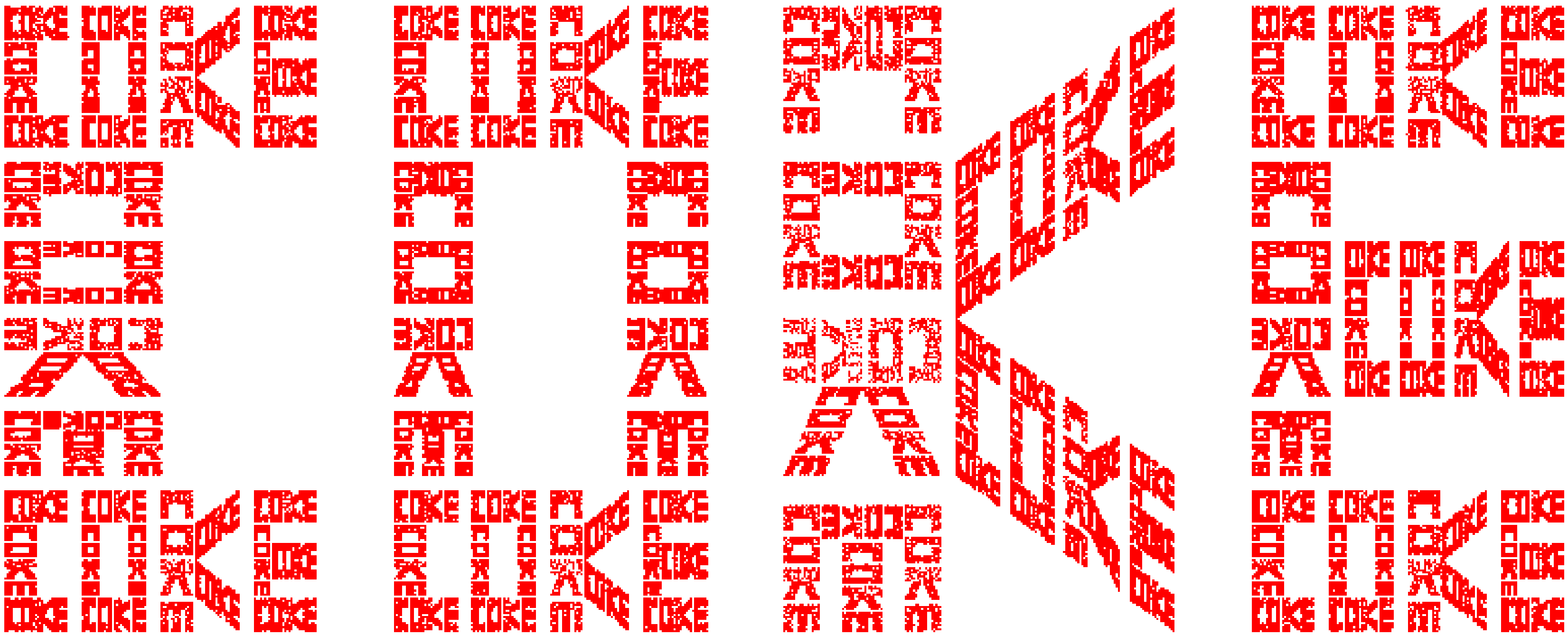
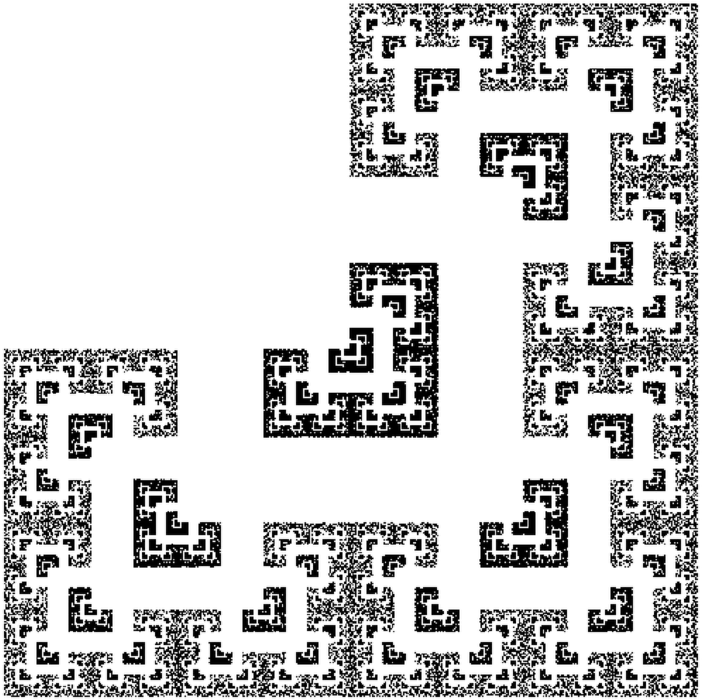


What are Fractals?

- Recursion made visible
- A self-similar shape created from a set of contractive transformations



Contractive Transformations

- A transformation $F(X)$ is contractive if, for all compact sets $X_1 \neq X_2$,

$$D_H(F(X_1), F(X_2)) < D_H(X_1, X_2)$$

- Transformations on sets?
- Distance between sets?

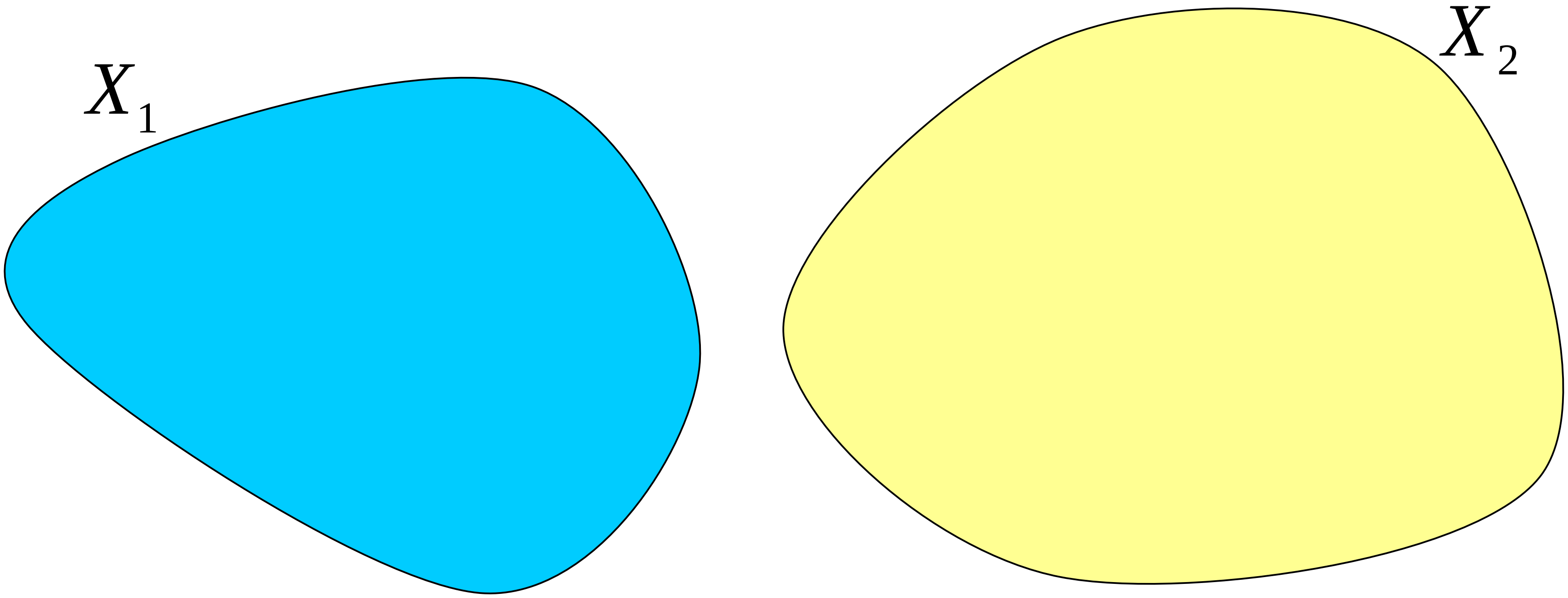
Transformations on Sets

- Given a transformation $F(x)$,

$$F(X) = \{F(x) \mid x \in X\}$$

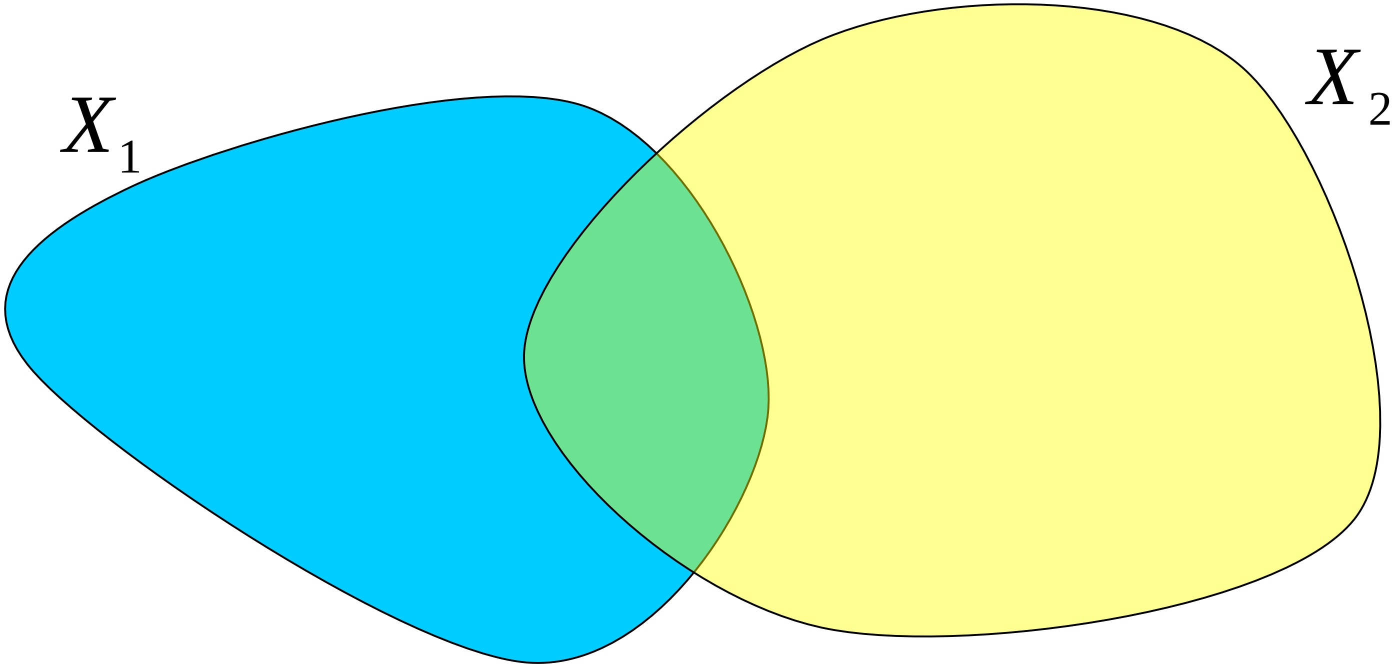
- In other words, $F(X)$ means apply the transformation to each point in the set X

Hausdorff Distance



$$D_H(X_1, X_2) = ?$$

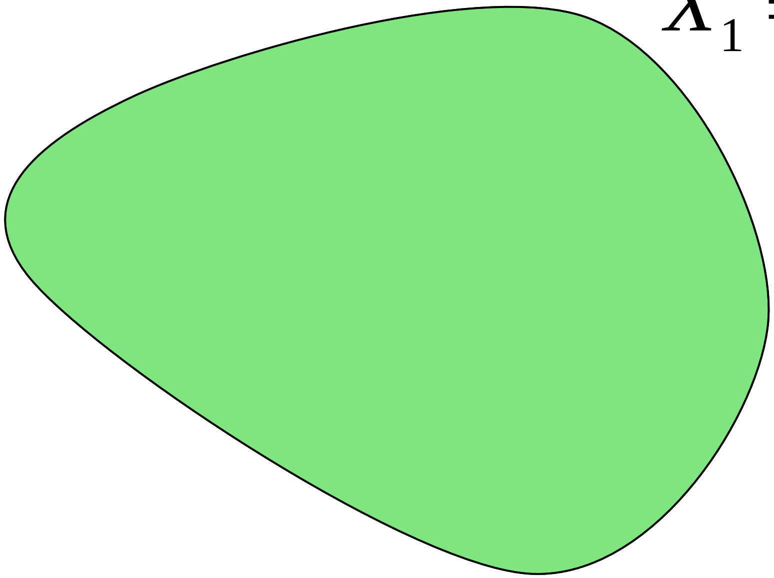
Hausdorff Distance



$$D_H(X_1, X_2) = ?$$

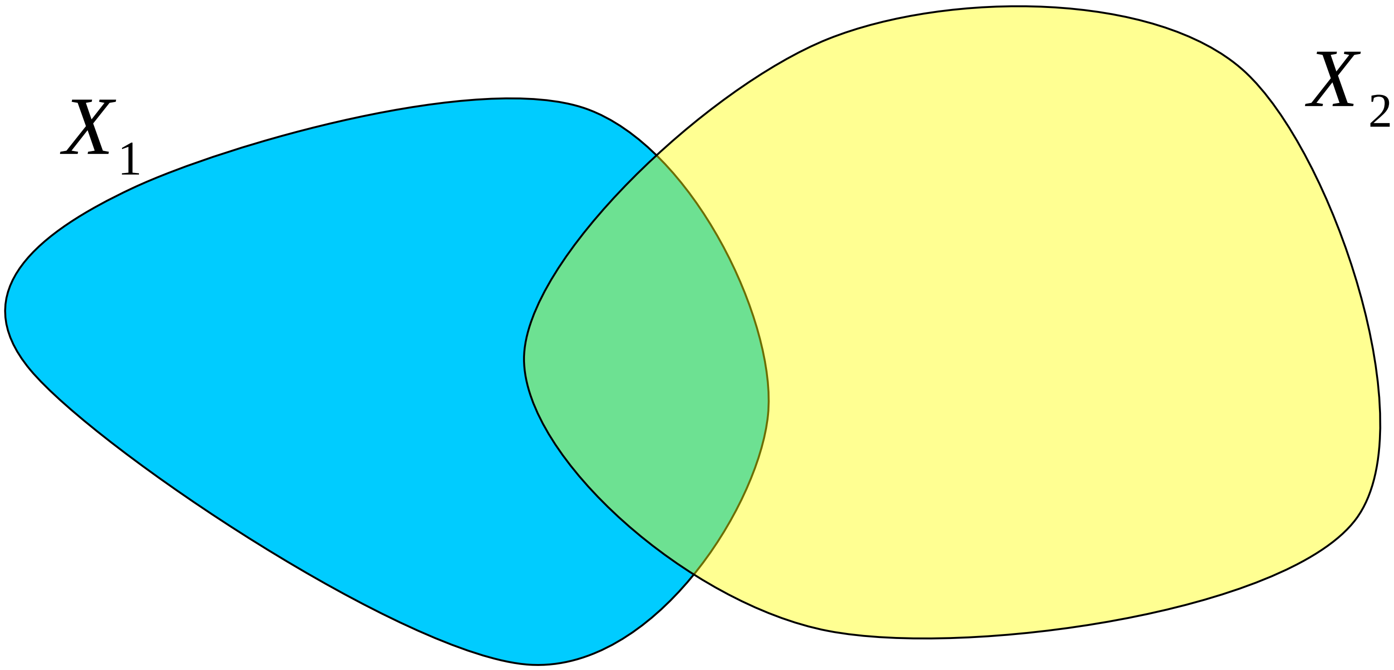
Hausdorff Distance

$X_1 = X_2$



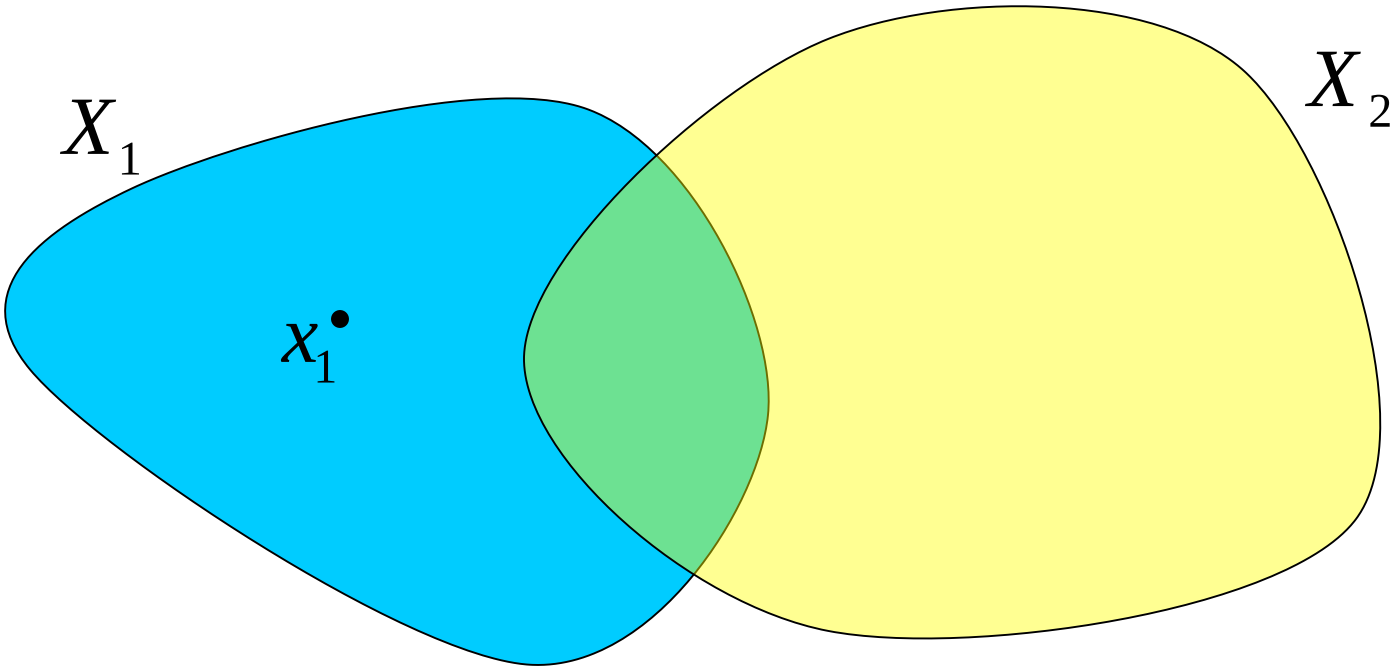
$$D_H(X_1, X_2) = 0$$

Hausdorff Distance



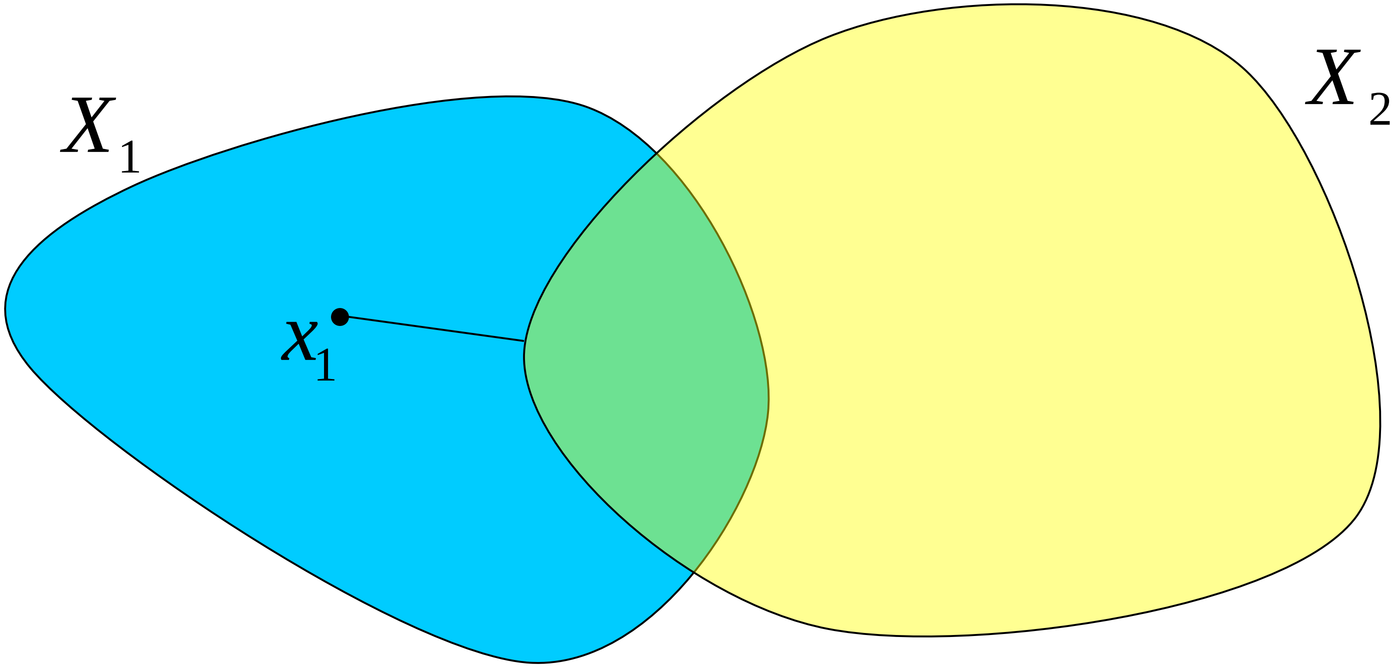
$$D_H(X_1, X_2) = D_H(X_2, X_1)$$

Hausdorff Distance



$$d_{X_1 \to X_2} = \max_{x_1 \in X_1}\left(\min_{x_2 \in X_2} |x_1 - x_2|\right)$$

Hausdorff Distance



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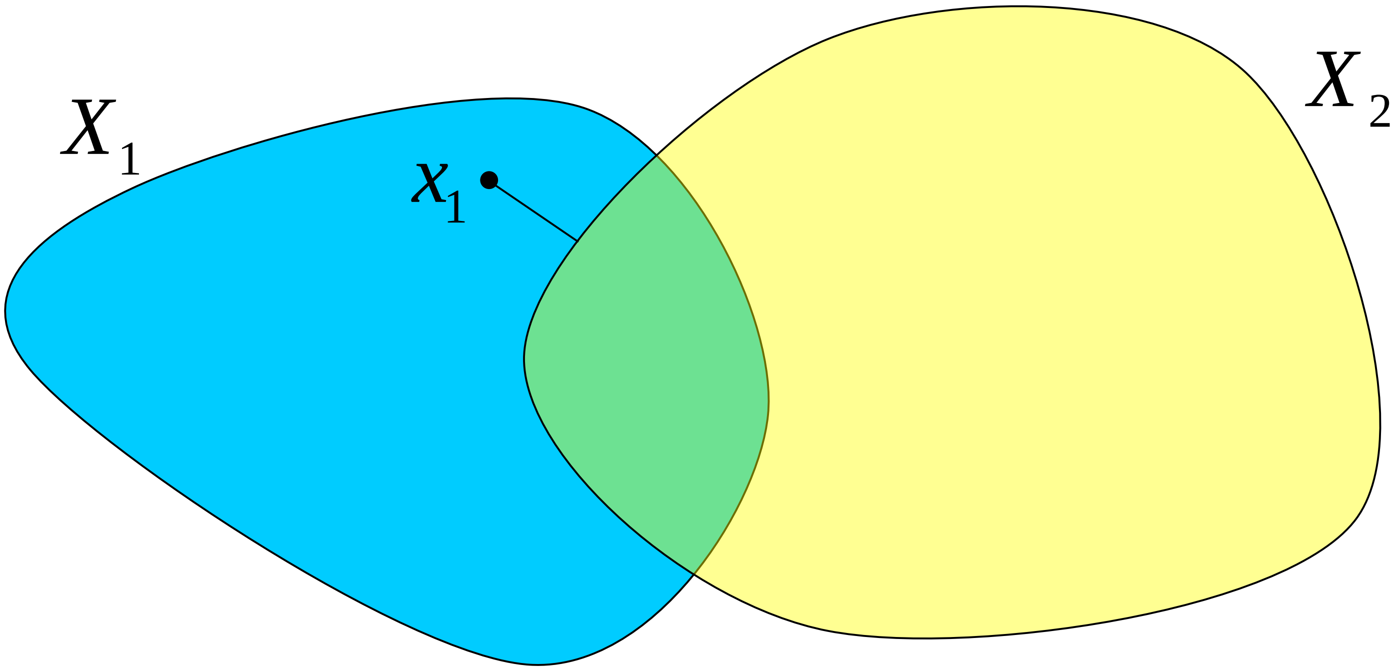
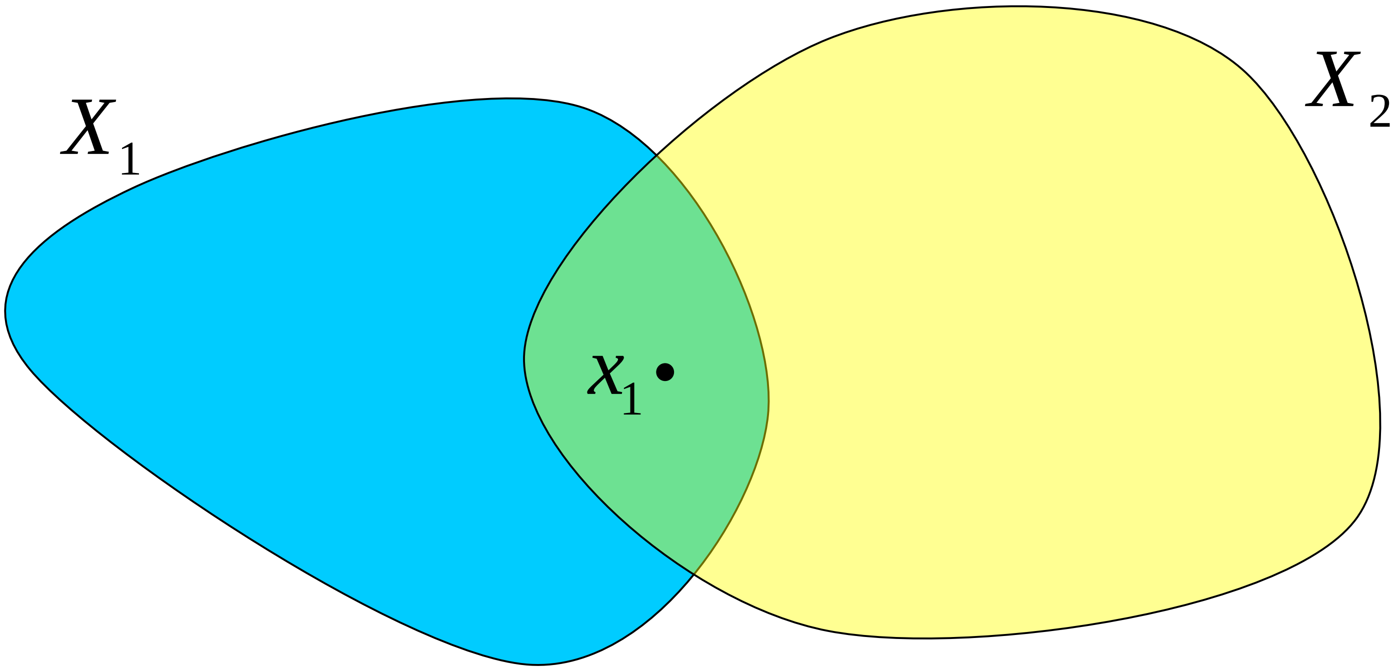


$$d_{X_1 \rightarrow X_2} = \max_{x_1 \in X_1}(\min_{x_2 \in X_2} | x_1 - x_2 |)$$

Hausdorff Distance



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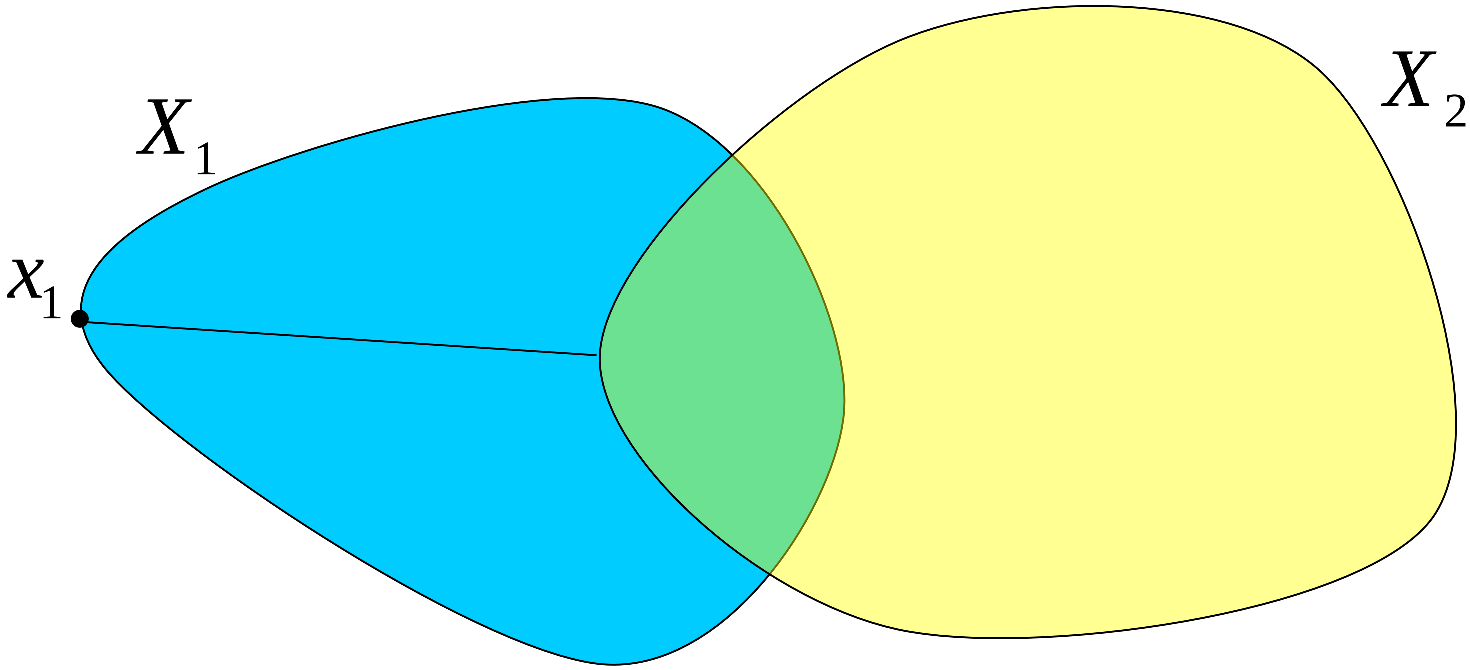

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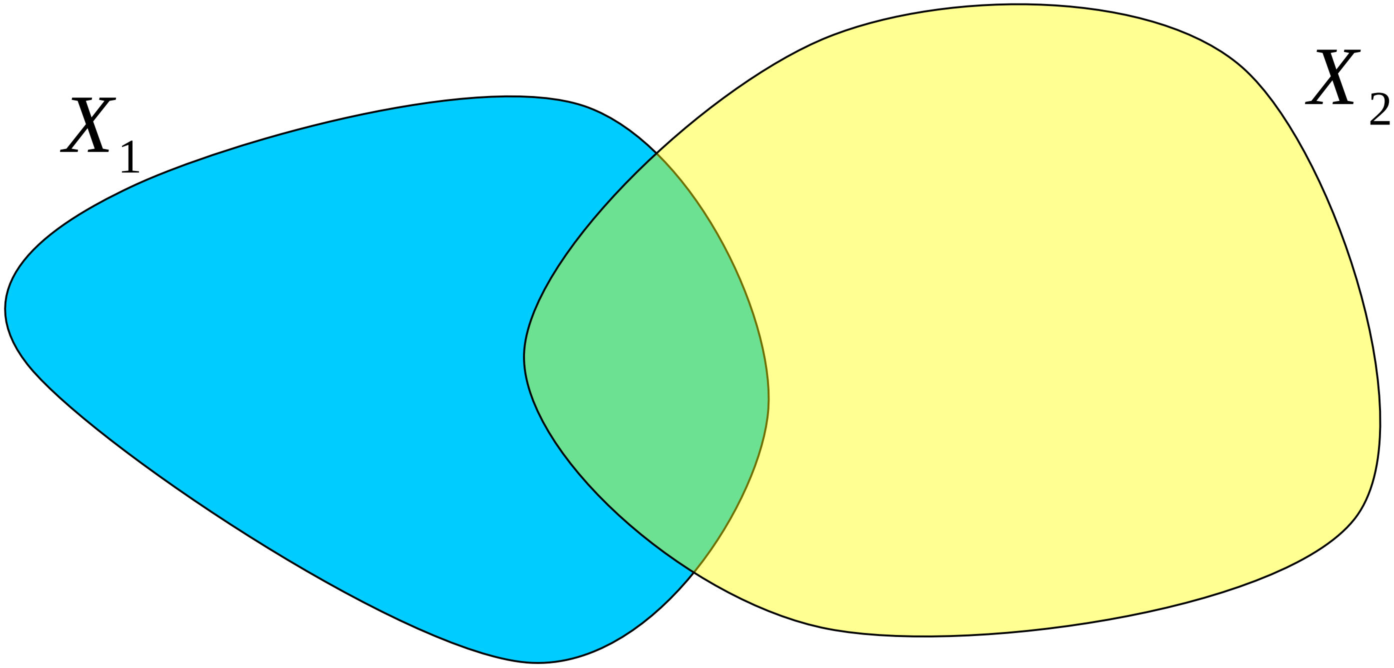

$$D_H(X_1, X_2) = \max(d_{X_1 \to X_2}, d_{X_2 \to X_1})$$

Contractive Transformations

- A transformation $F(X)$ is contractive if, for all compact sets $X_1 \neq X_2$,

$$D_H(F(X_1), F(X_2)) < D_H(X_1, X_2)$$

Iterated Affine Transformations

- Special class of fractals where each transformation is an affine transformation

$$\{F_1(x) = M_1 x, F_2(x) = M_2 x, F_3(x) = M_3 x, \ldots\}$$

$$M_i = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

Rendering Fractals

Given starting set X_0

$$X_{i+1} = \bigcup_j F_j(X_i)$$

- Attractor is X_∞

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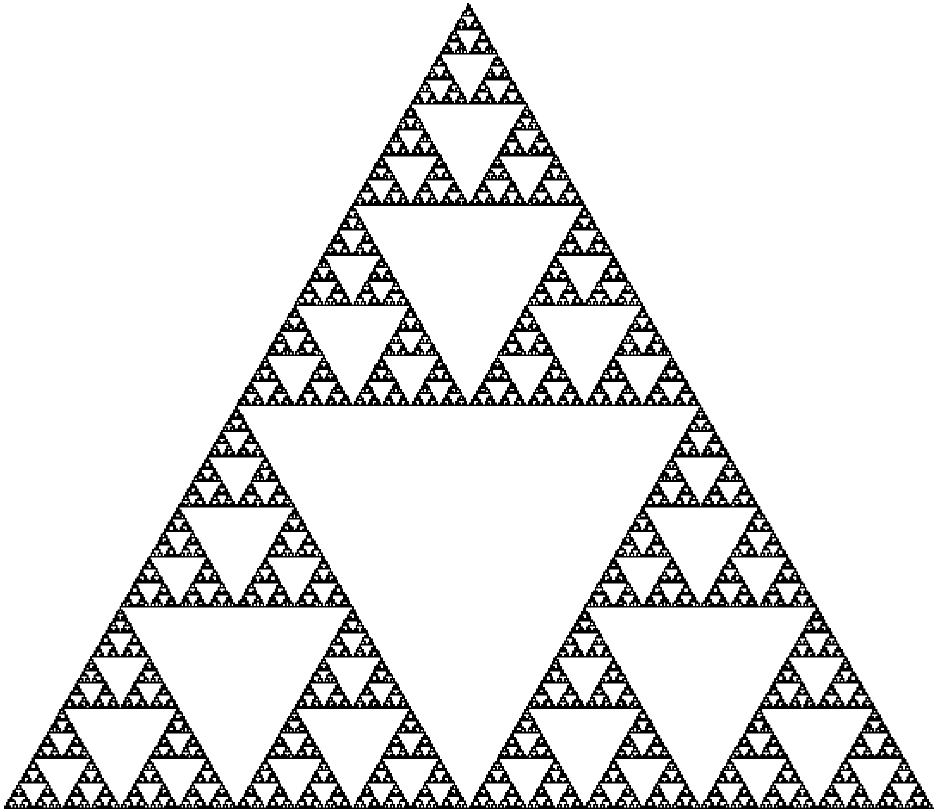

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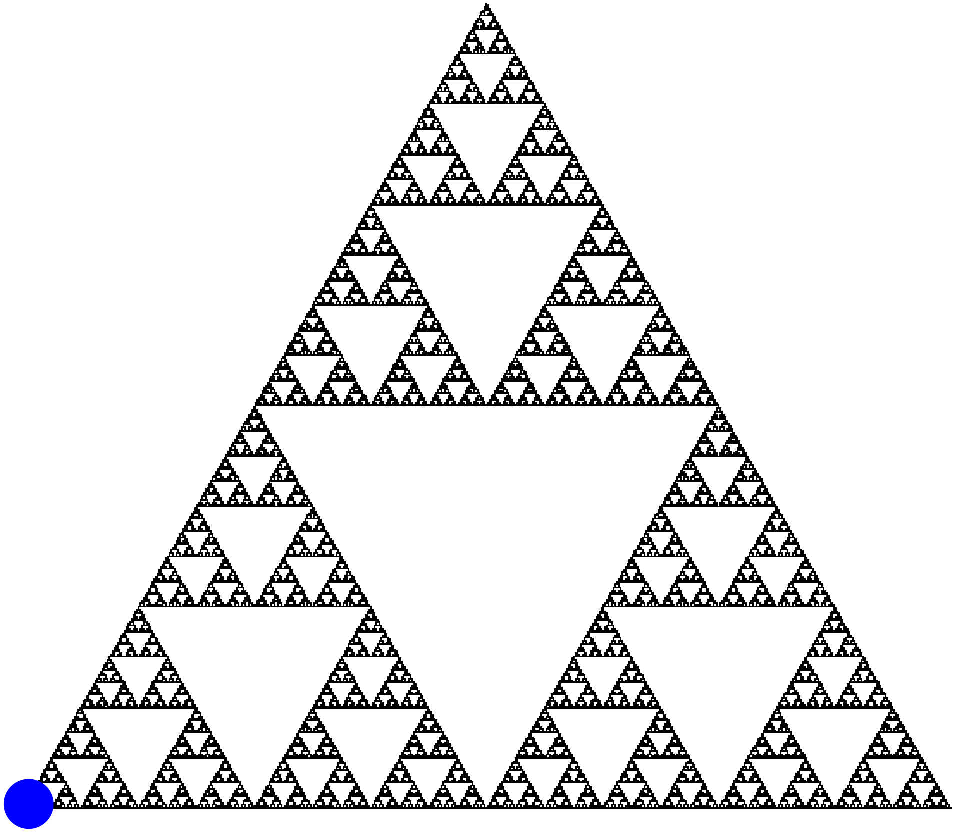

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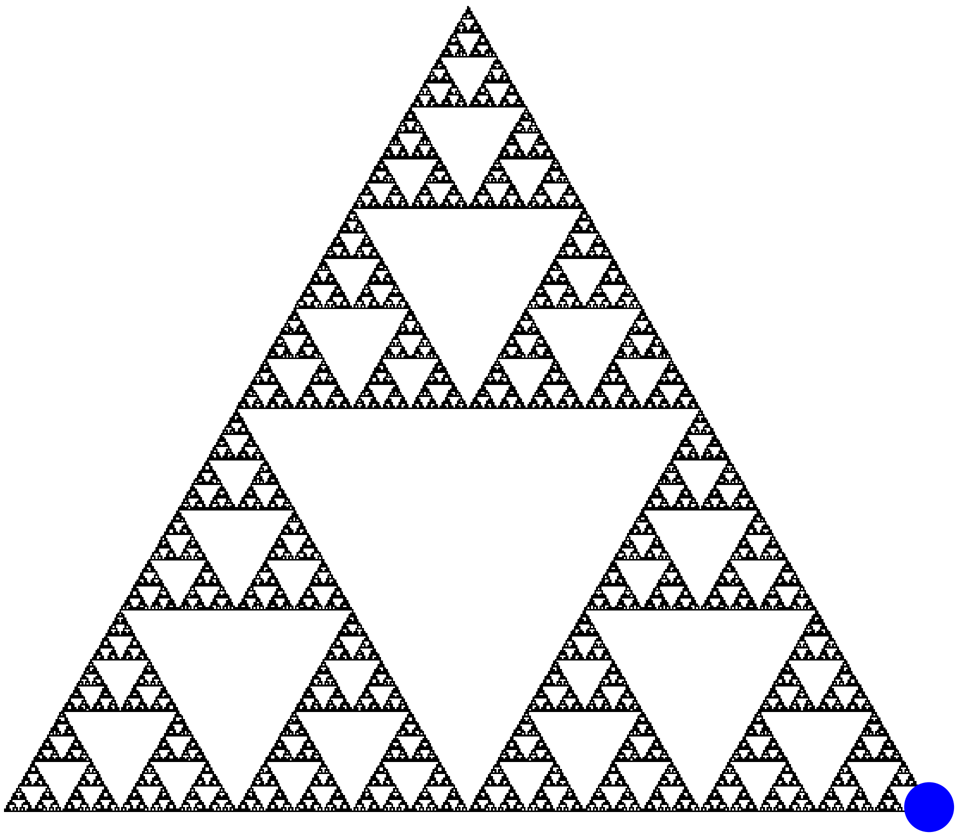

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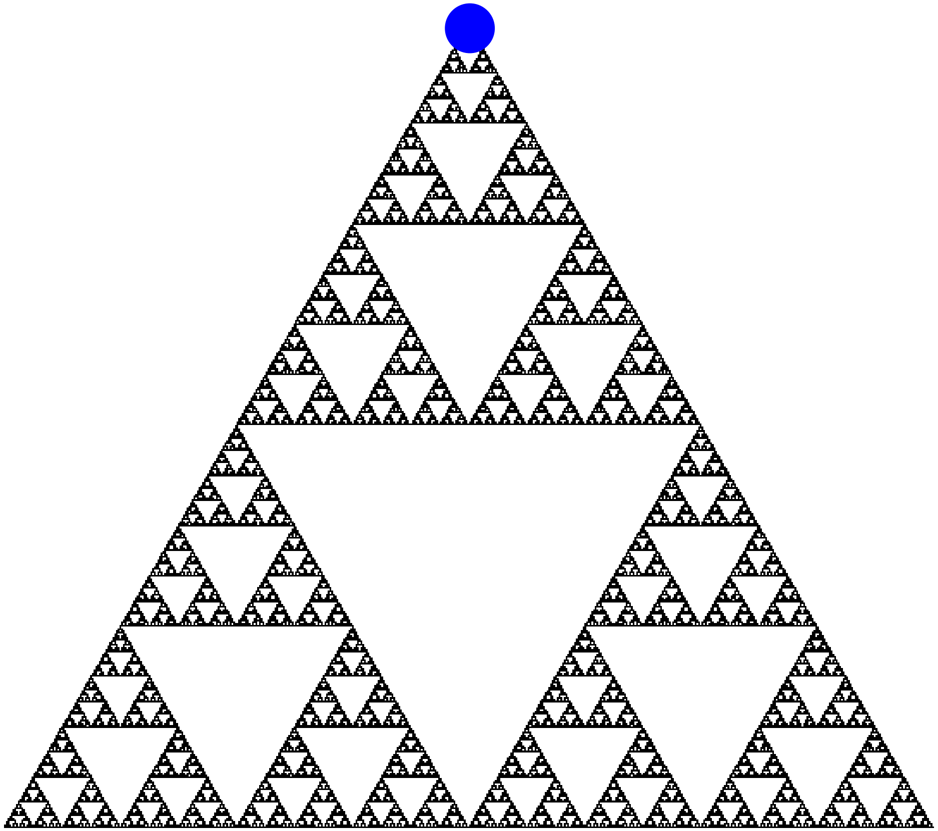

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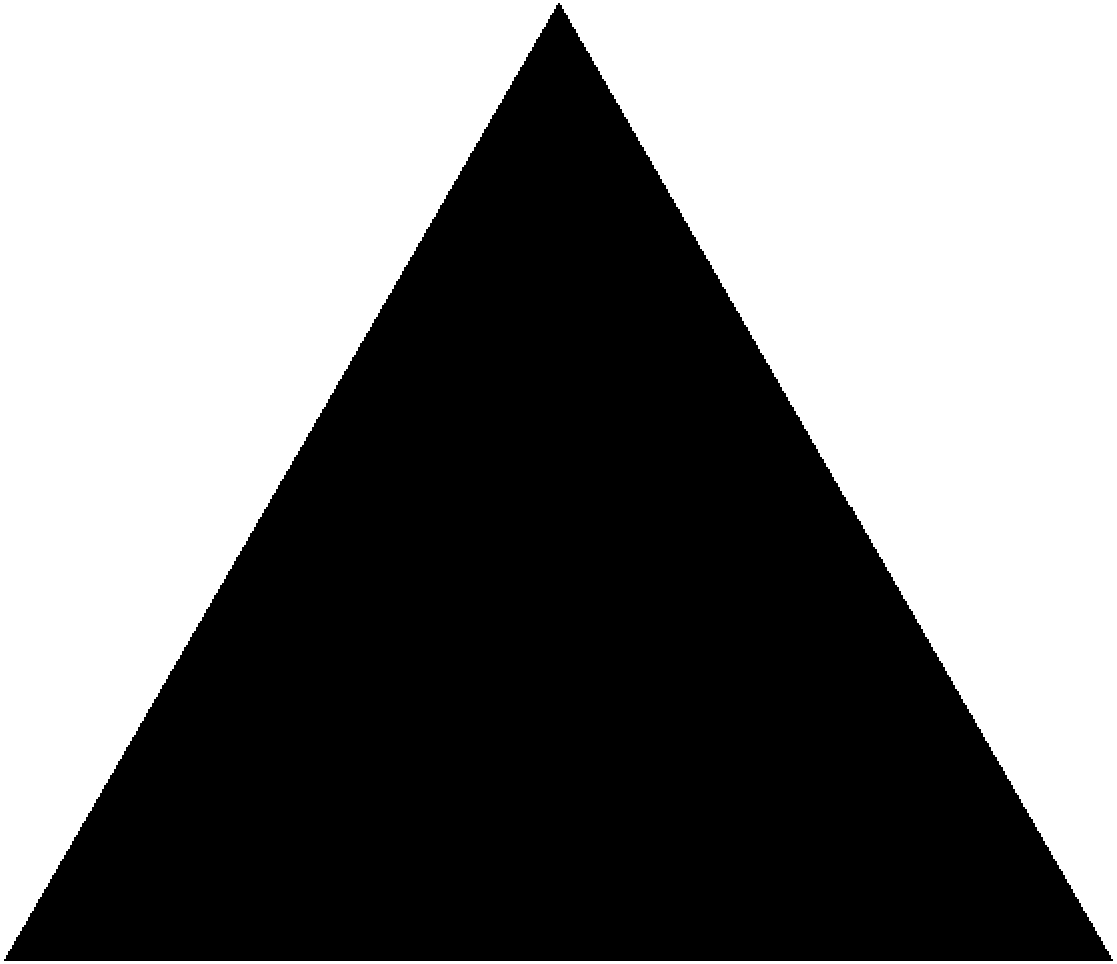

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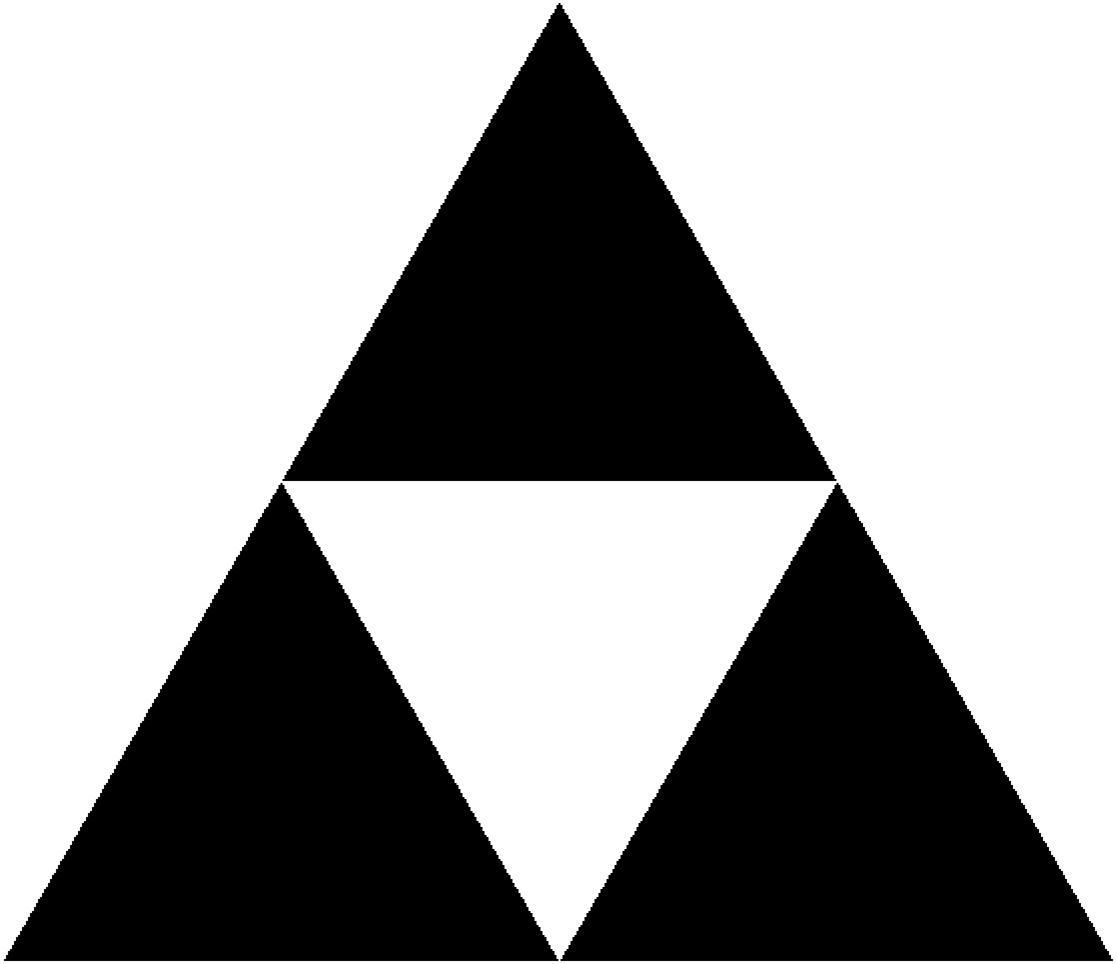

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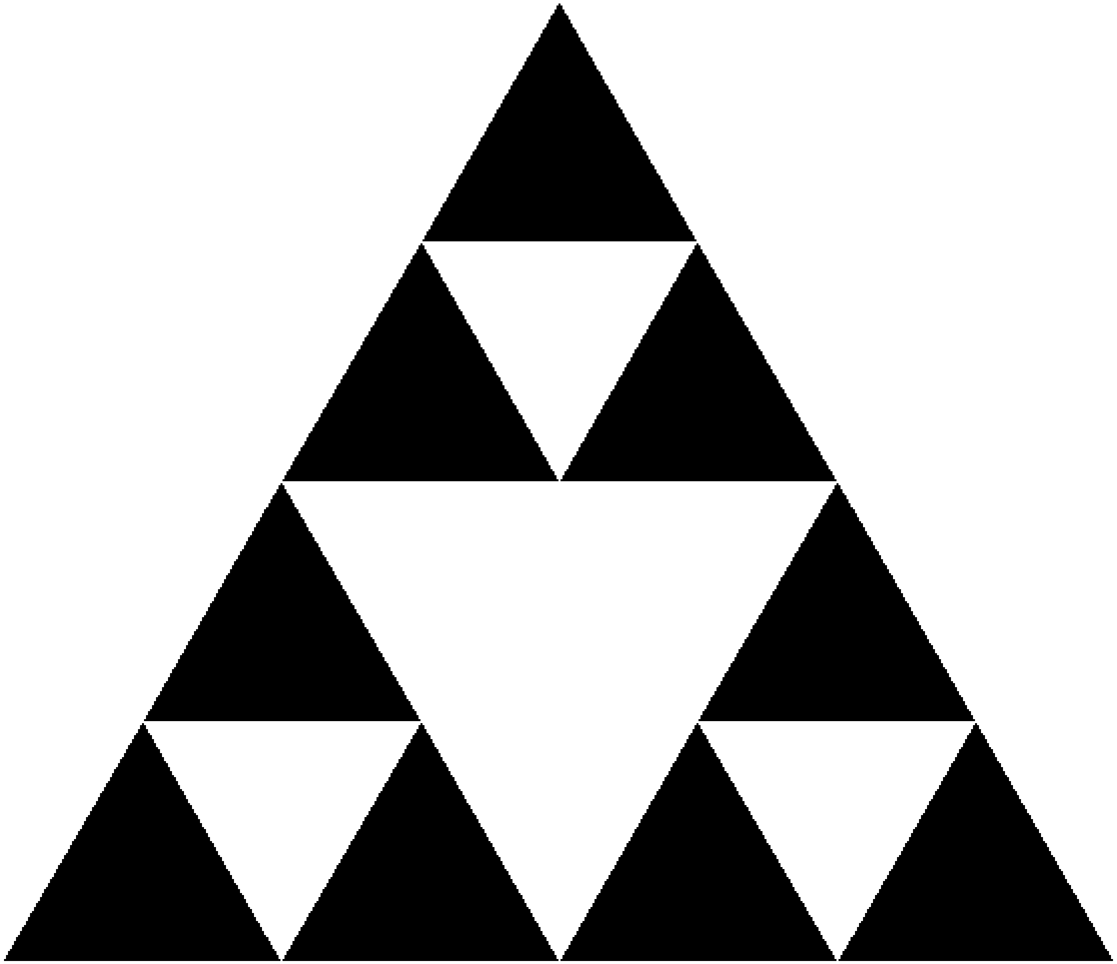

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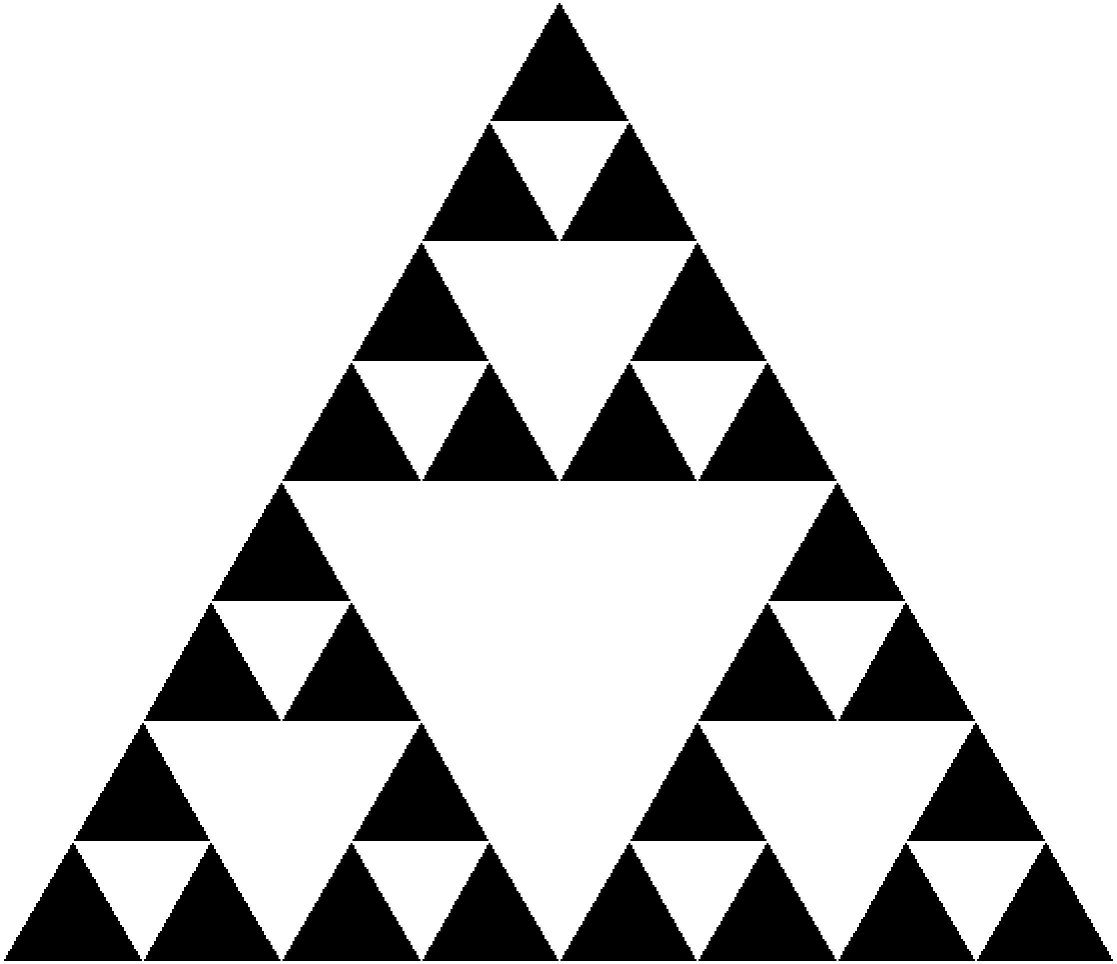

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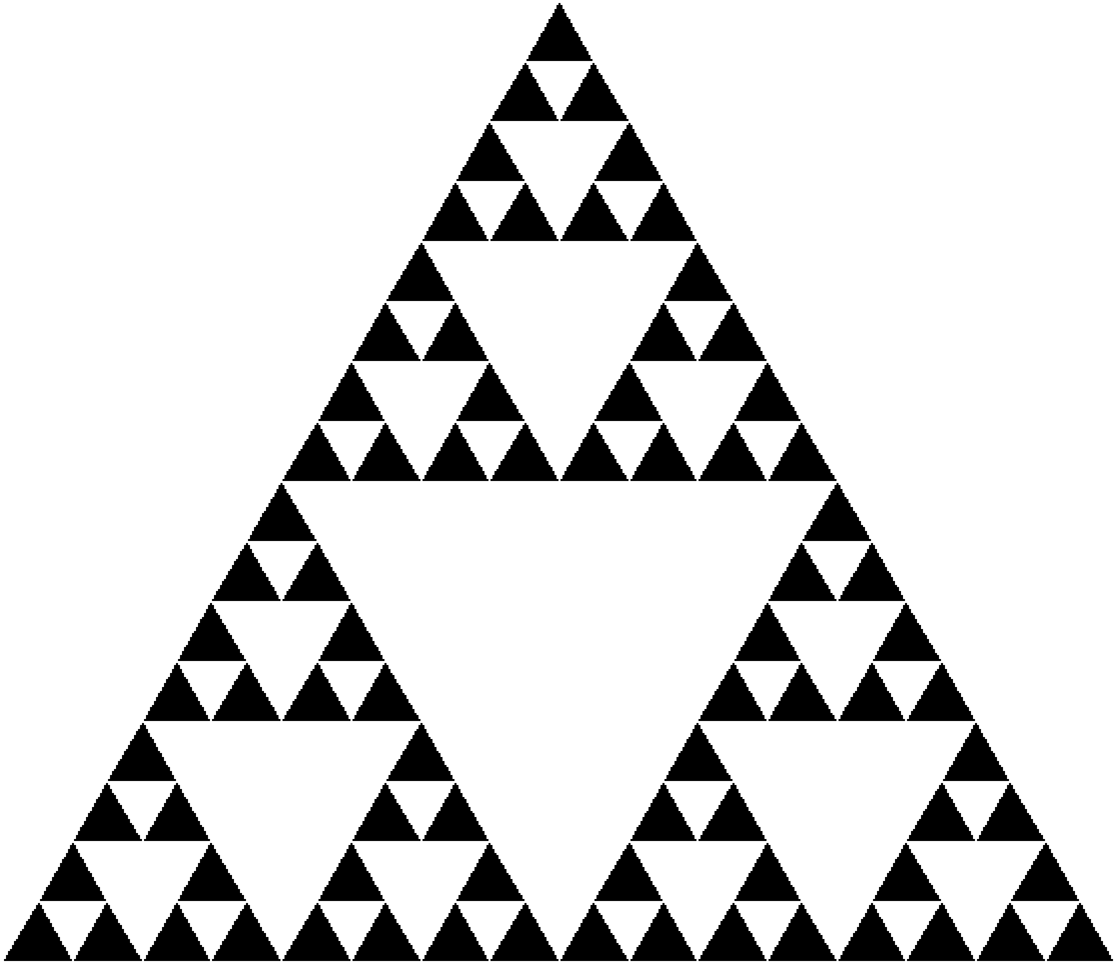

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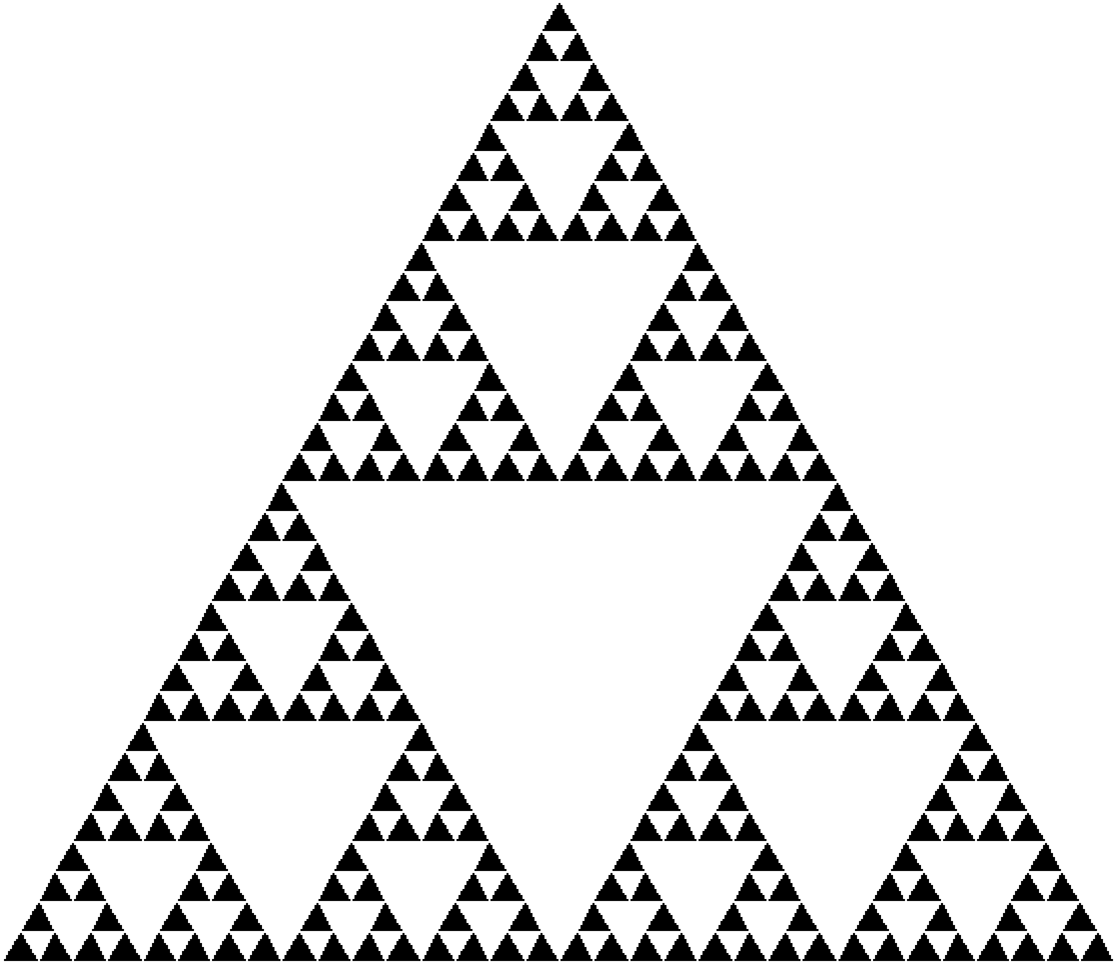

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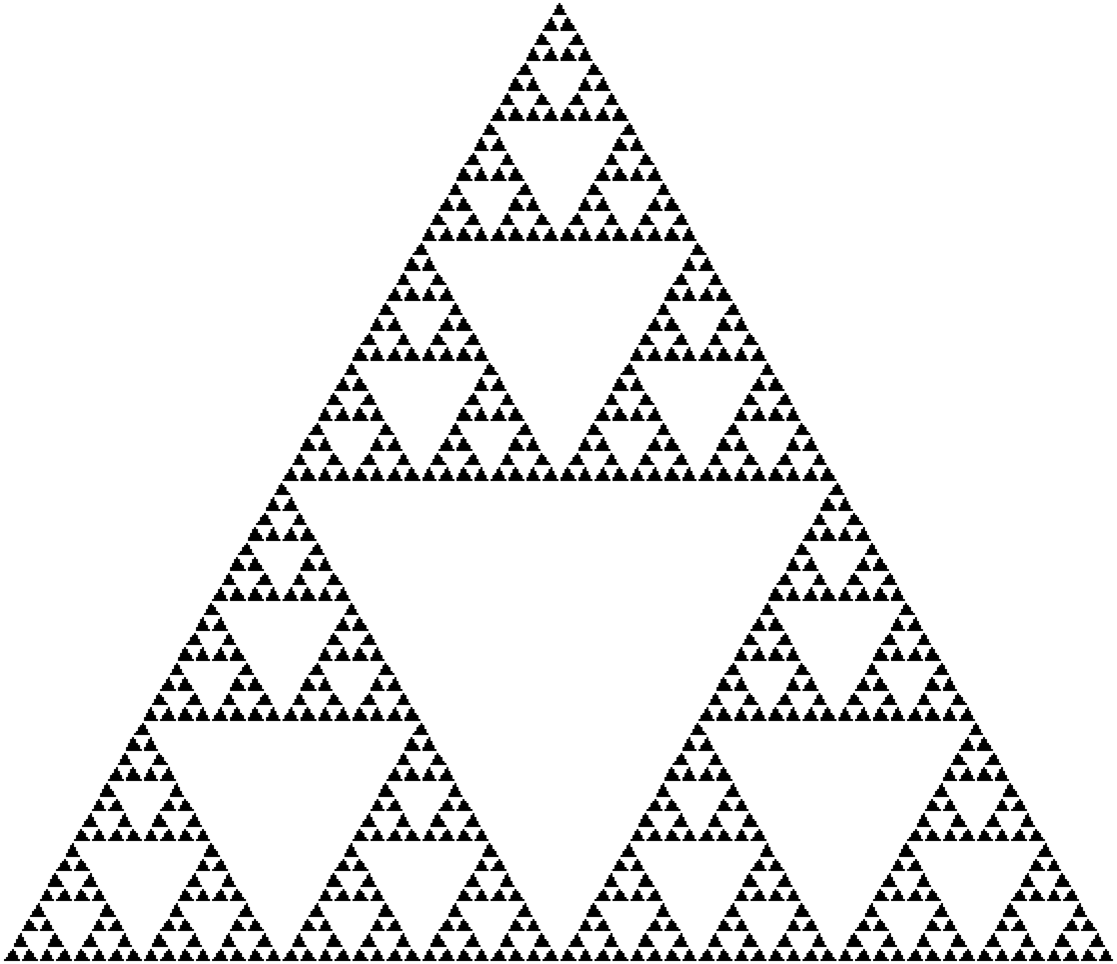

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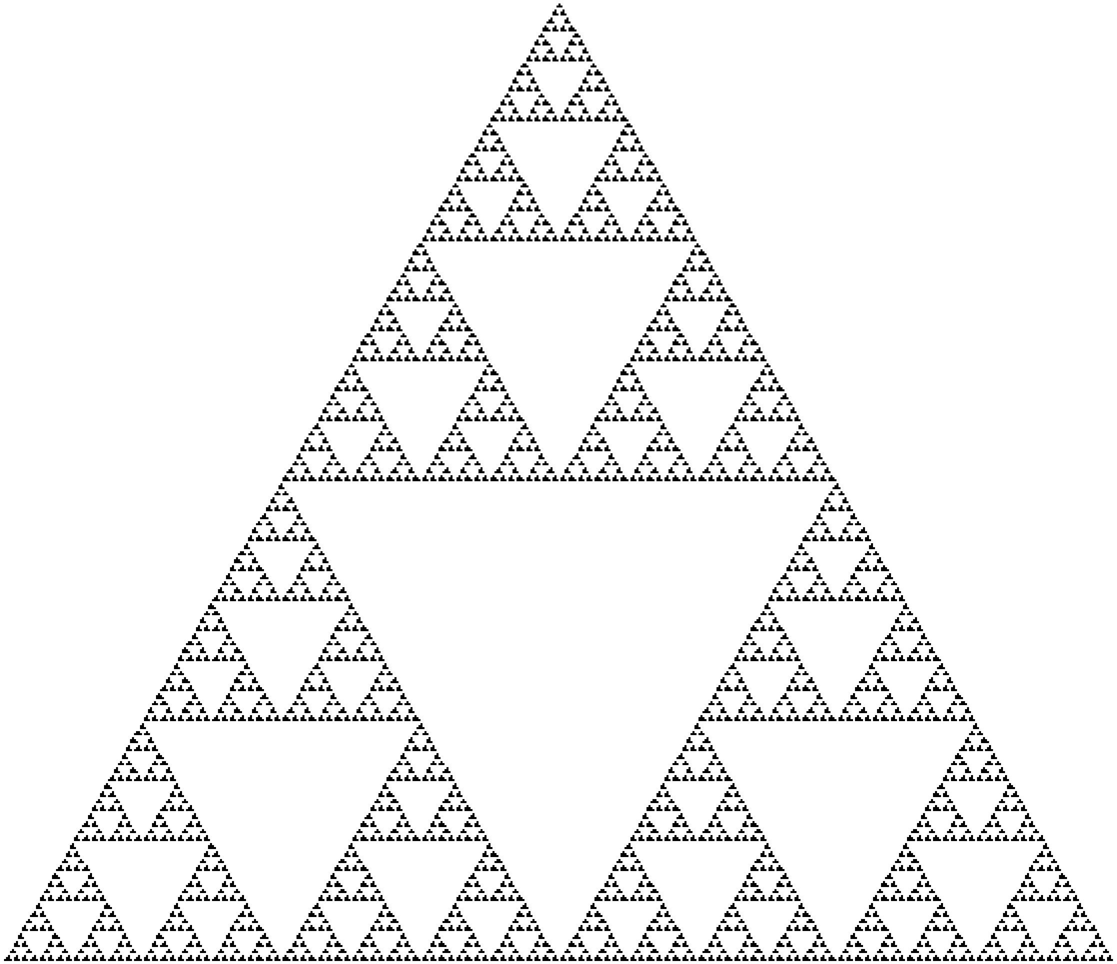

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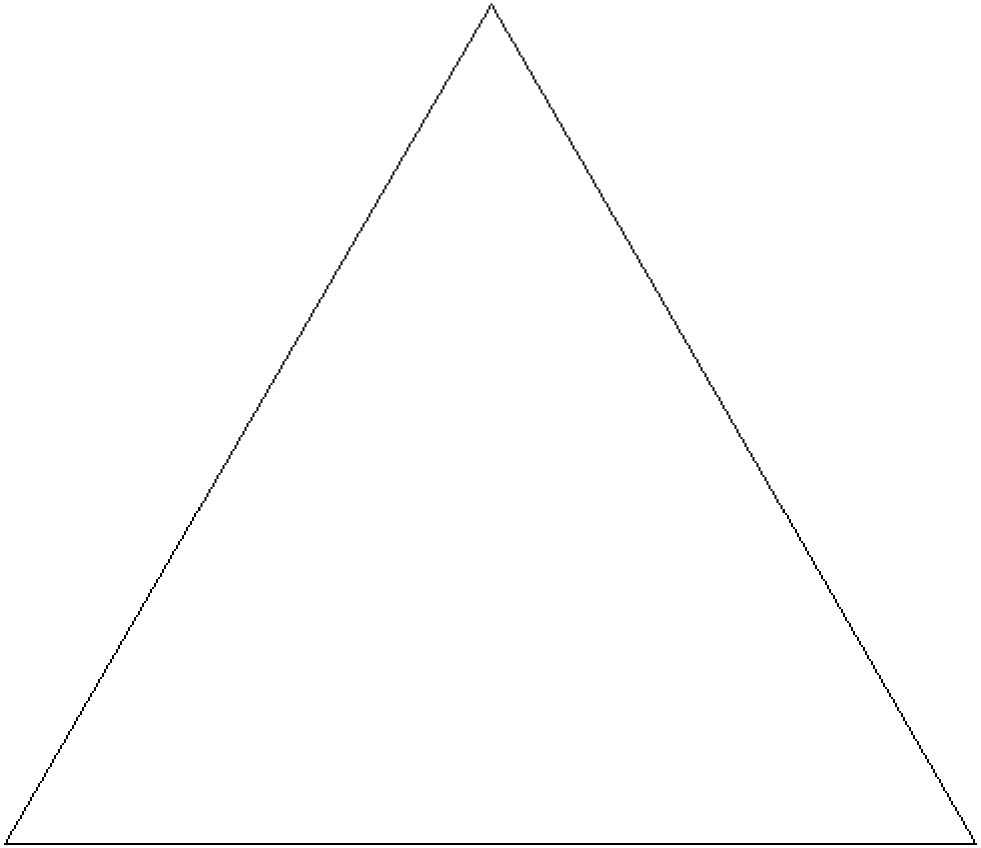

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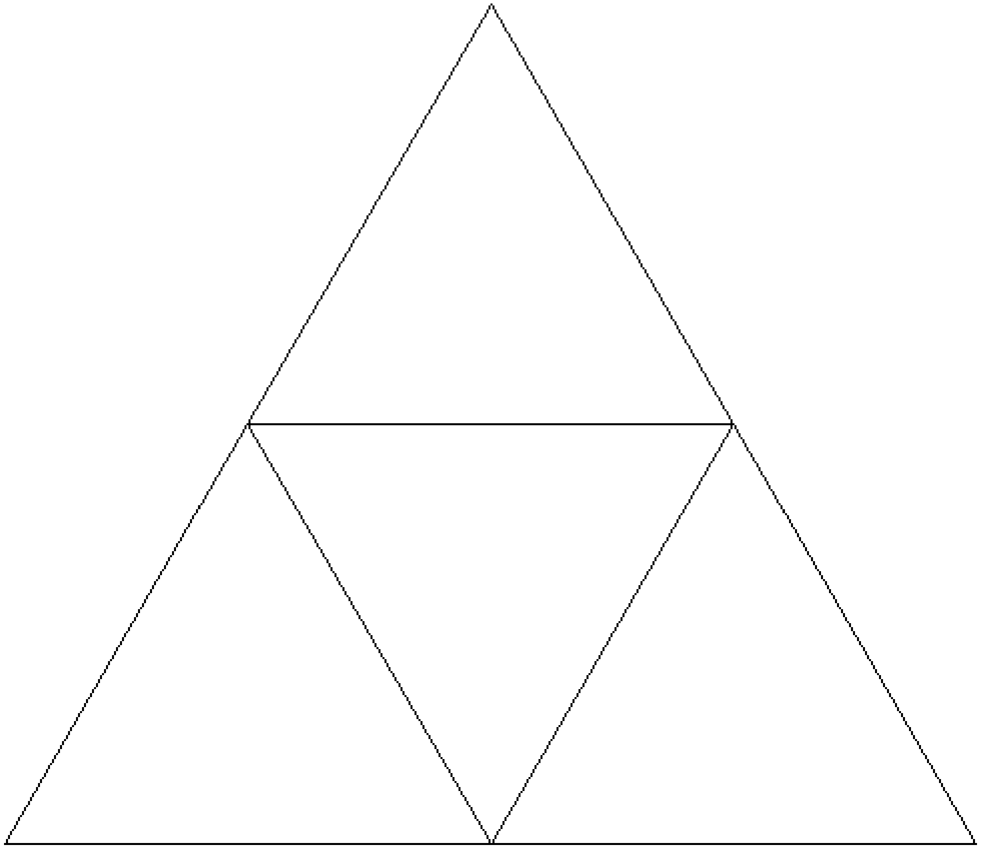

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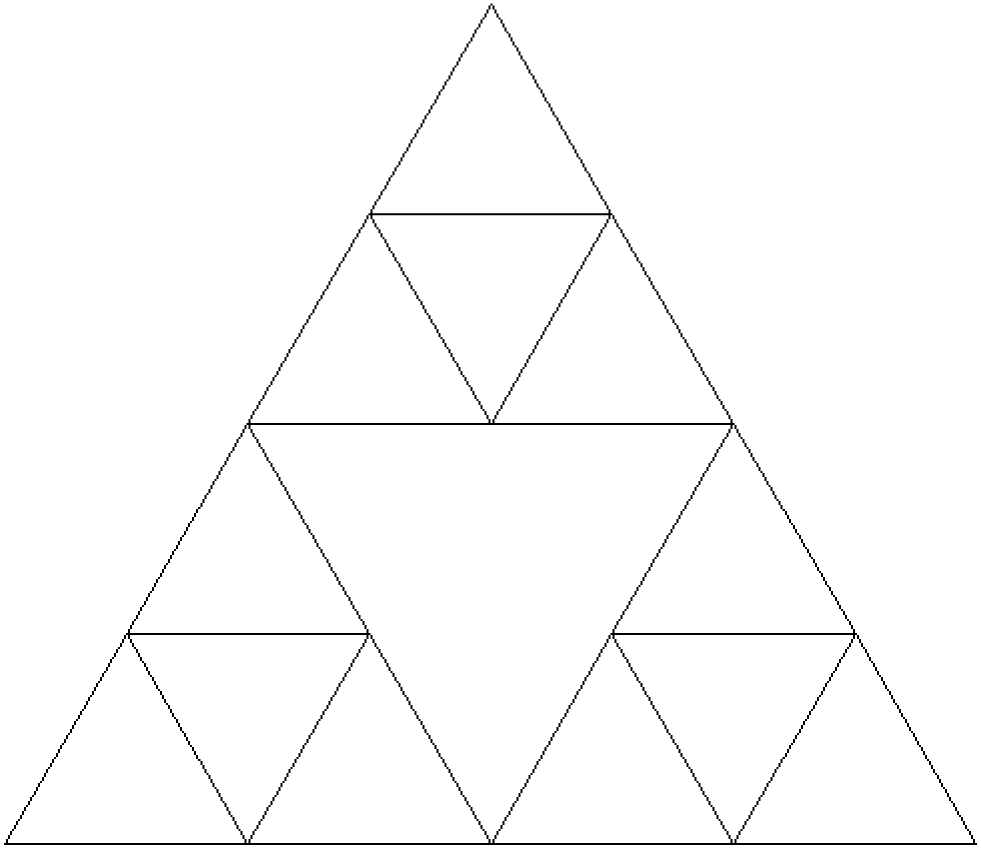

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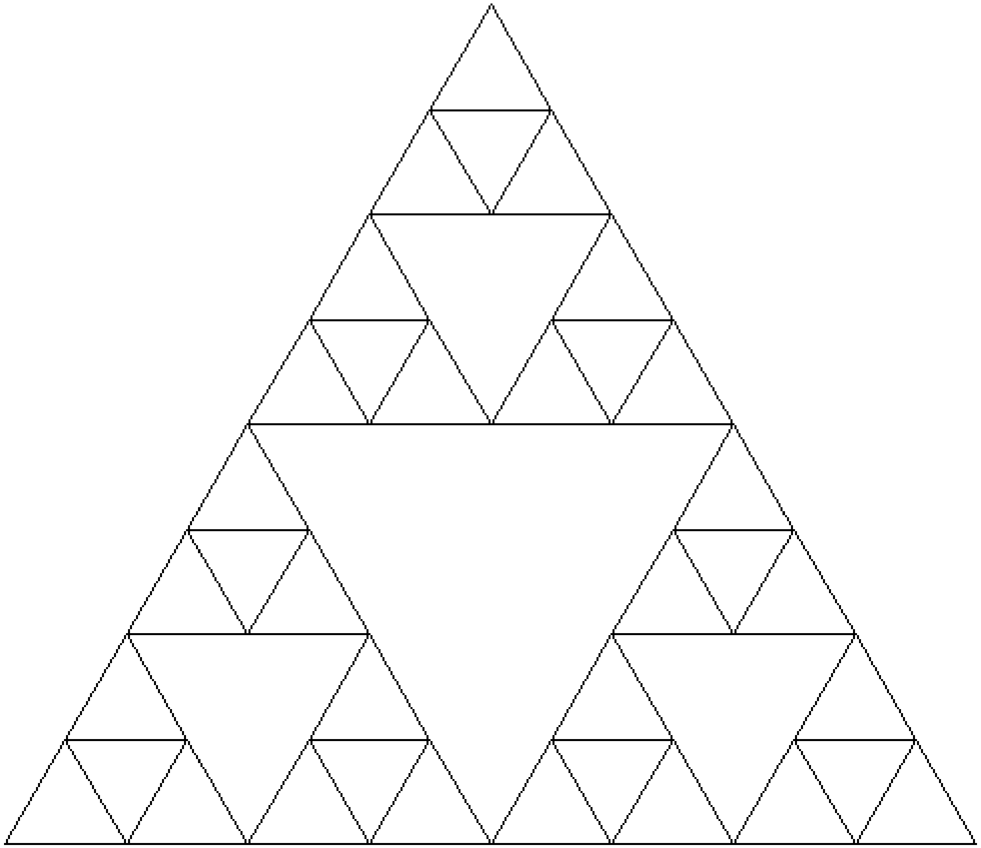

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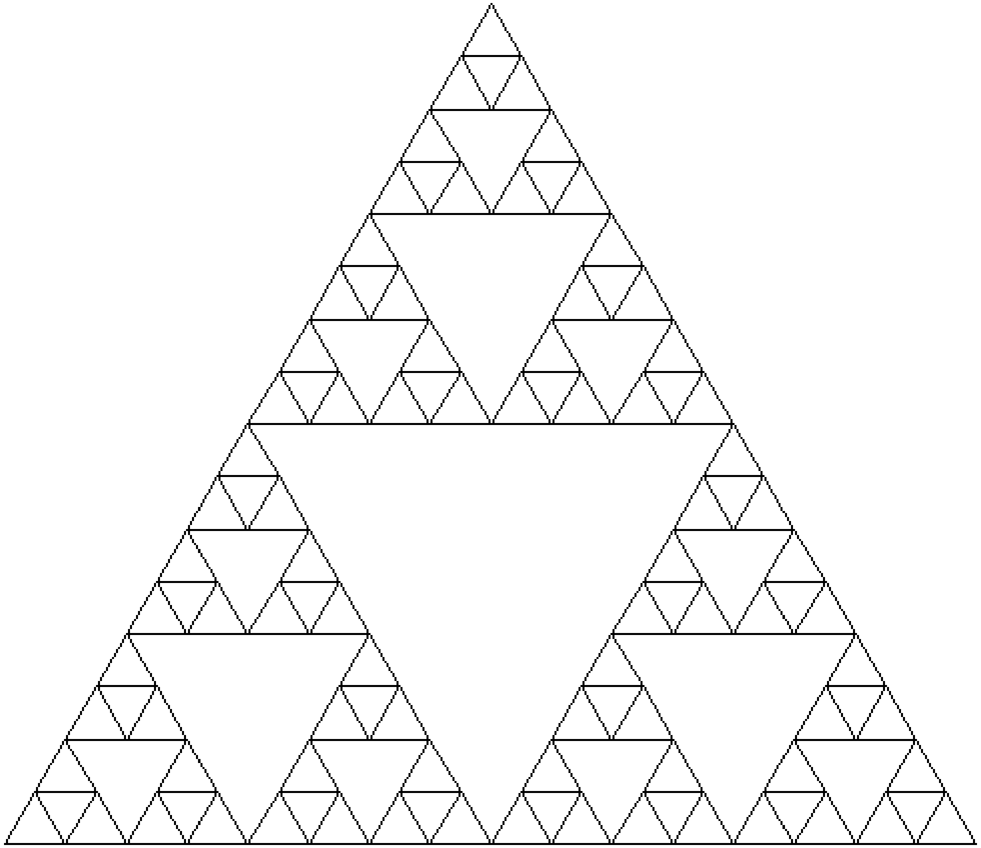

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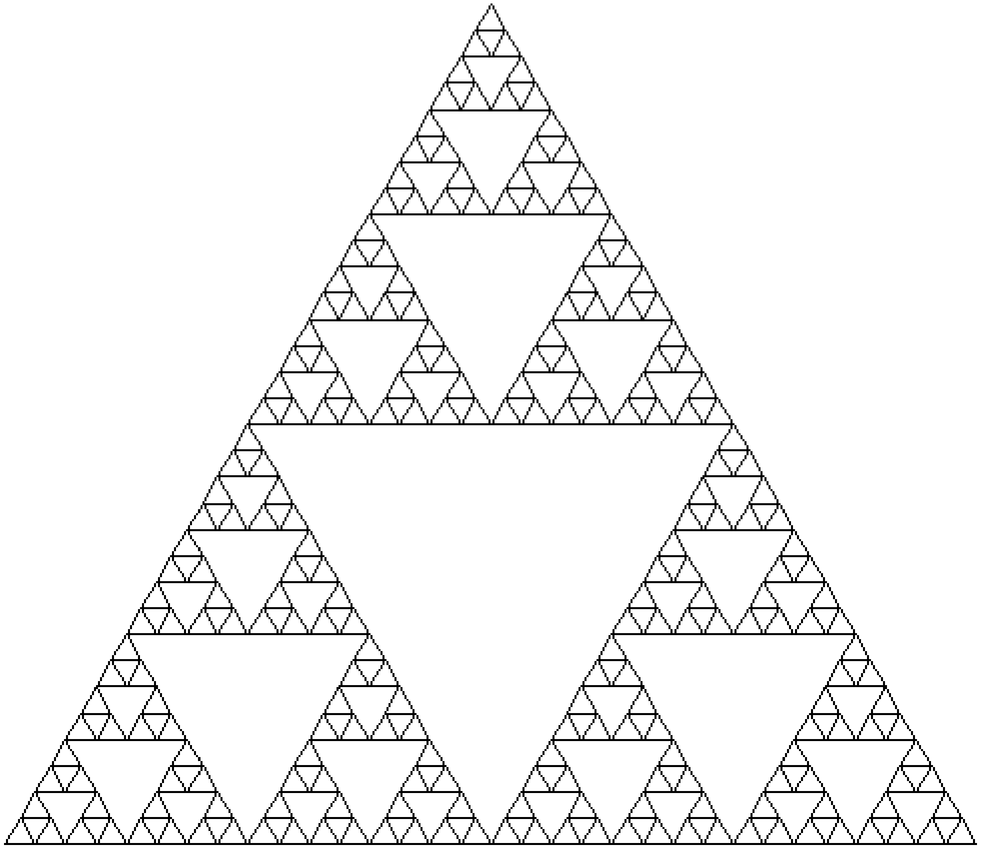

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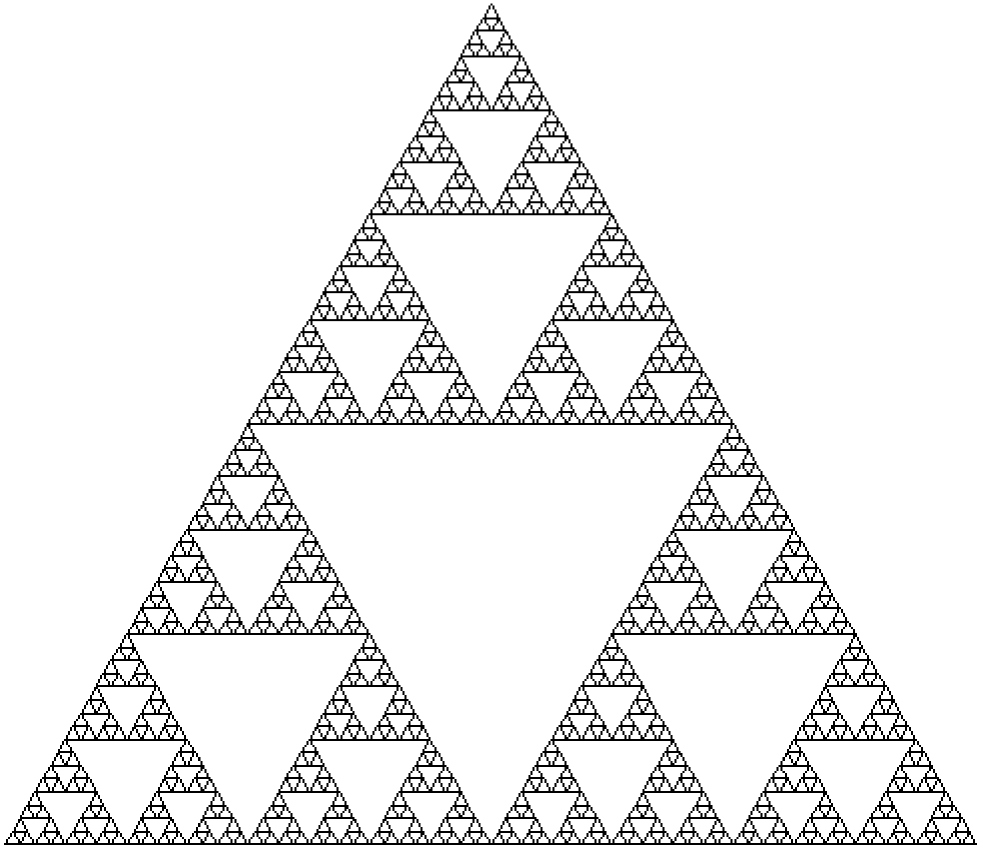

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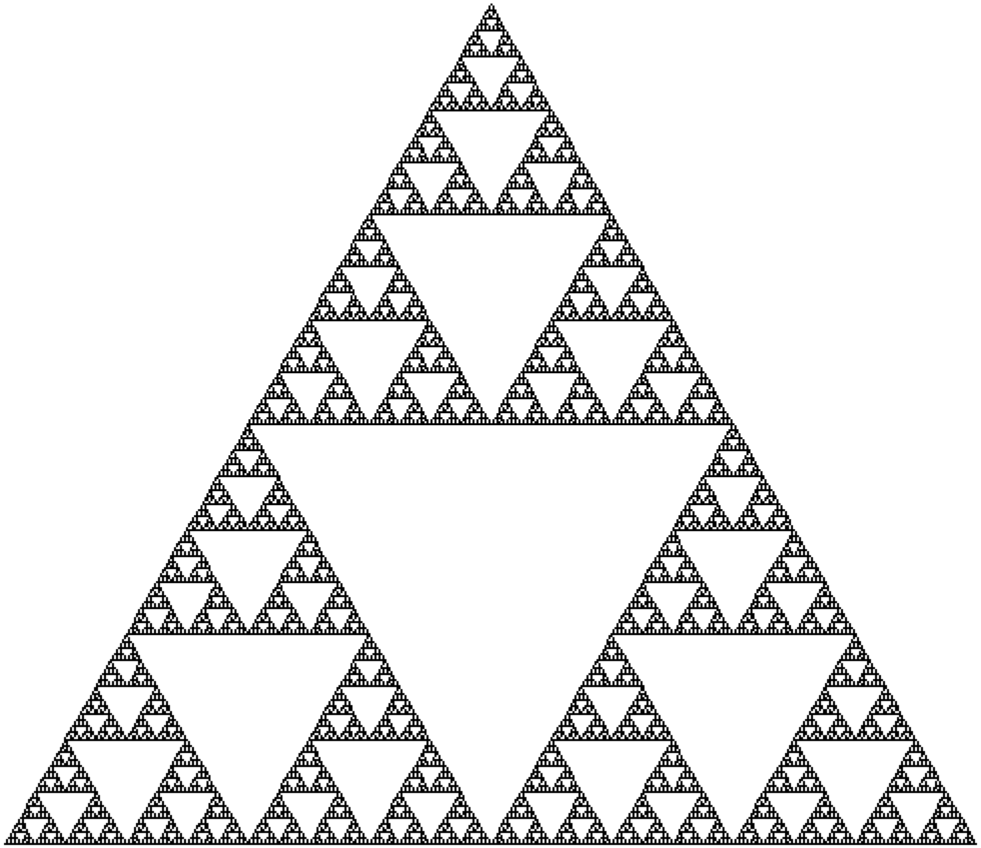

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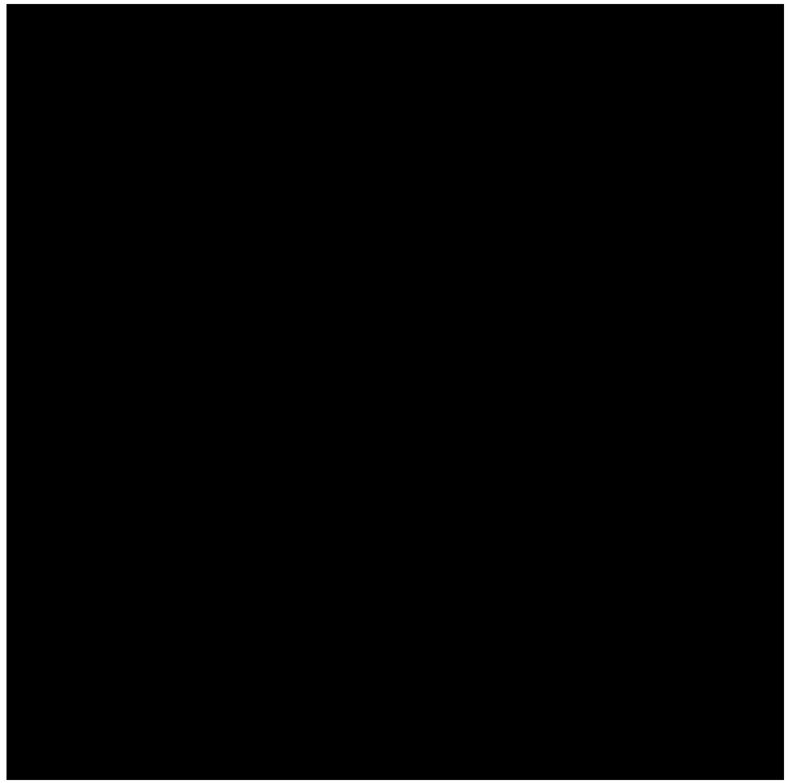

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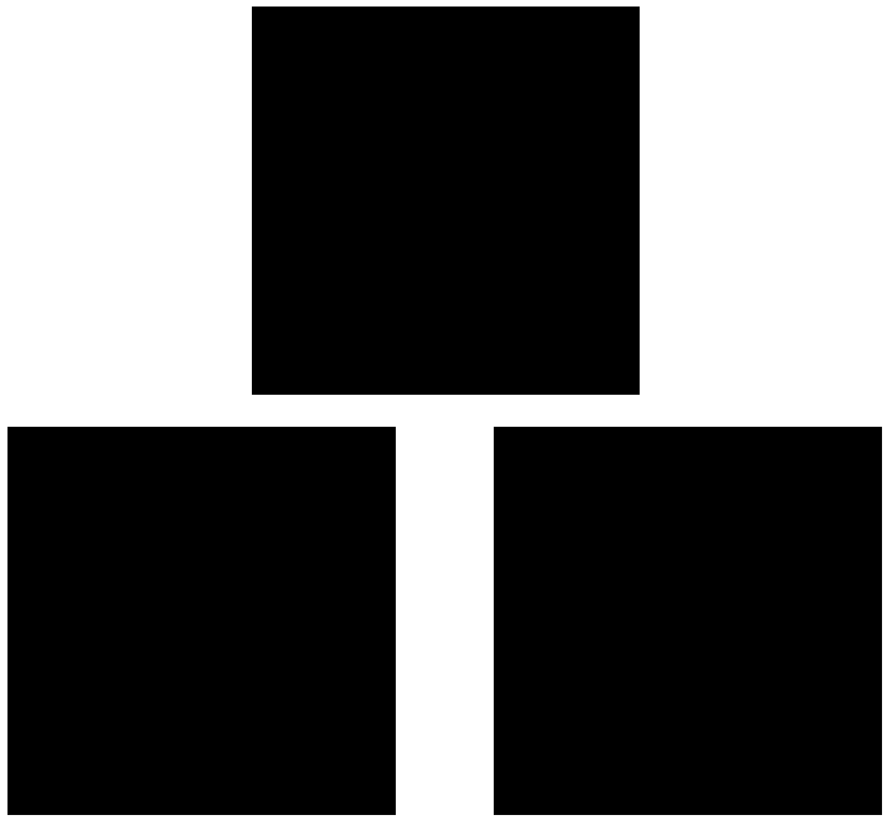

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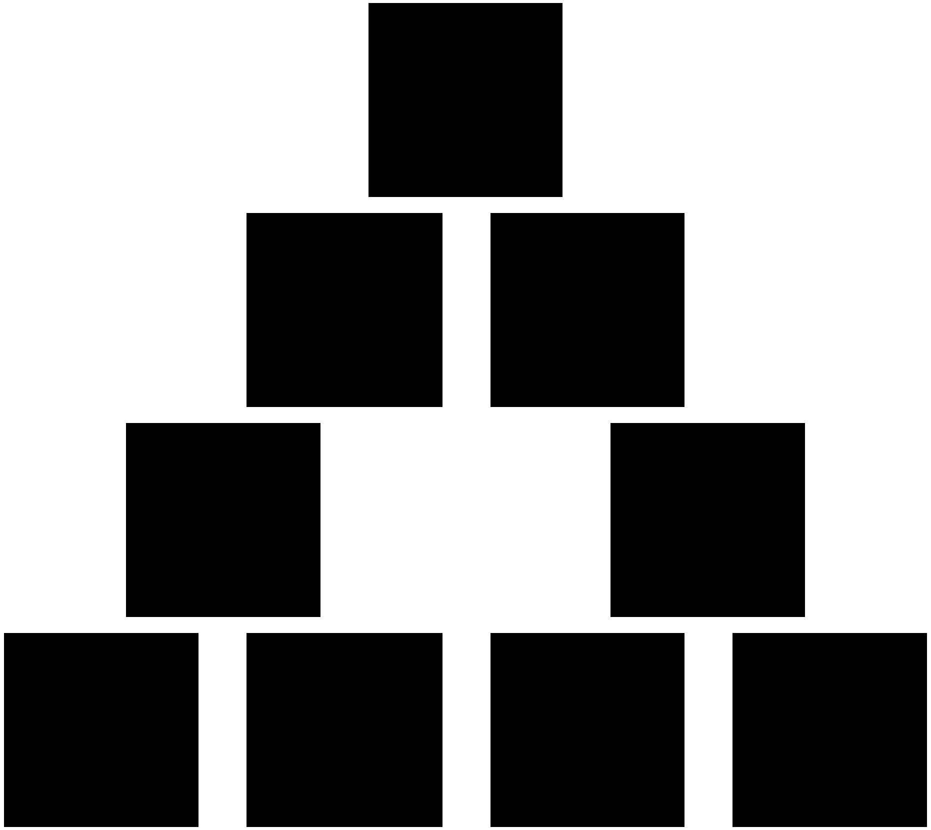

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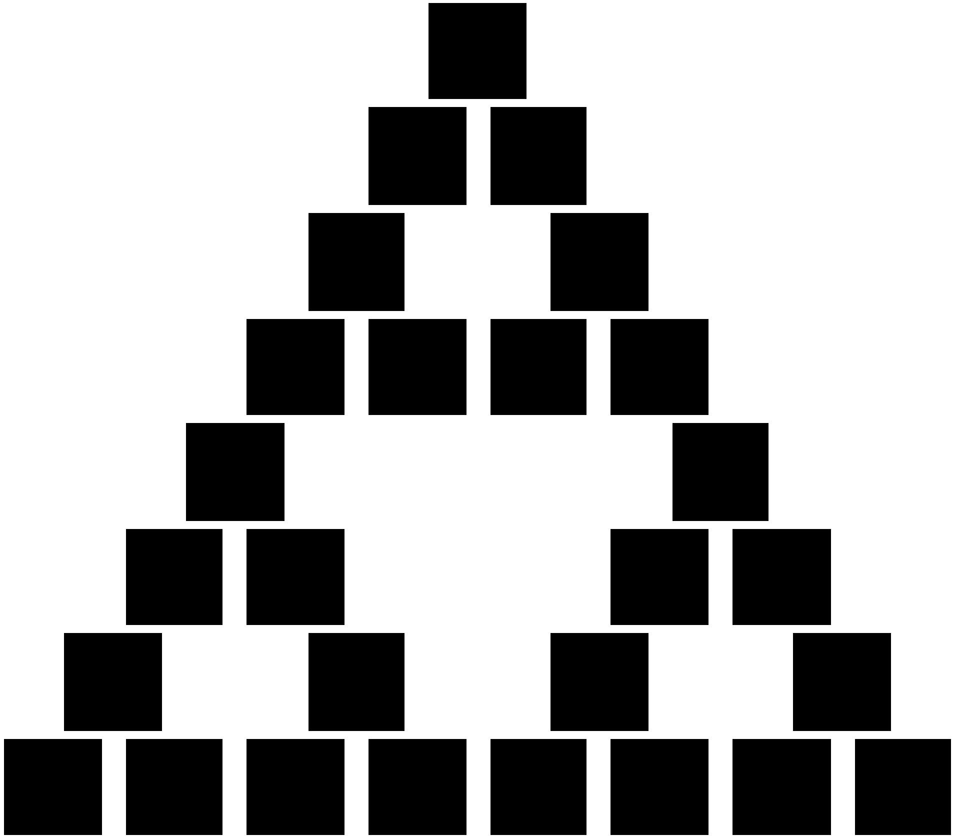

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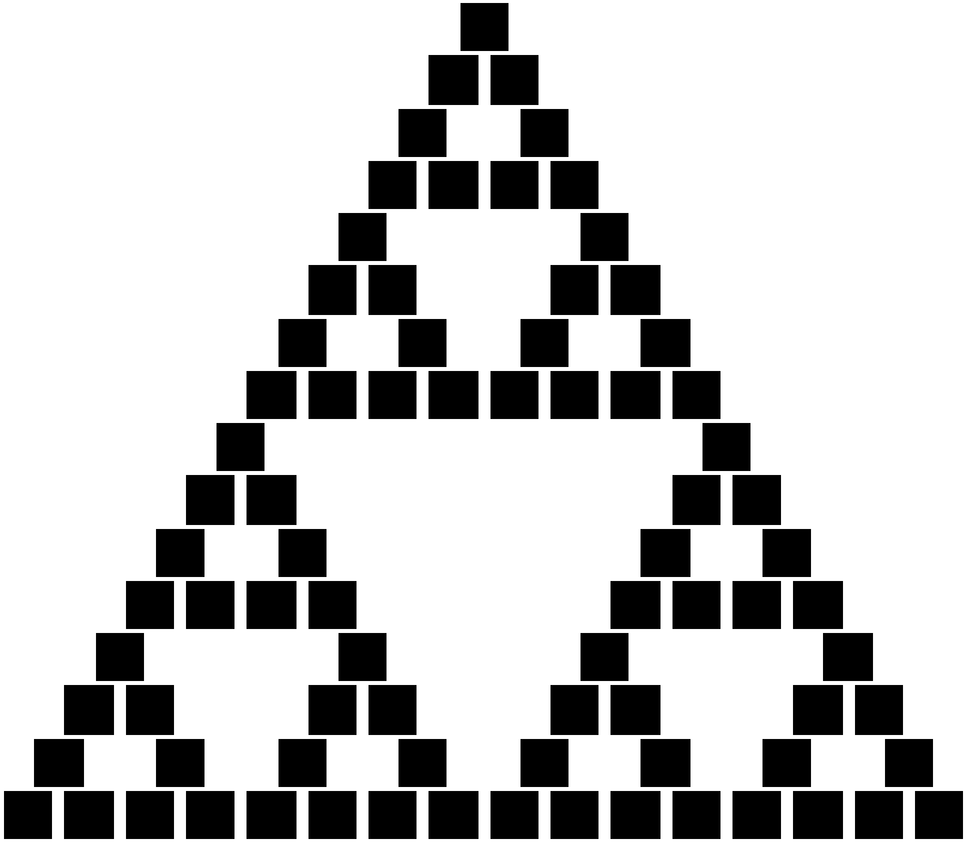

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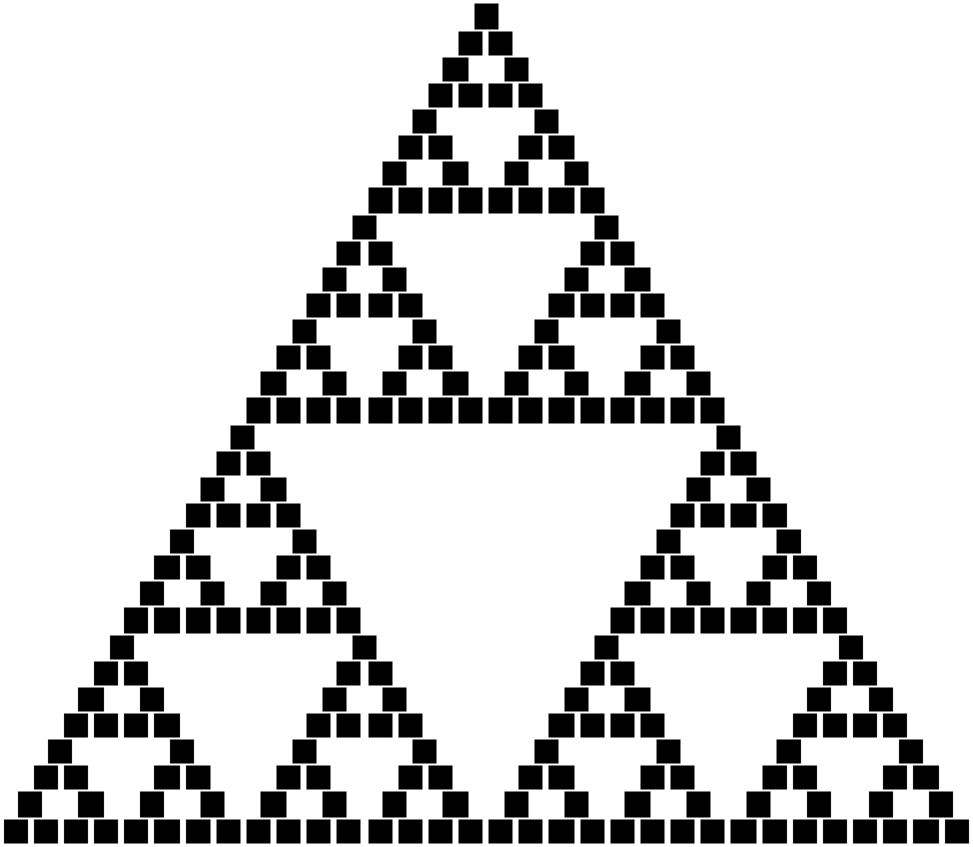

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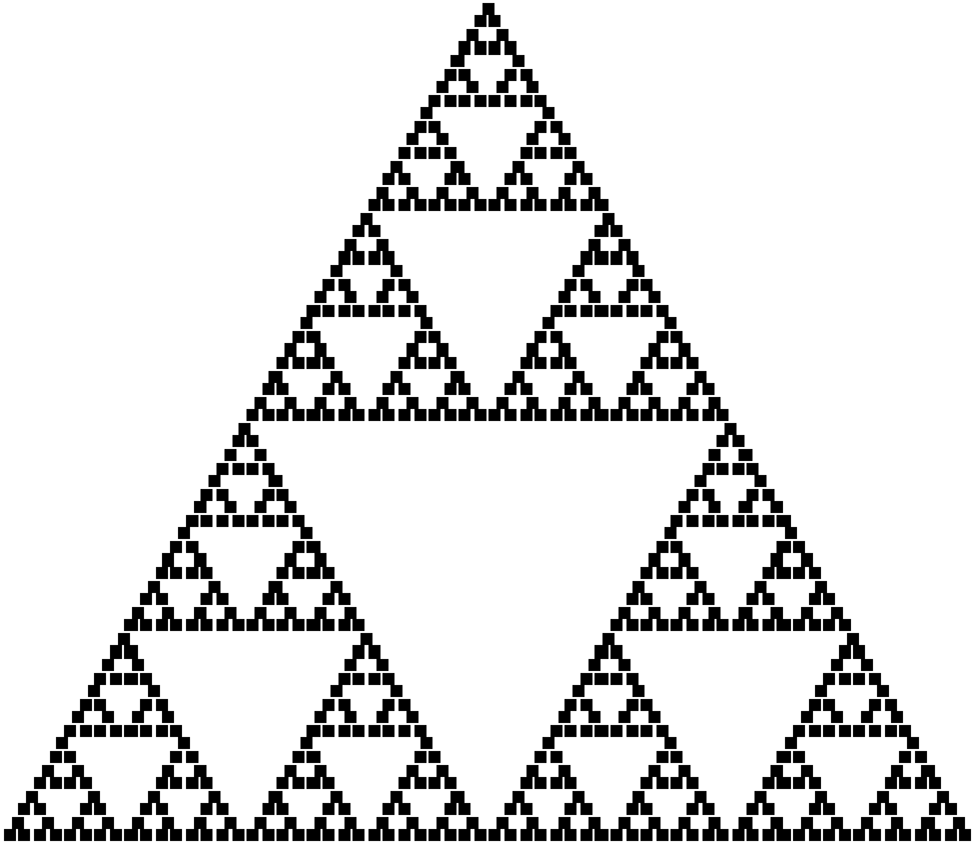

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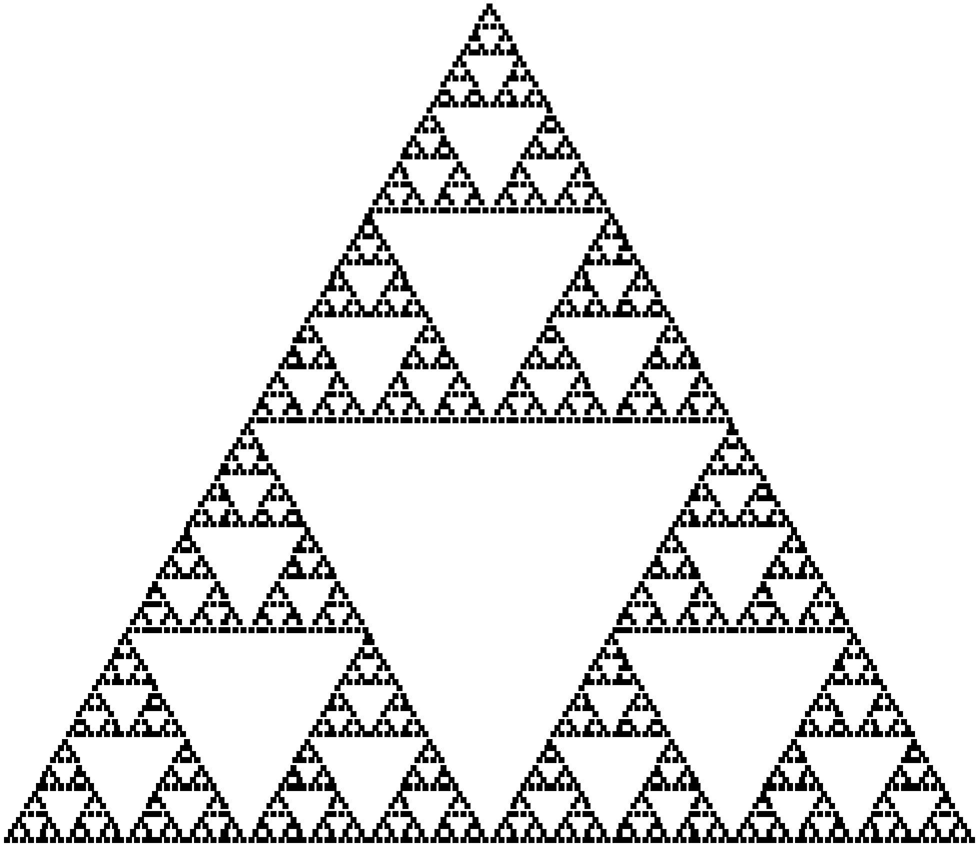


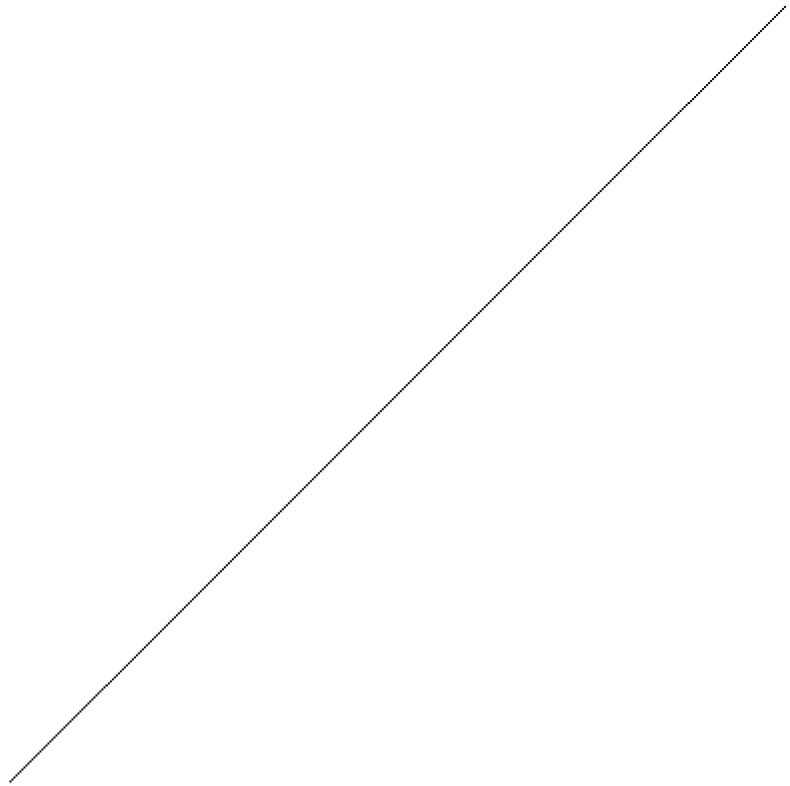
Contractive Transformations

- A transformation $F(X)$ is contractive if, for all compact sets $X_1 \neq X_2$,

$$D_H(F(X_1), F(X_2)) < D_H(X_1, X_2)$$

- A set of transformations has a unique attractor if all transformations are contractive
  - That attractor is independent of the starting shape!!!

Rendering Fractals



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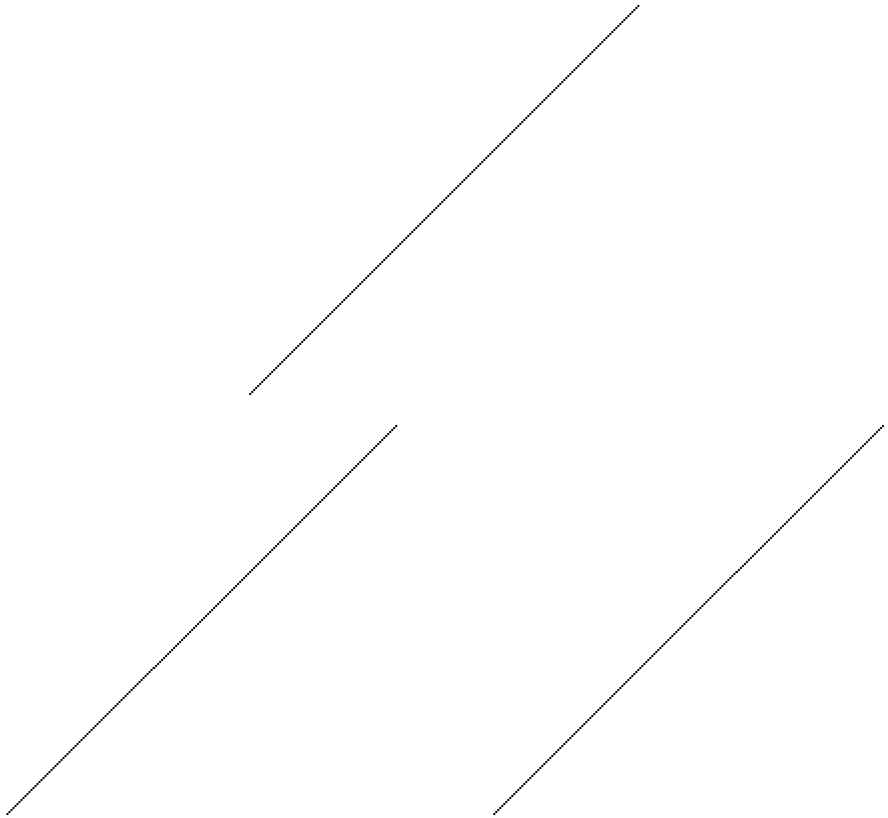

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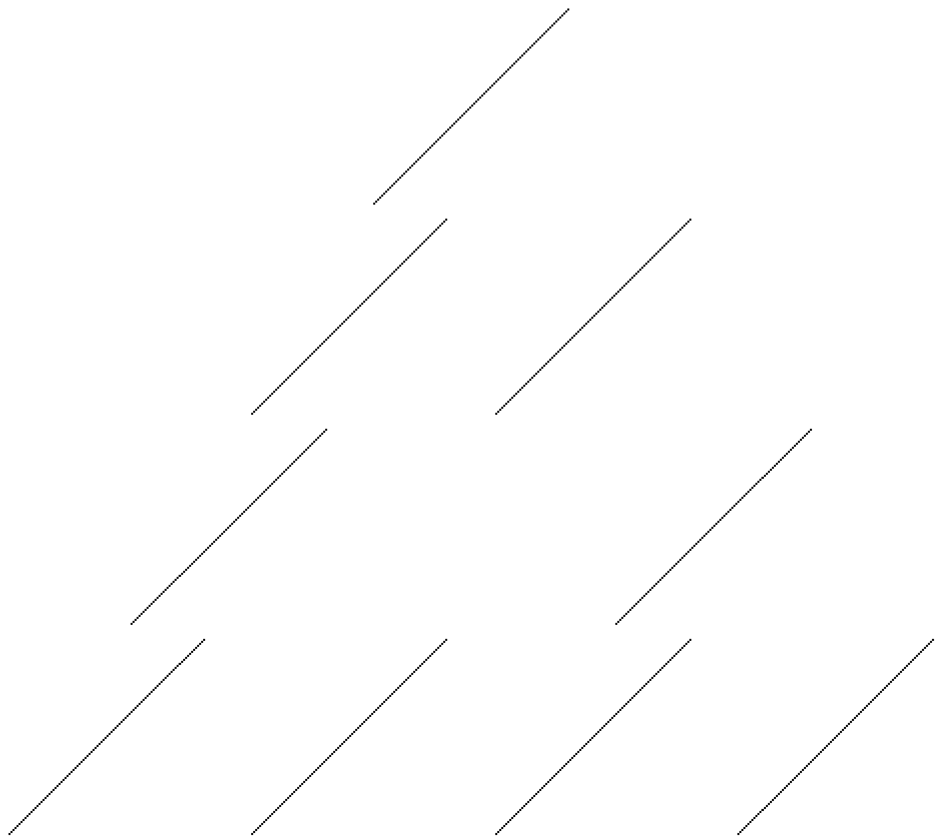

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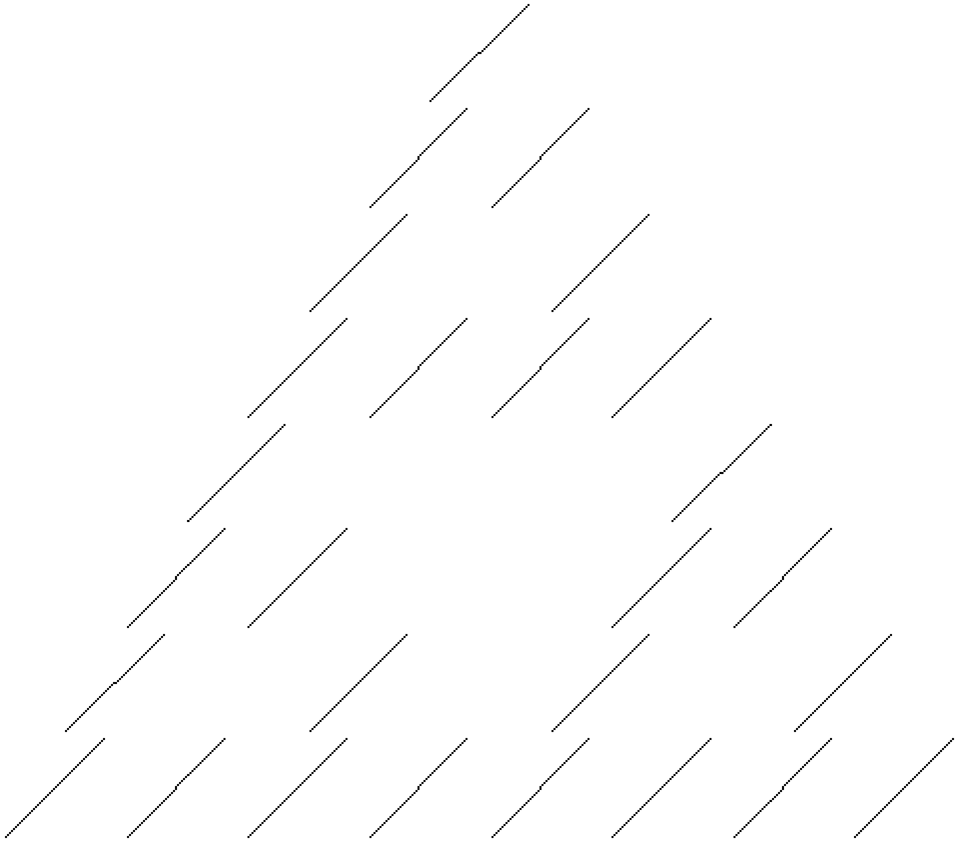

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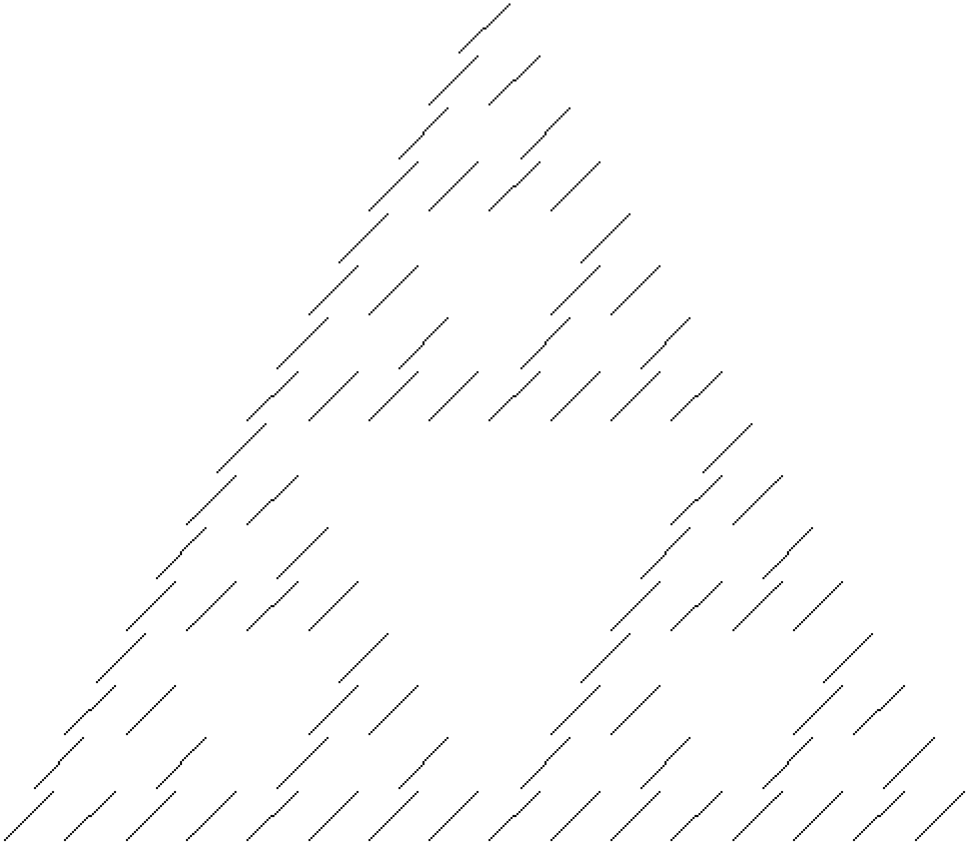

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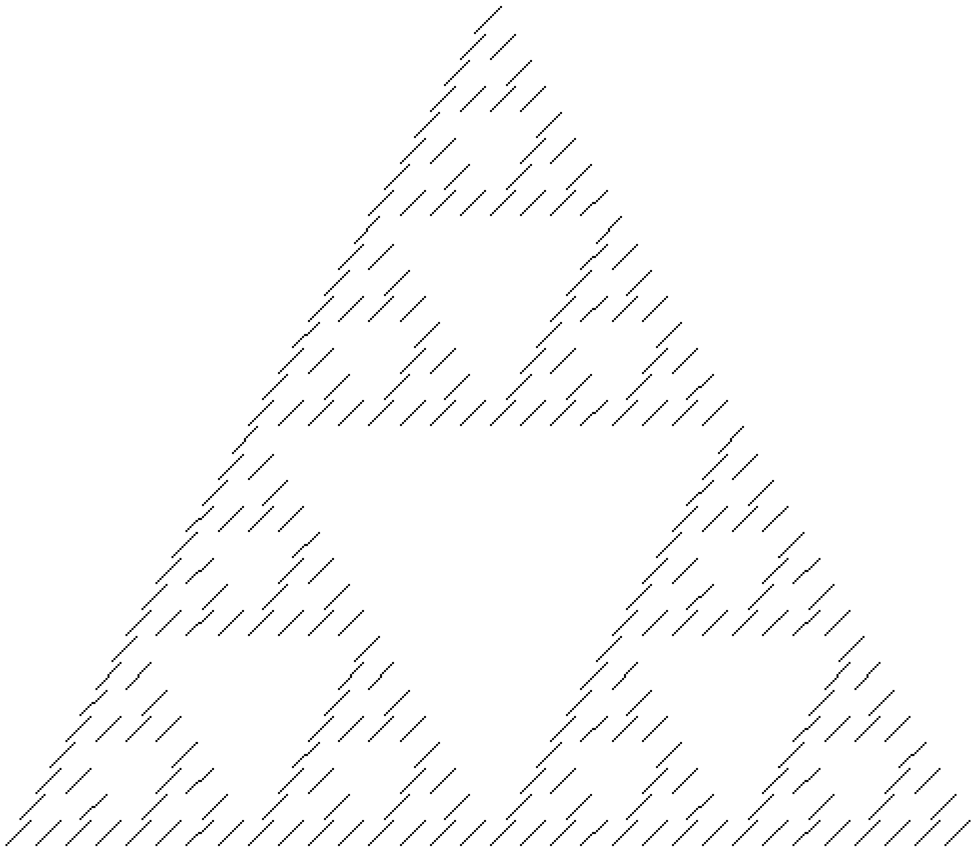

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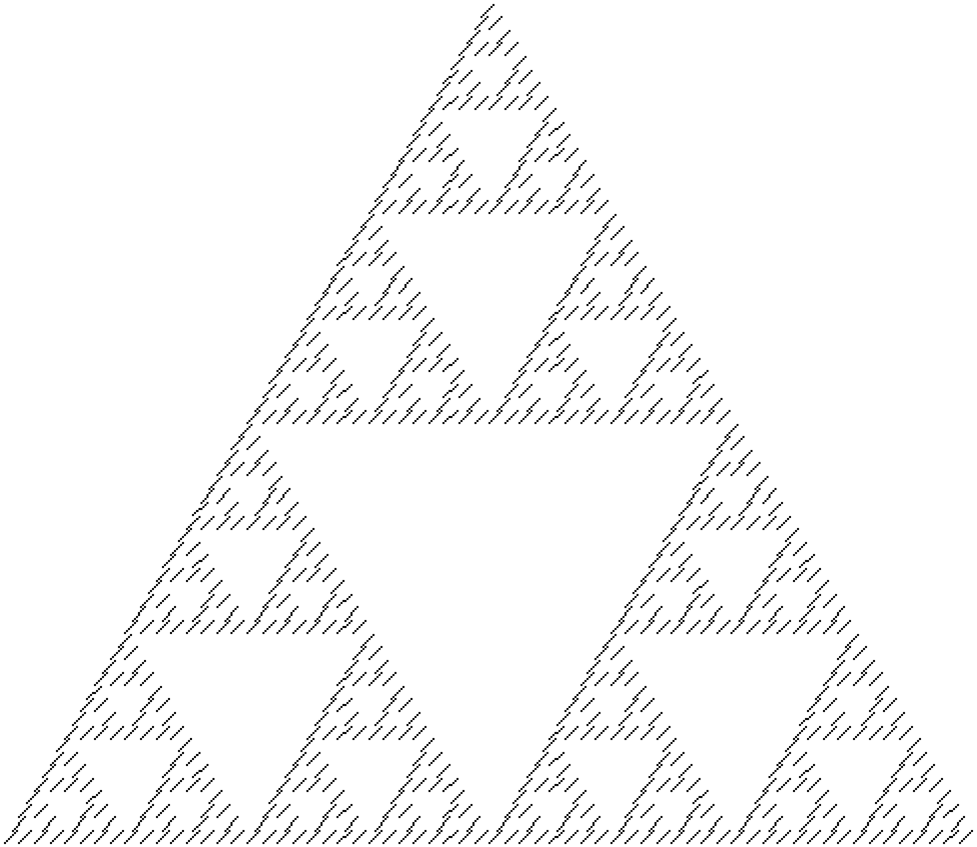

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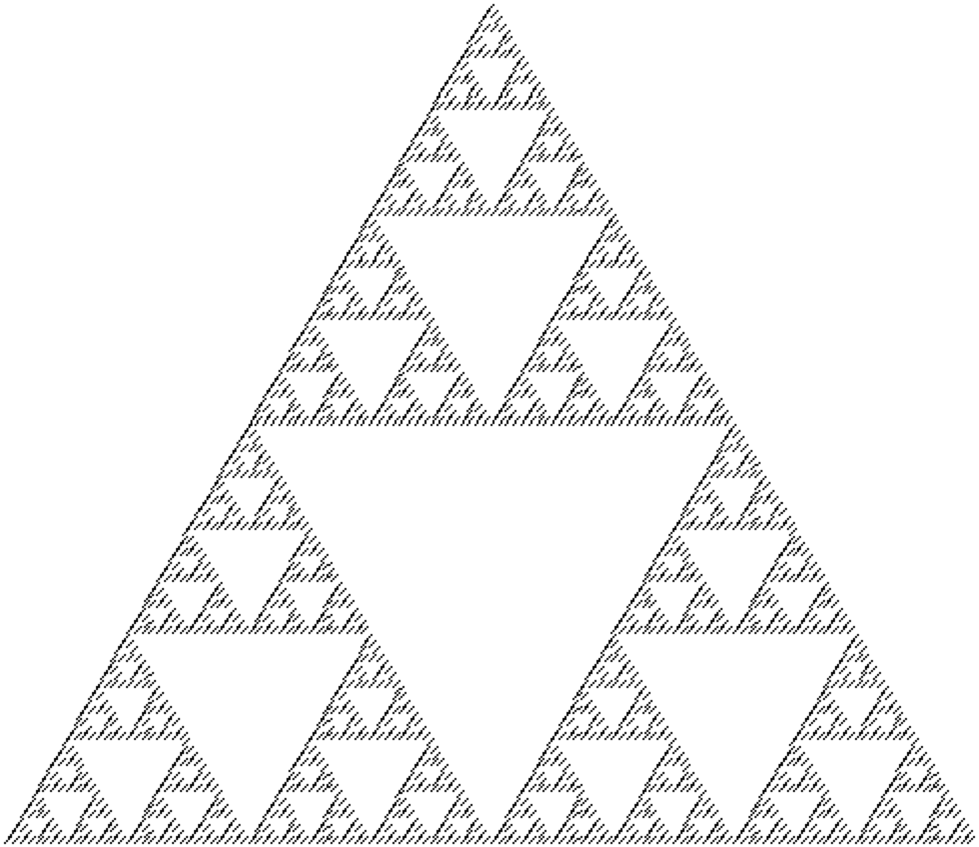

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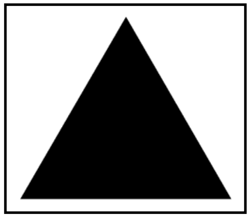

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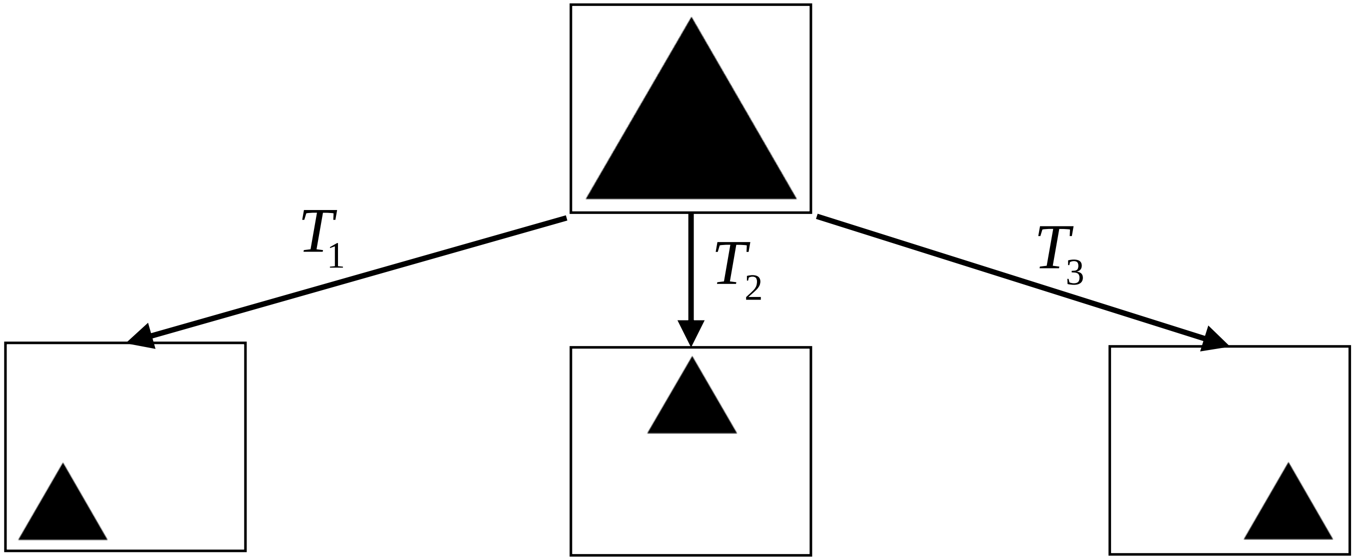

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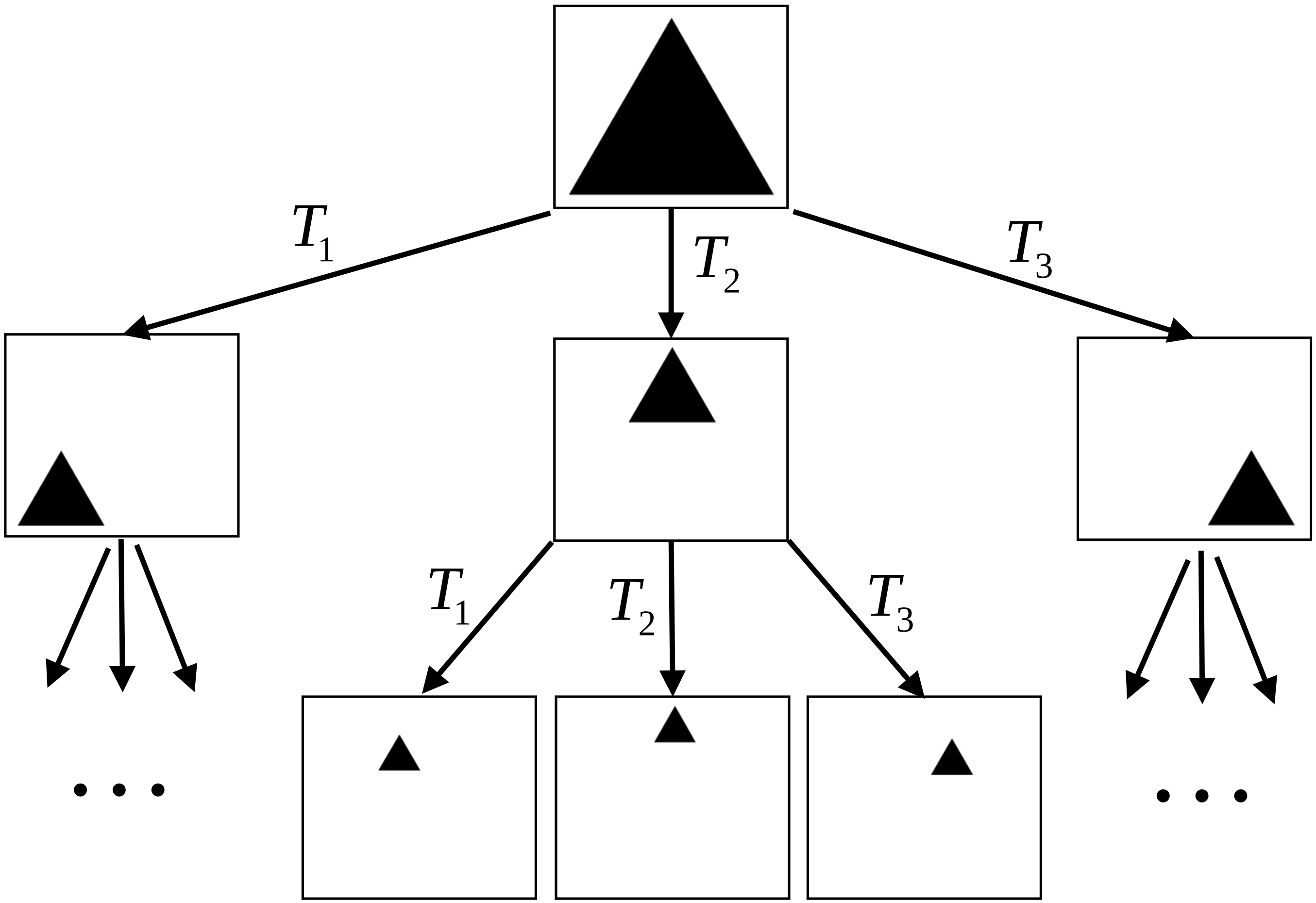

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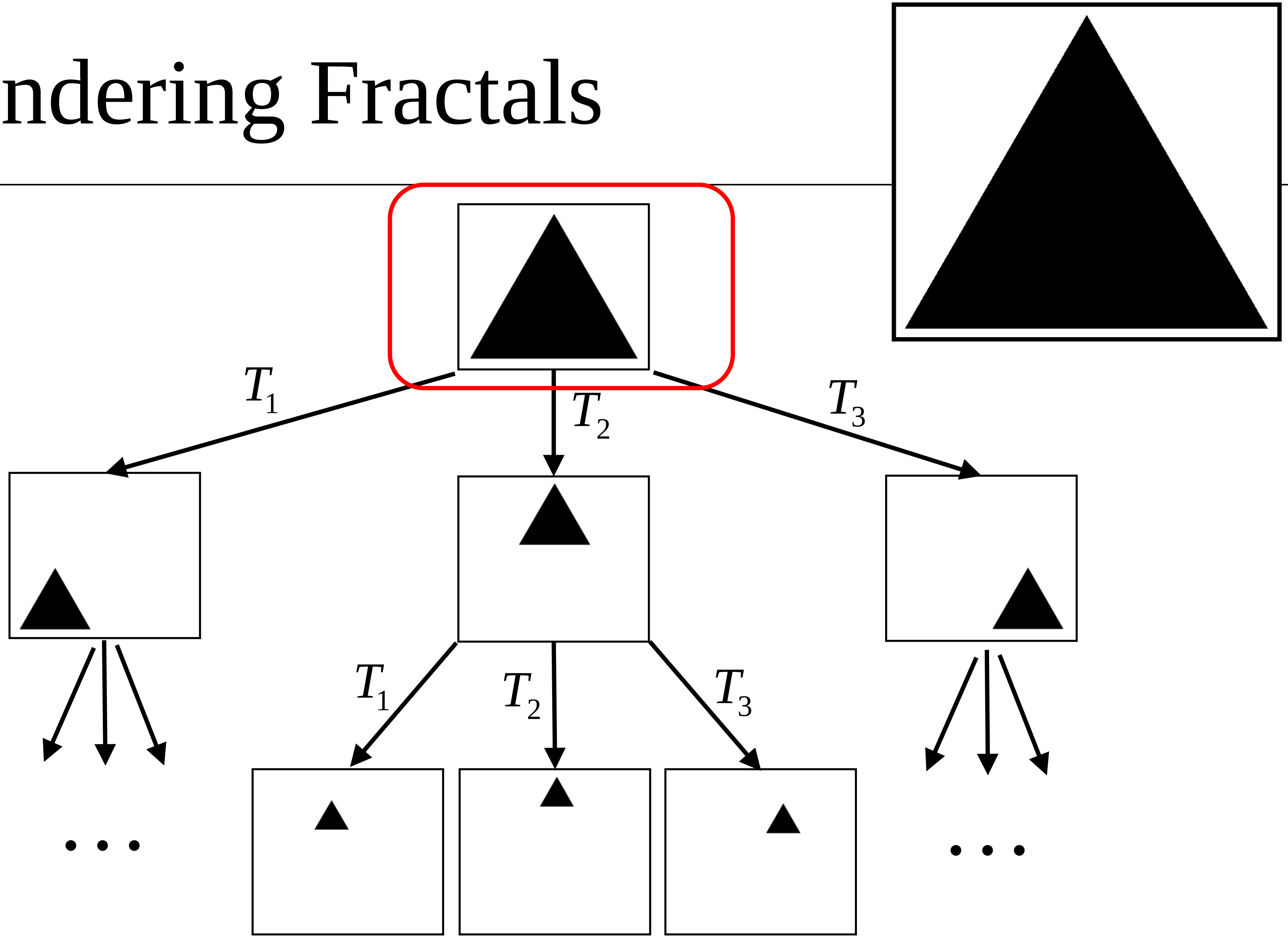

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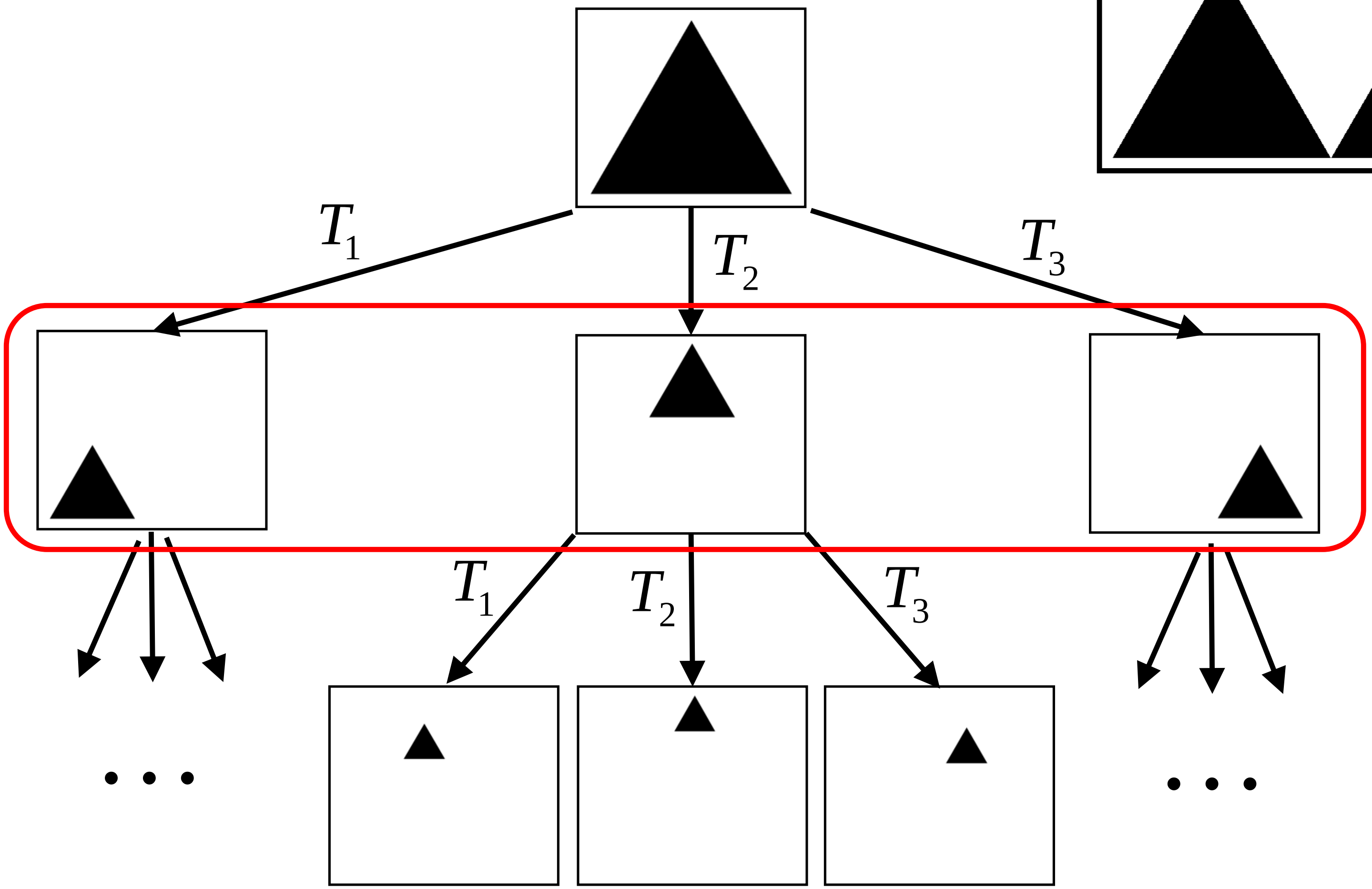

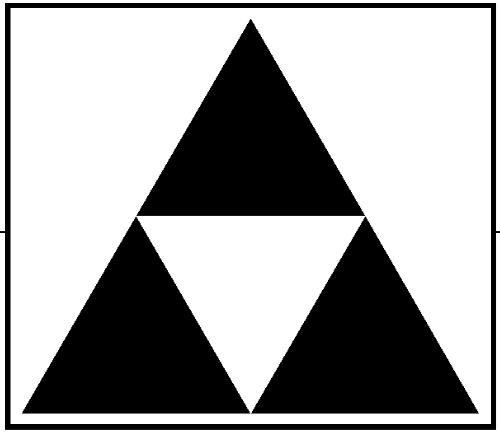

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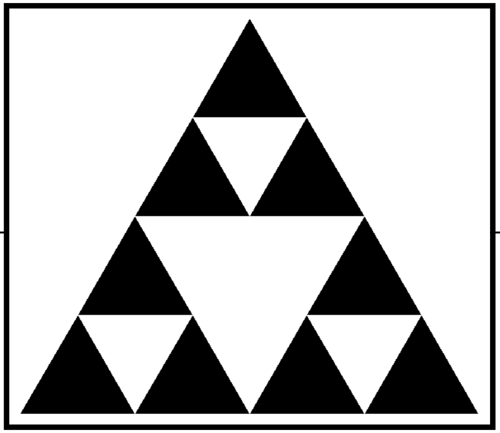
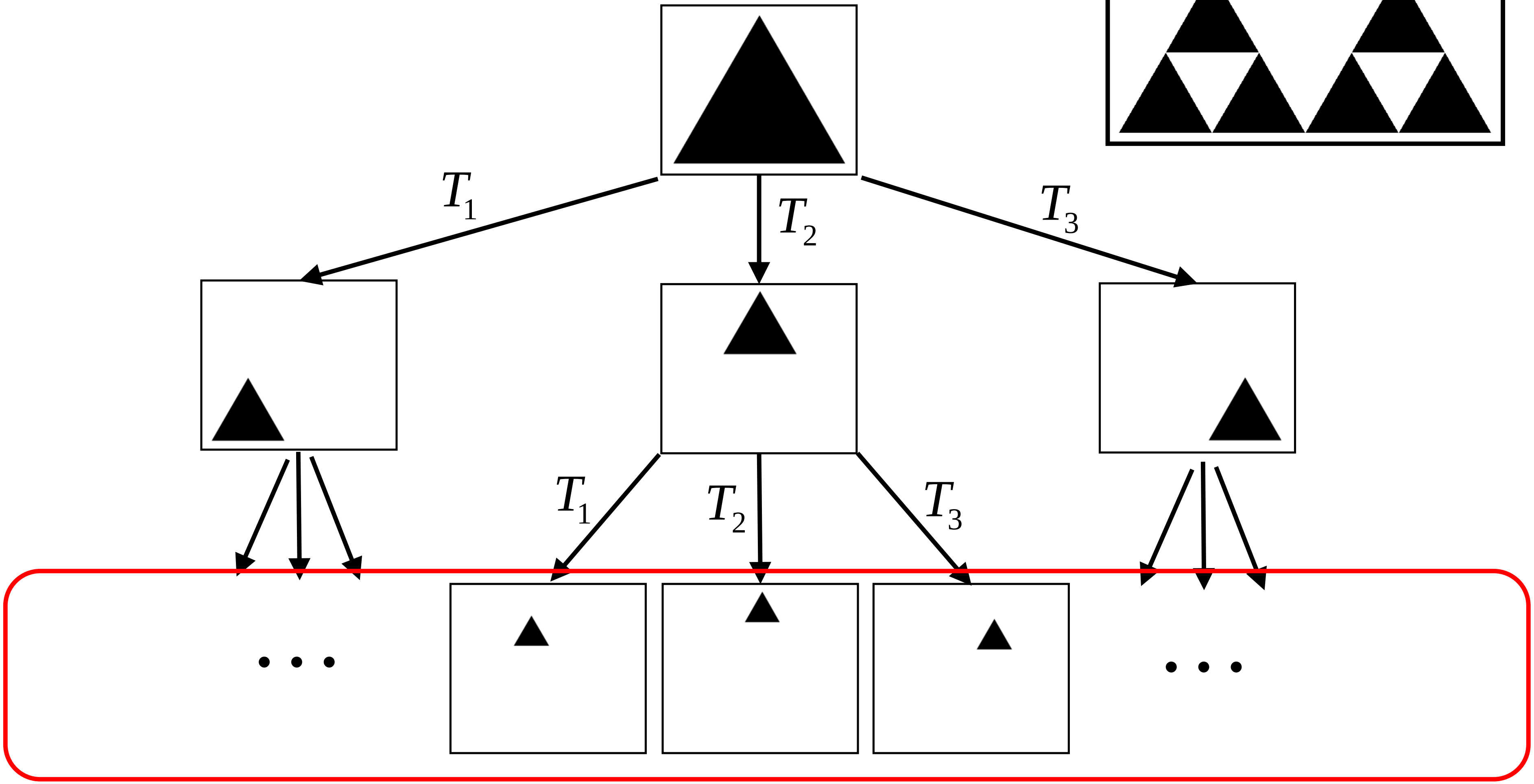
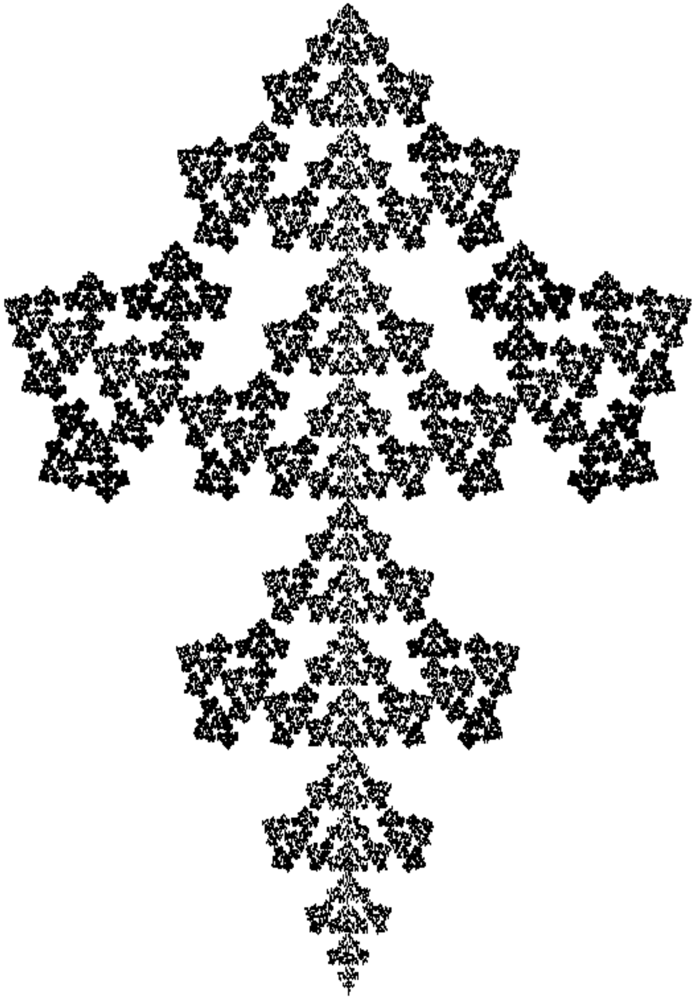




Finding Transformations



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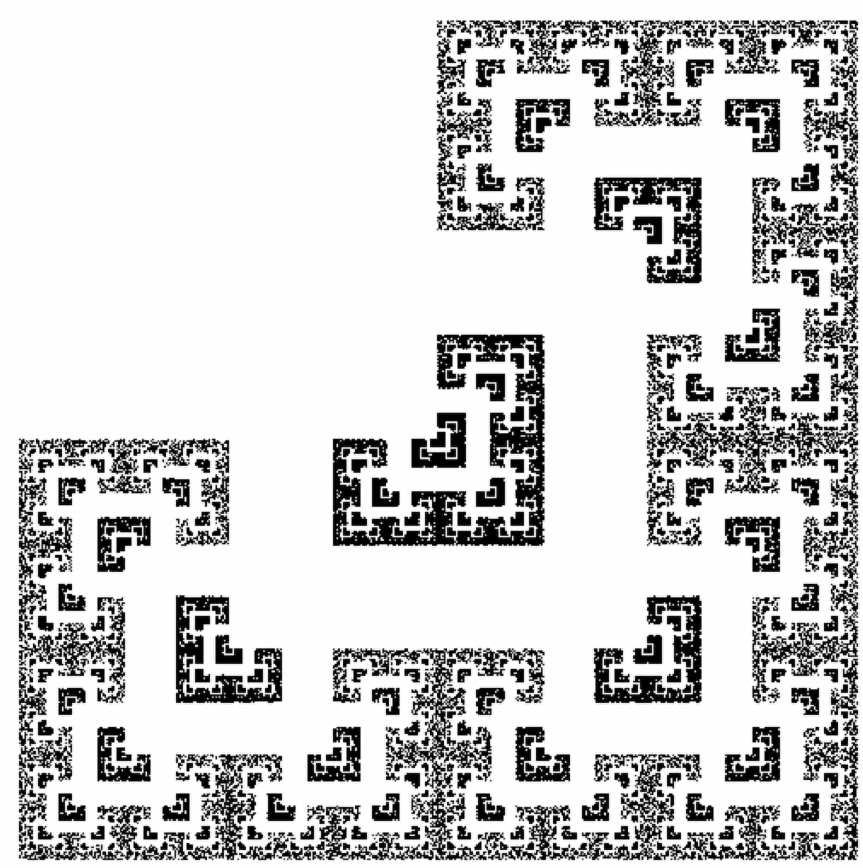

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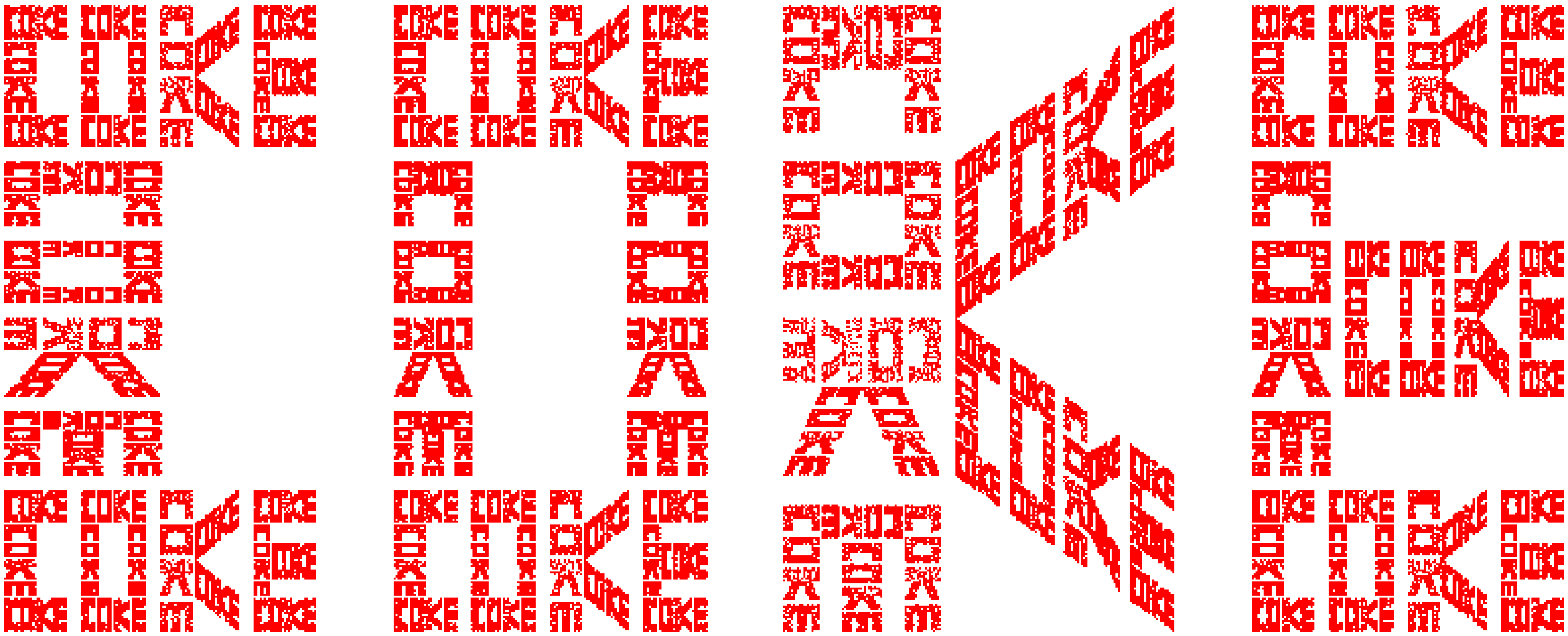

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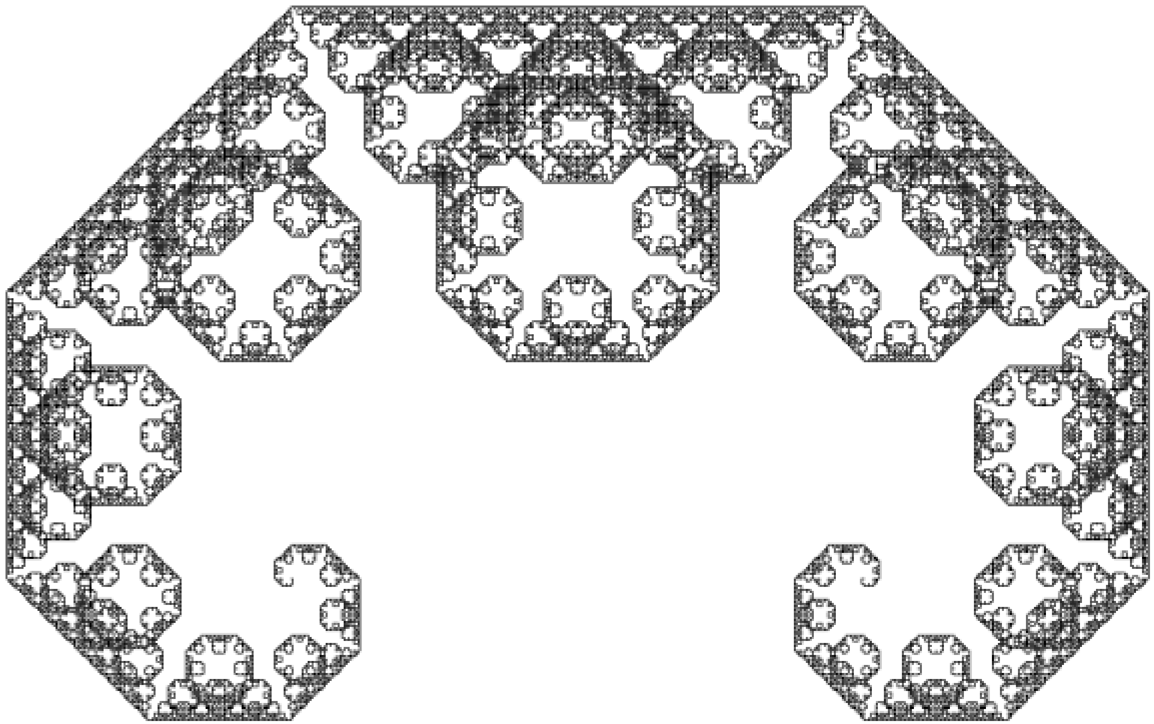


Fractal Tennis

Start with any point x

For (i=1; i<100; i++)

$x = M_{random} * x$

For (i=1; i<100000; i++)

draw(x)

$x = M_{random} * x$

Fractal Tennis

Start with any point x

For (i=1; i<100; i++)

$x = M_{random} * x$

Gets a point on the fractal

For (i=1; i<100000; i++)

draw(x)

$x = M_{random} * x$

Fractal Tennis

Start with any point x

For (i=1; i<100; i++)

$x = M_{random} * x$

For (i=1; i<100000; i++)

draw(x)

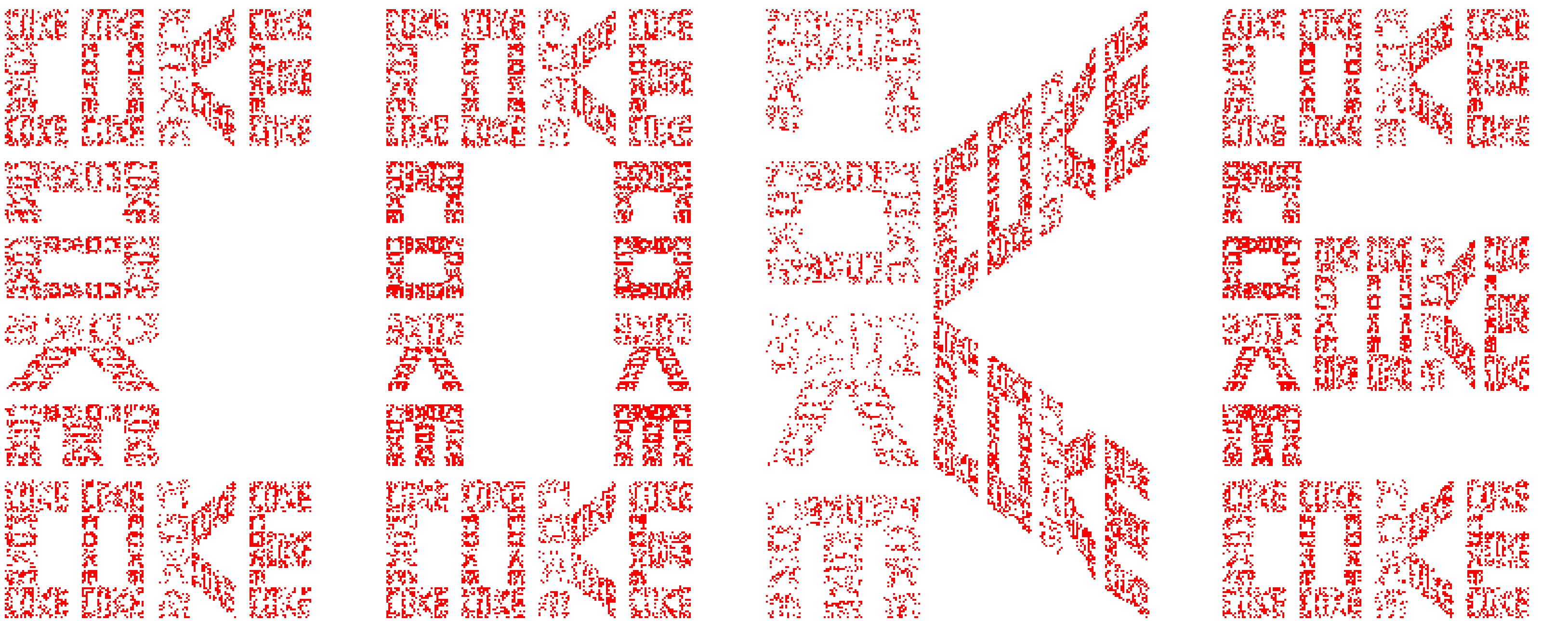
$x = M_{random} * x$

Creates new points on the fractal

Fractal Tennis – Example

25,000 Points

Fractal Tennis – Example



50,000 Points

Fractal Tennis – Example

75,000 Points

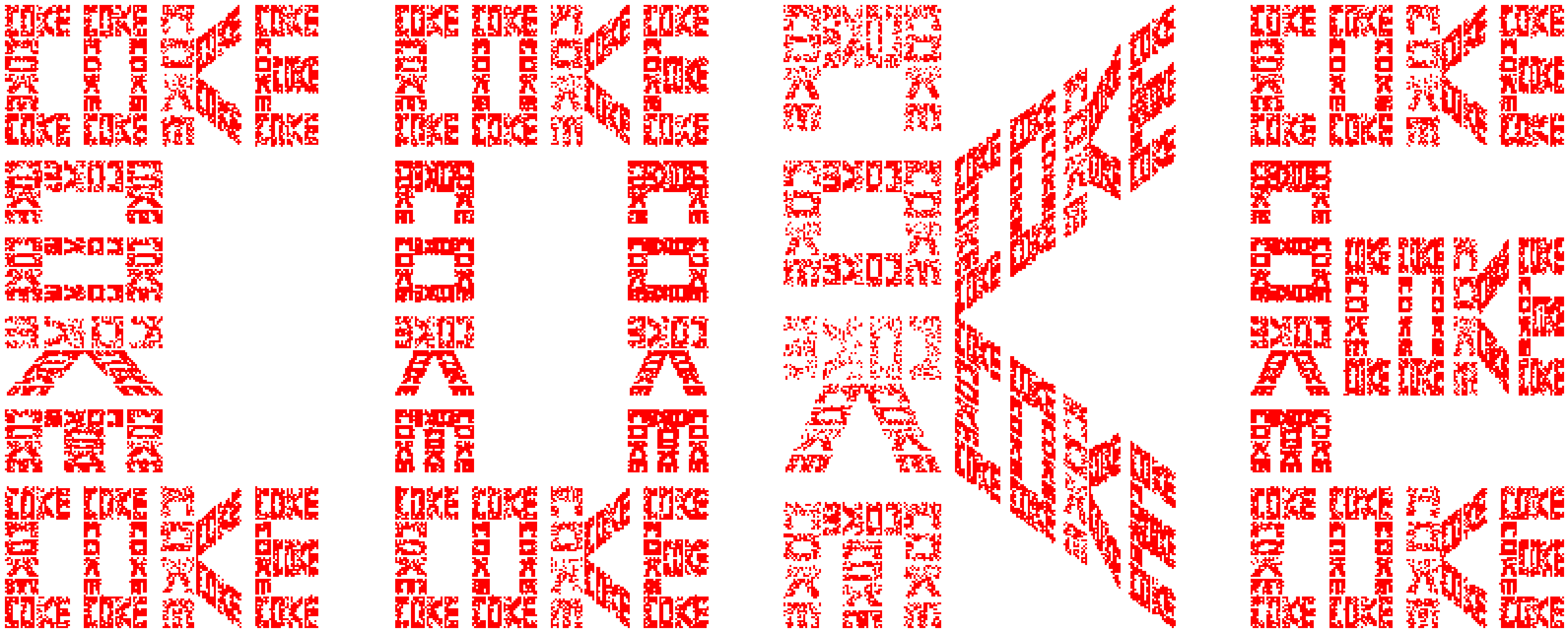
Fractal Tennis – Example

100,000 Points

Fractal Tennis – Example

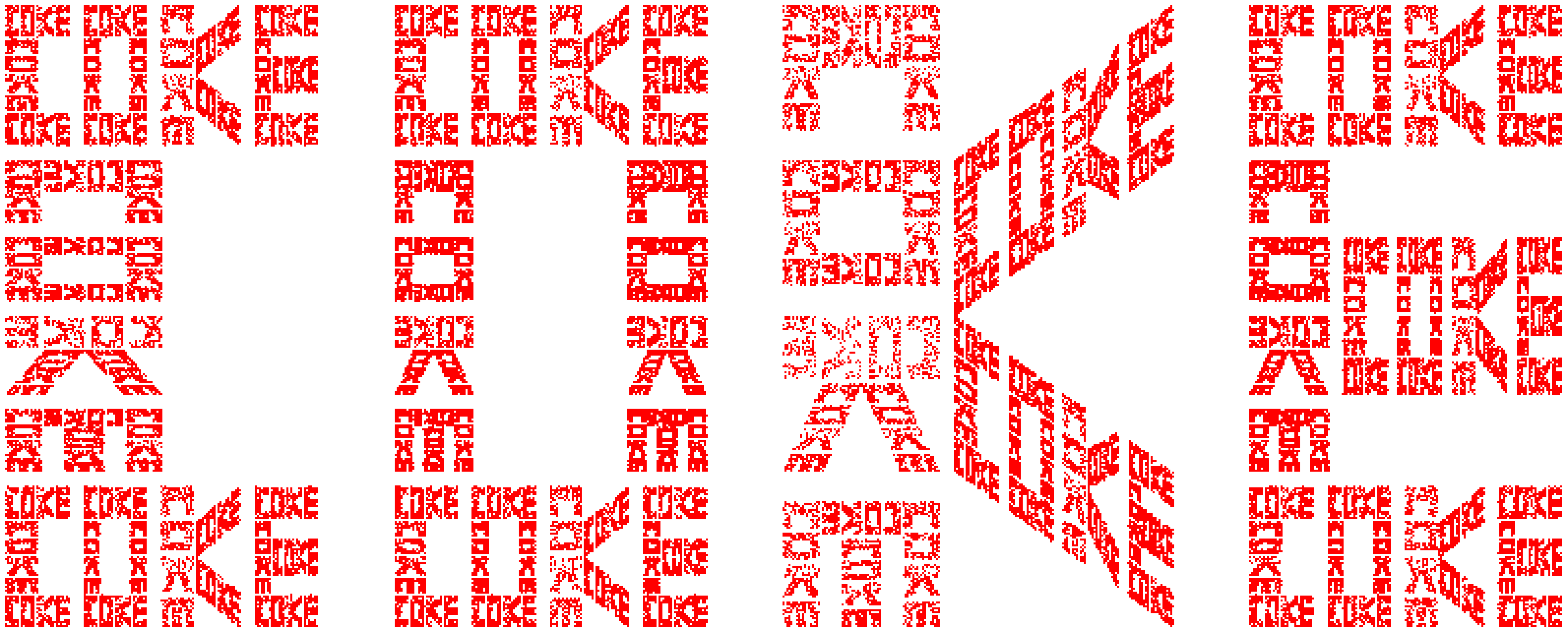
125,000 Points

Fractal Tennis – Example



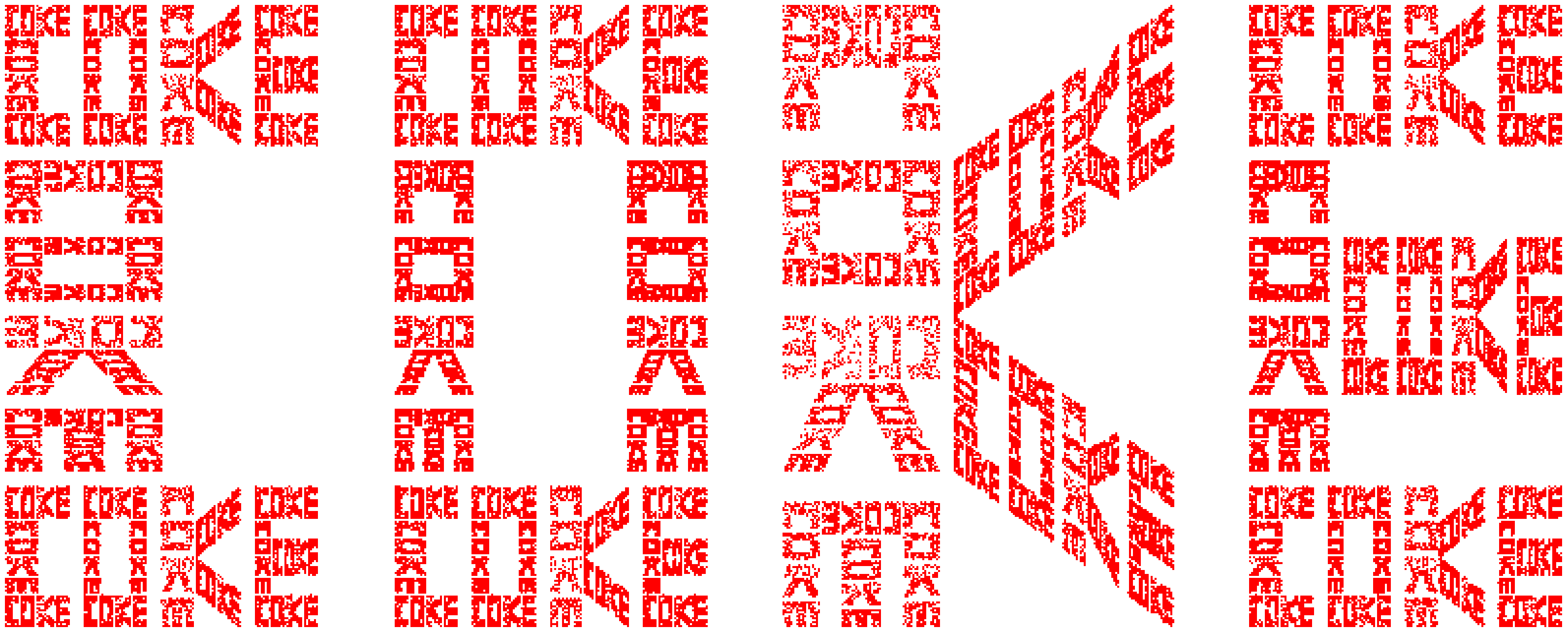
150,000 Points

Fractal Tennis – Example



175,000 Points

Fractal Tennis – Example



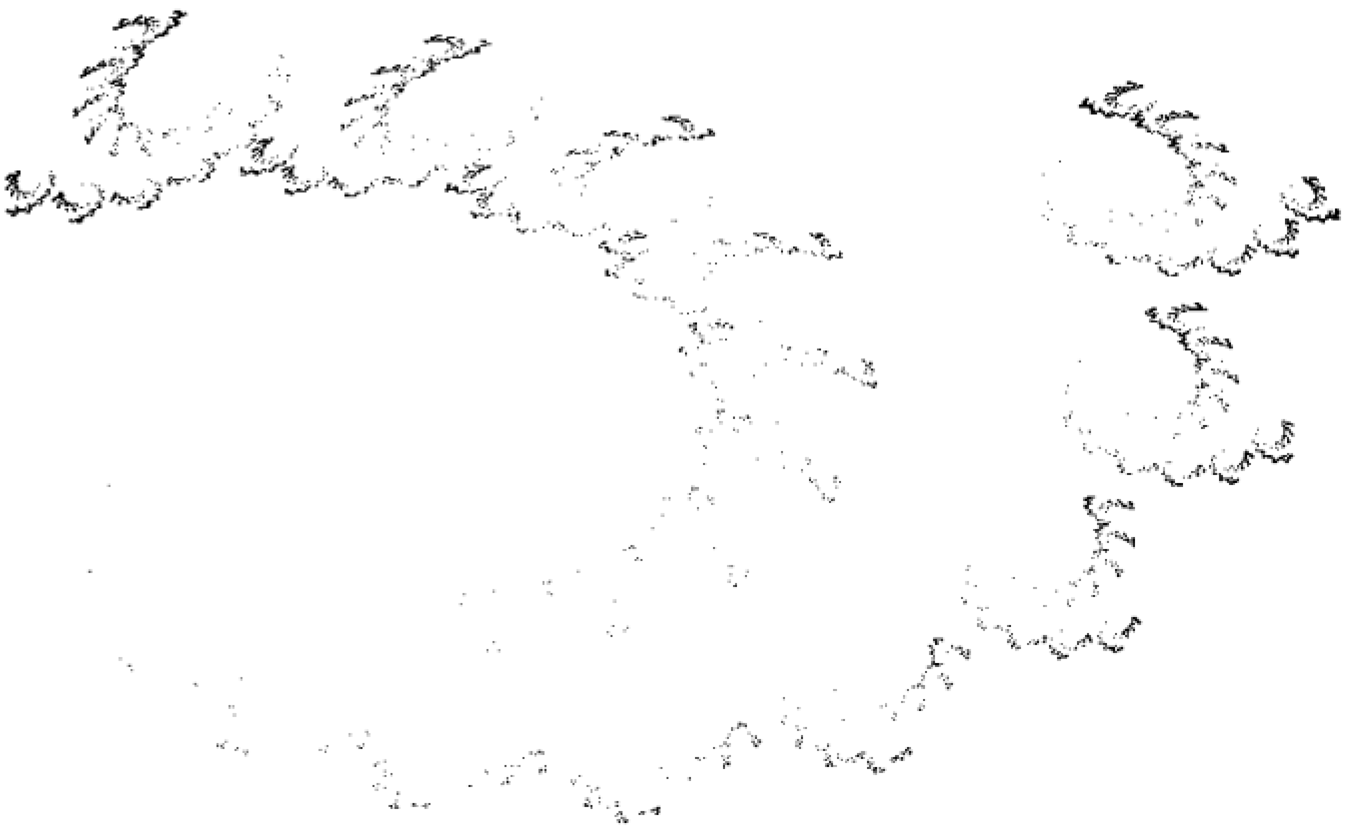
200,000 Points

Modified Fractal Tennis

Modified Fractal Tennis

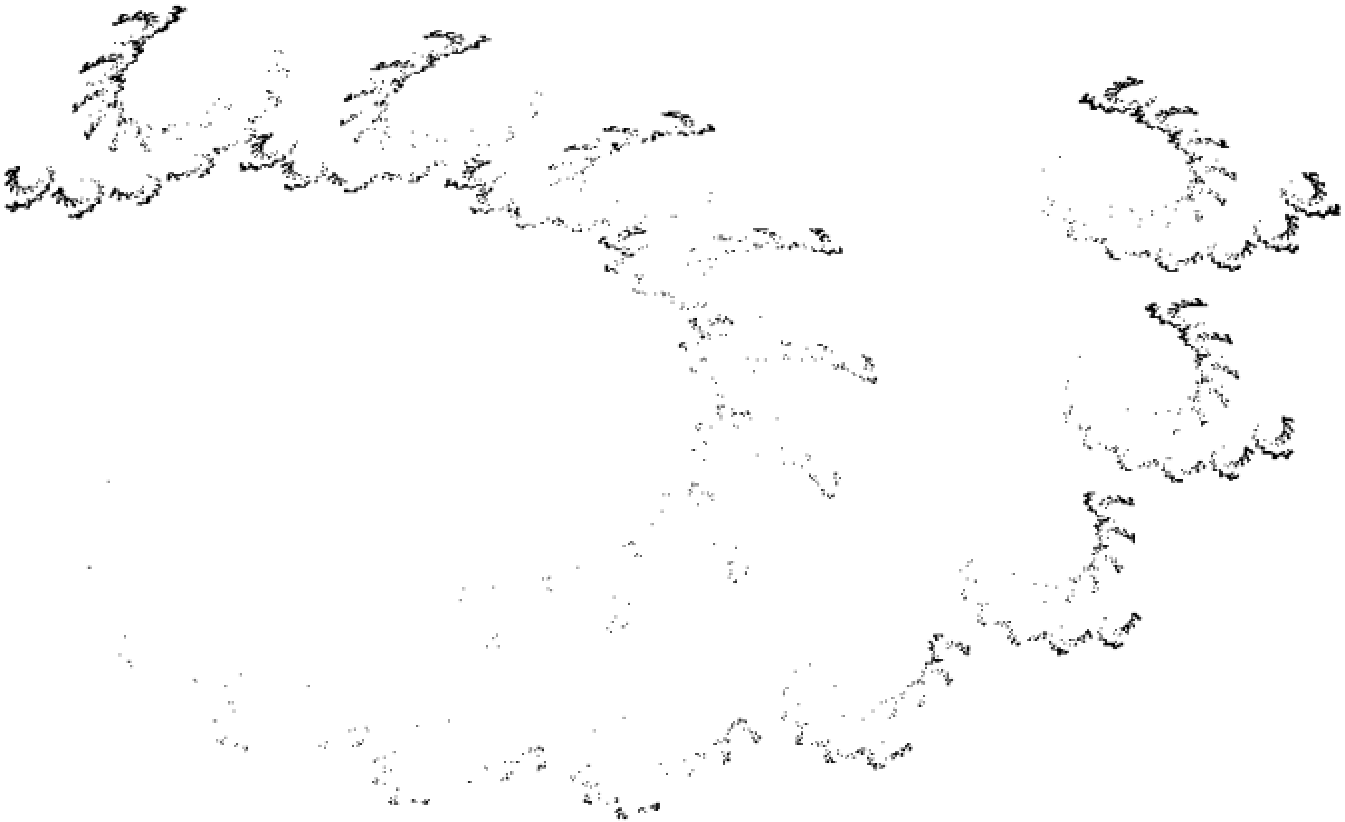
25,000 Points

Modified Fractal Tennis



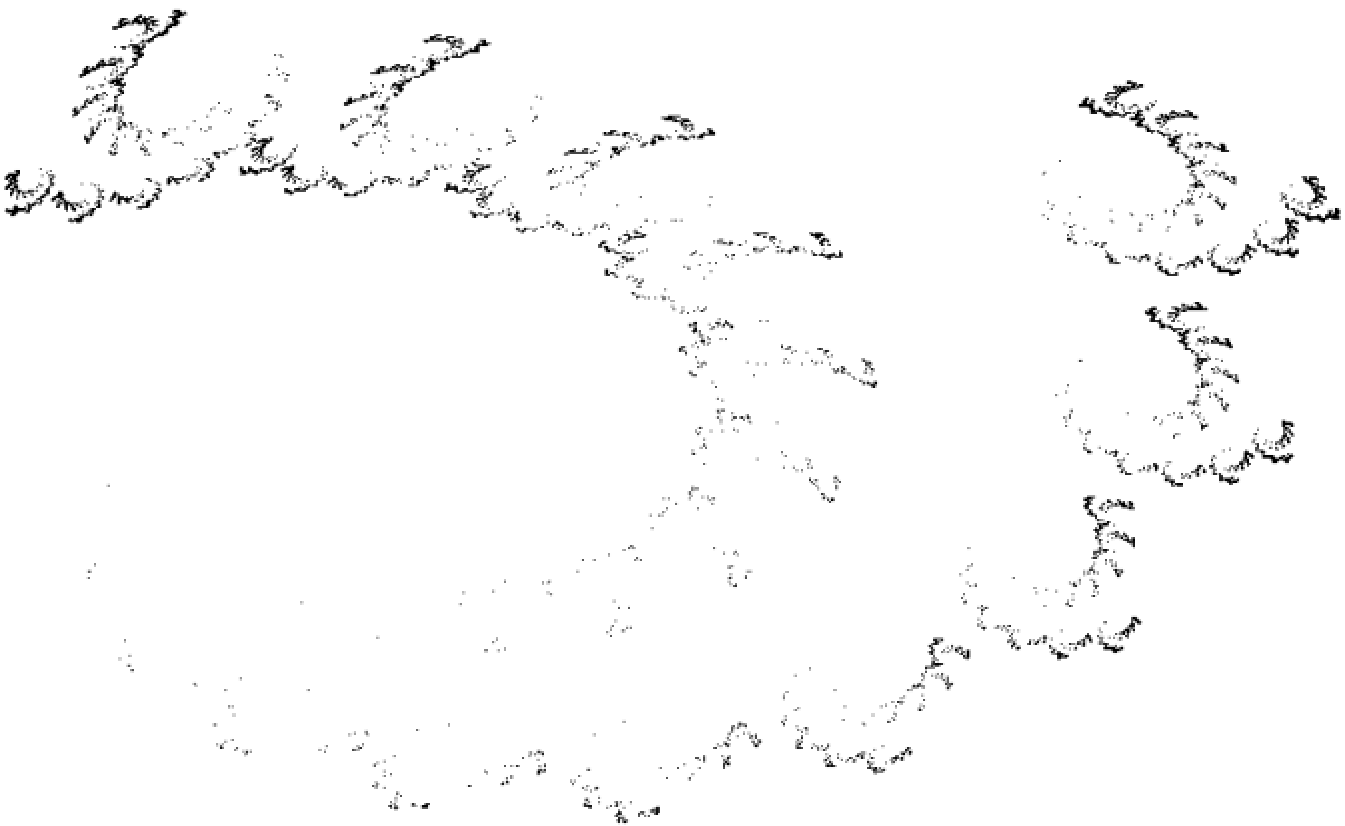
50,000 Points

Modified Fractal Tennis



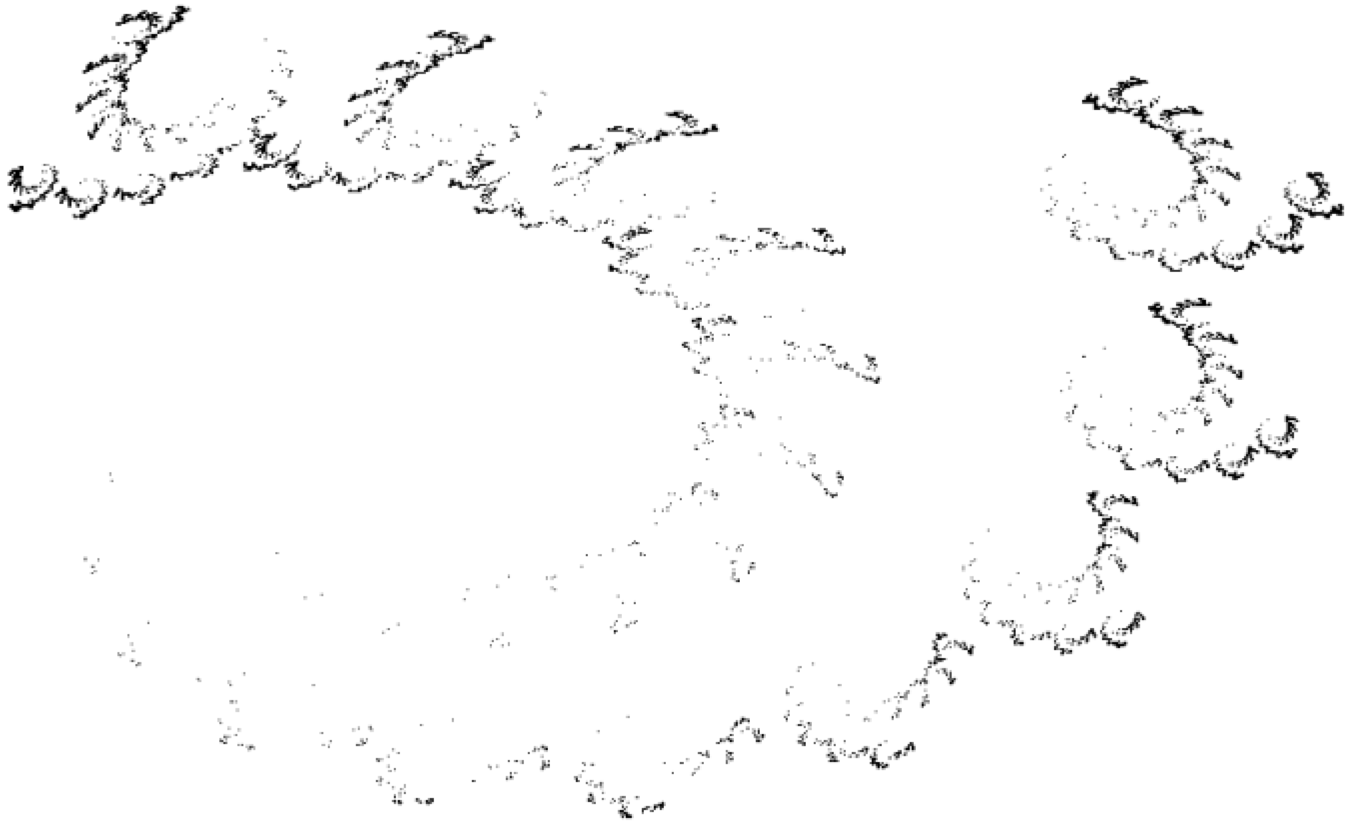
75,000 Points

Modified Fractal Tennis



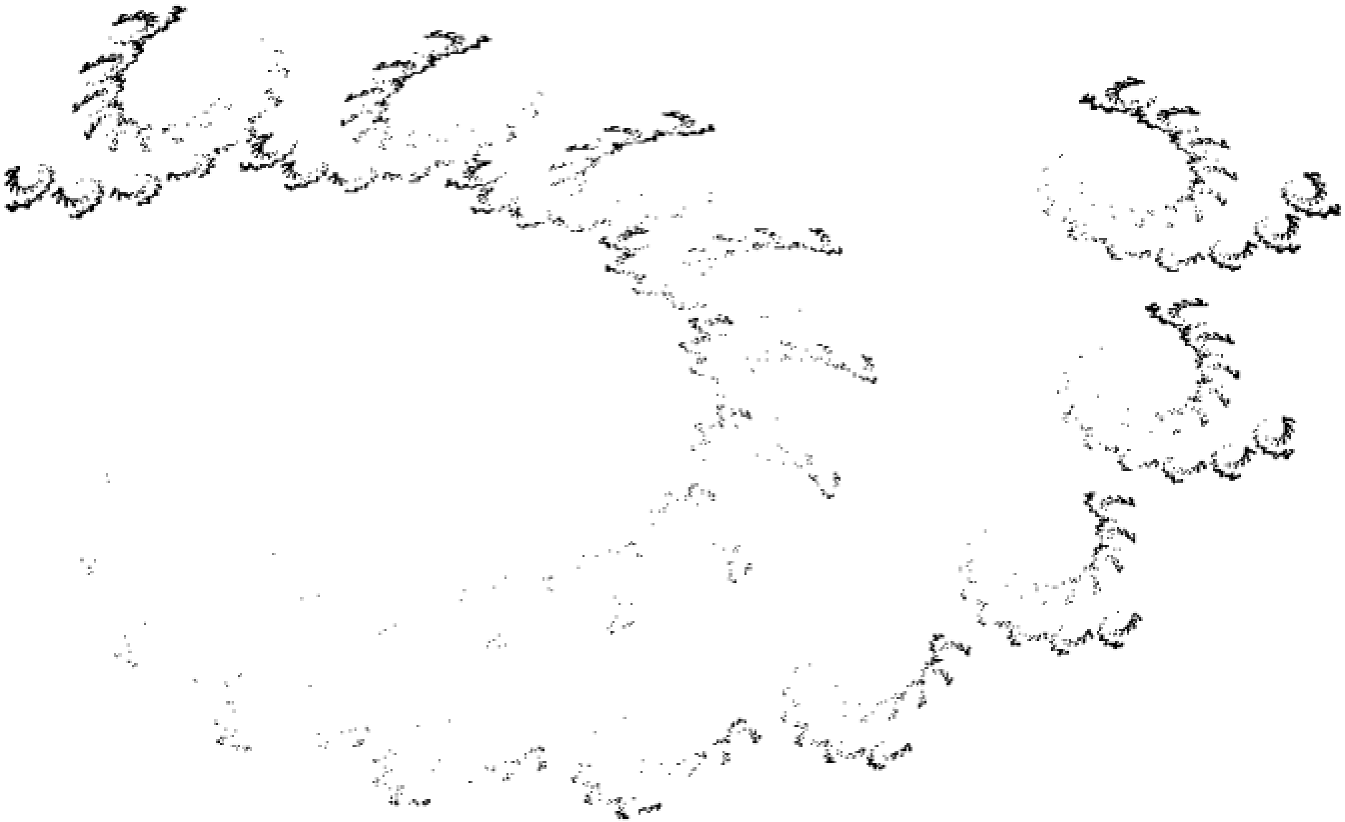
100,000 Points

Modified Fractal Tennis



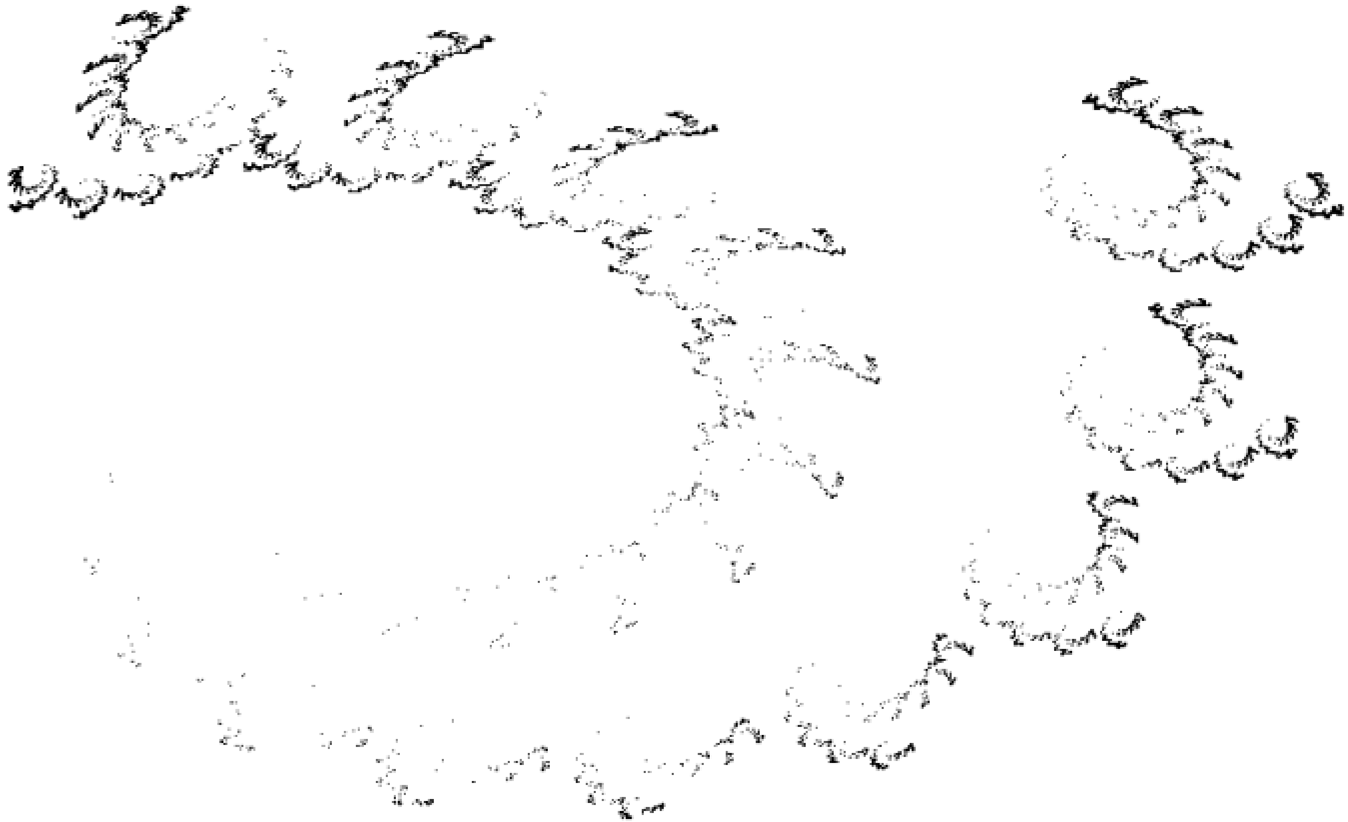
125,000 Points

Modified Fractal Tennis



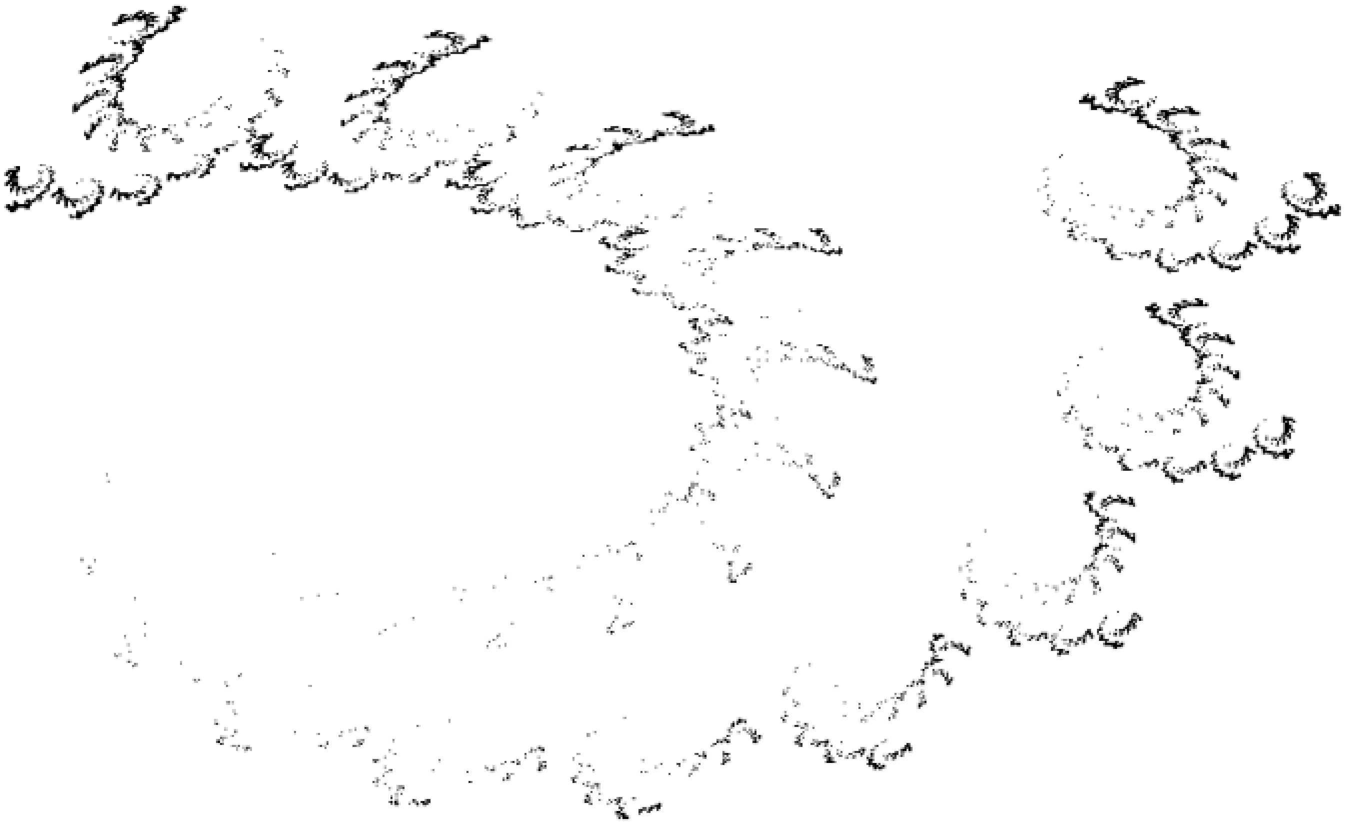
150,000 Points

Modified Fractal Tennis



175,000 Points

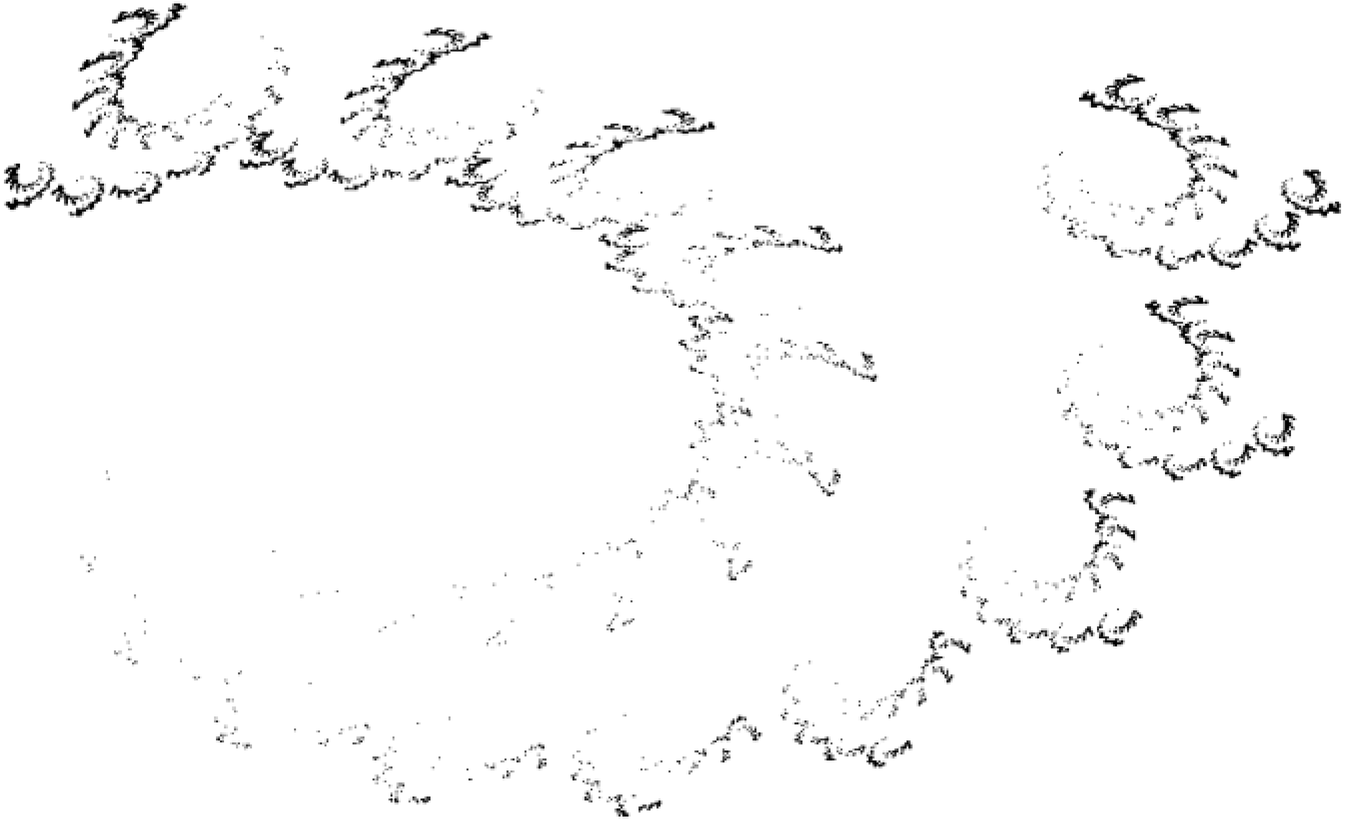
Modified Fractal Tennis



200,000 Points

Modified Fractal Tennis

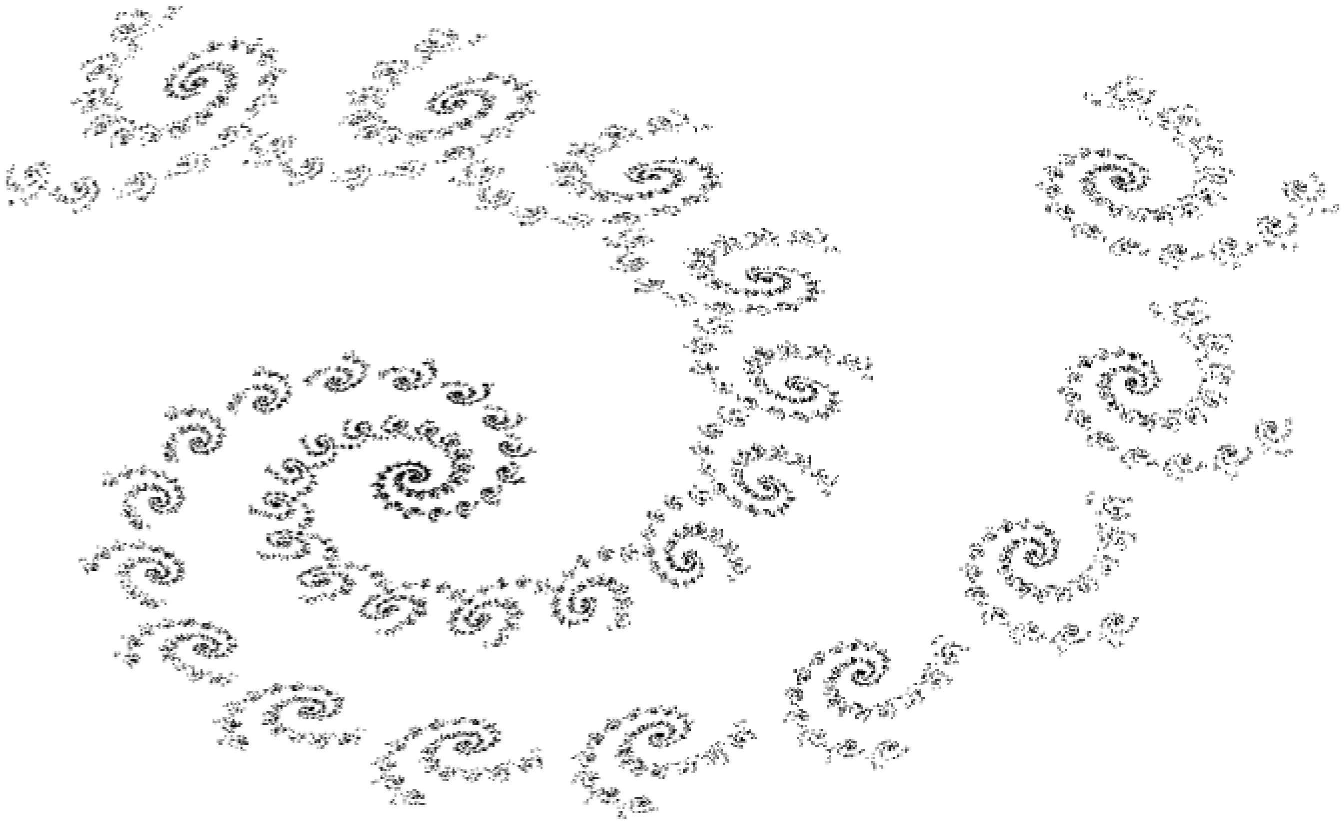
- Weight probabilities based on area



200,000 Points

Modified Fractal Tennis

- Weight probabilities based on area



25,000 Points

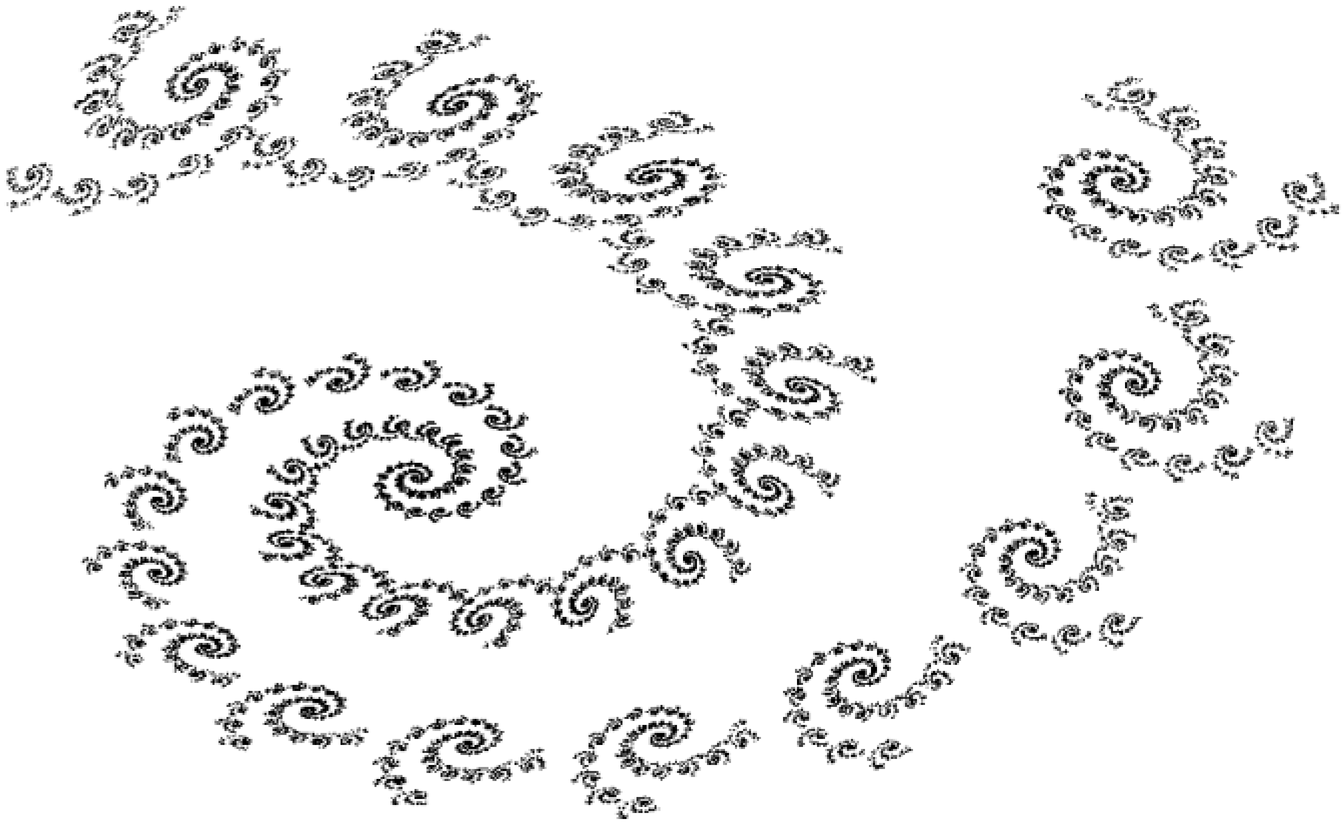
Modified Fractal Tennis

- Weight probabilities based on area

50,000 Points

Modified Fractal Tennis

- Weight probabilities based on area



75,000 Points

Modified Fractal Tennis

- Weight probabilities based on area

100,000 Points

Modified Fractal Tennis

- Weight probabilities based on area

125,000 Points

Modified Fractal Tennis

- Weight probabilities based on area

150,000 Points

Modified Fractal Tennis

- Weight probabilities based on area

175,000 Points

Modified Fractal Tennis

- Weight probabilities based on area

200,000 Points

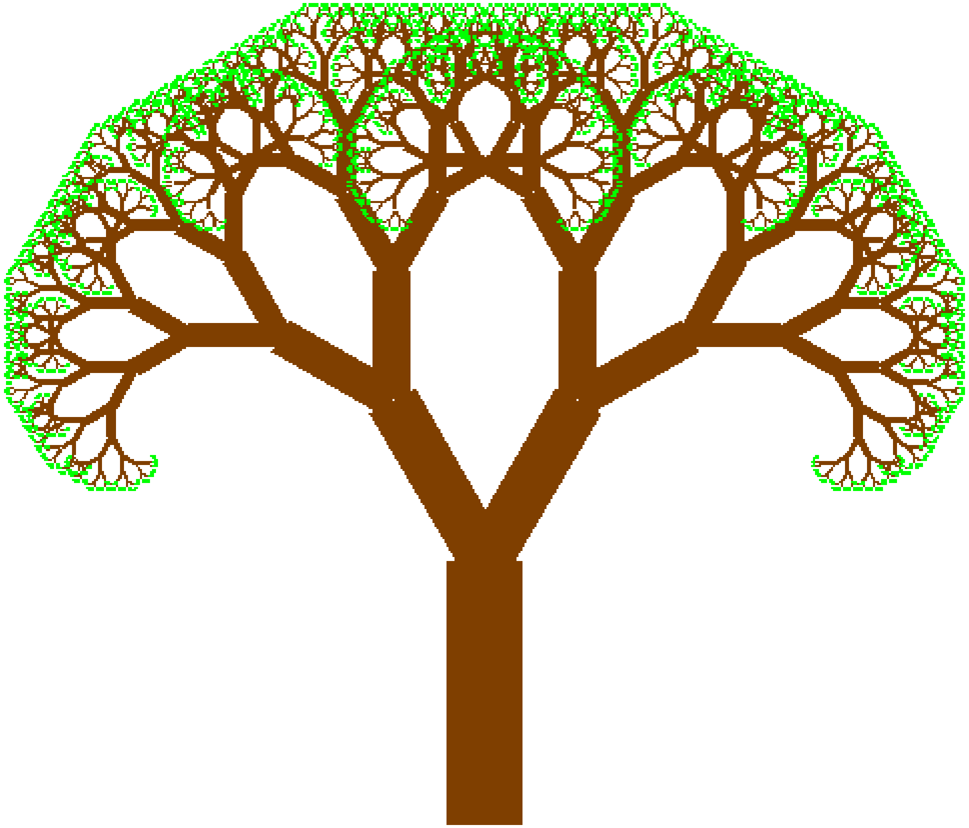
Condensation Sets

- Given a condensation set C

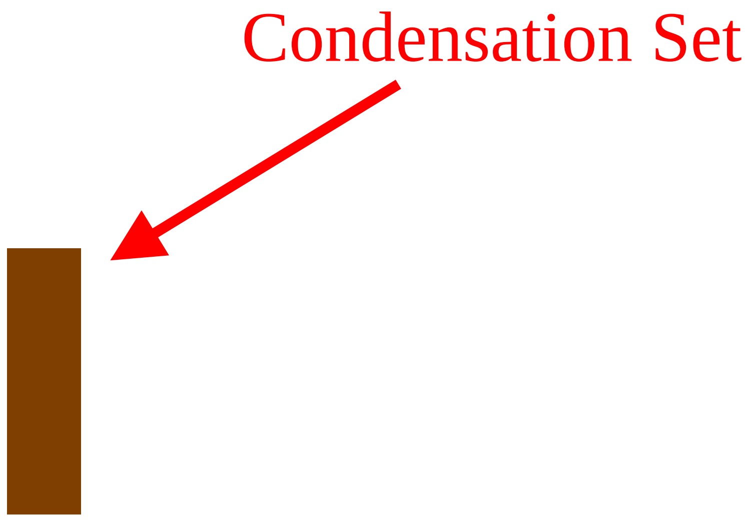
$$X_0 = C$$

$$X_{i+1} = \left(\bigcup_j F_j(X_i) \right) \bigcup C$$

Condensation Set Example

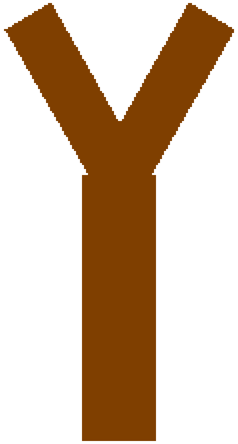


Condensation Sets – Example



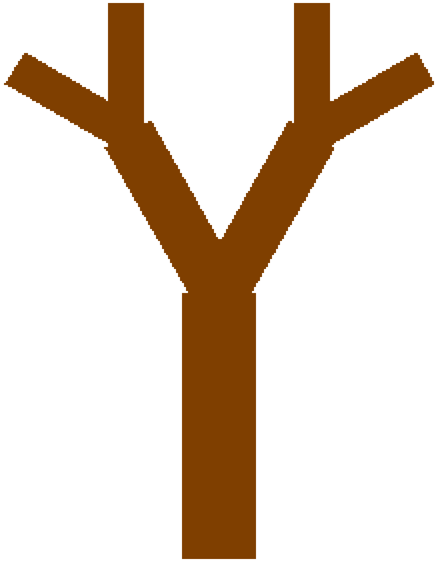
Condensation Sets – Example

Condensation Sets – Example

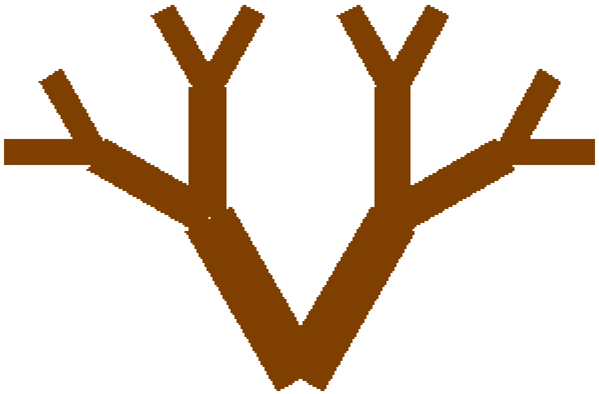


Condensation Sets – Example

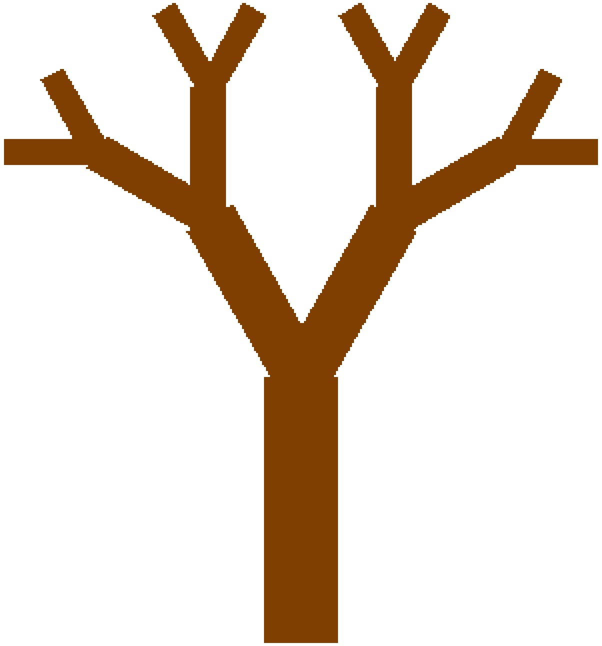
Condensation Sets – Example



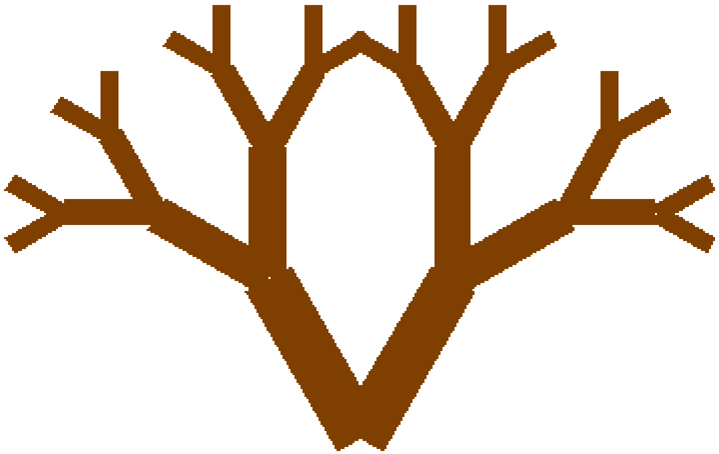
Condensation Sets – Example



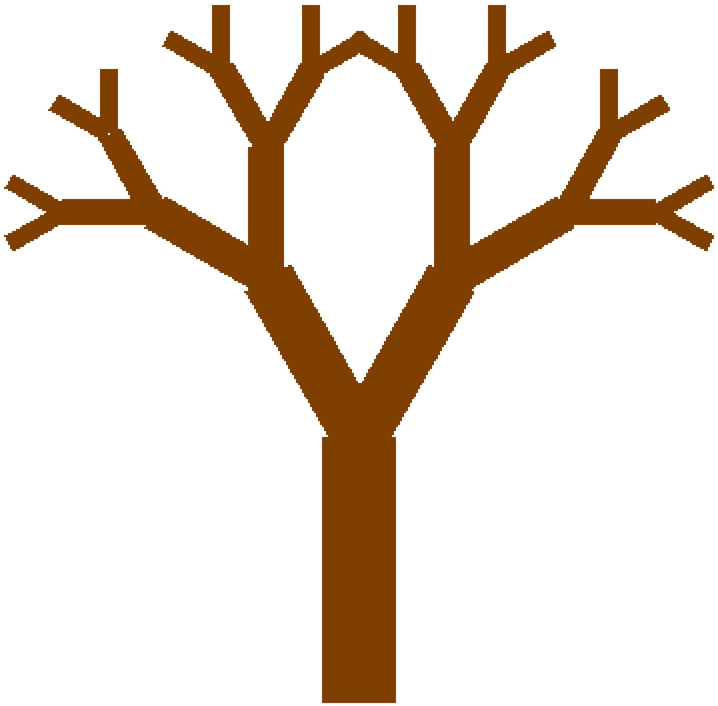
Condensation Sets – Example



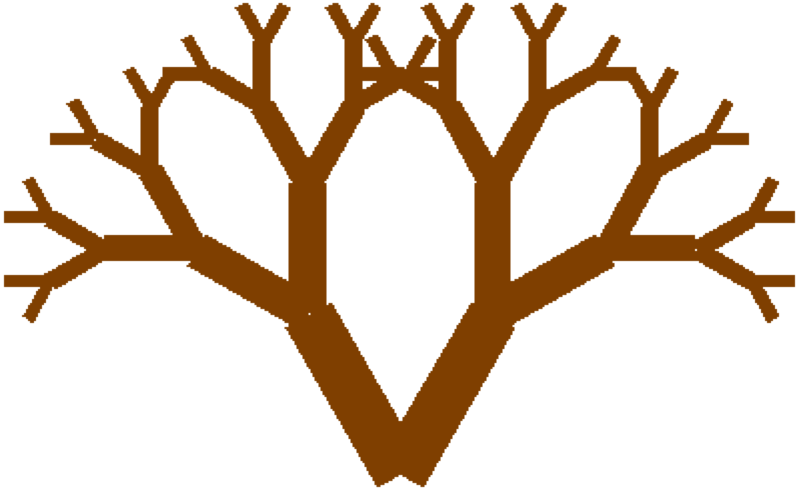
Condensation Sets – Example



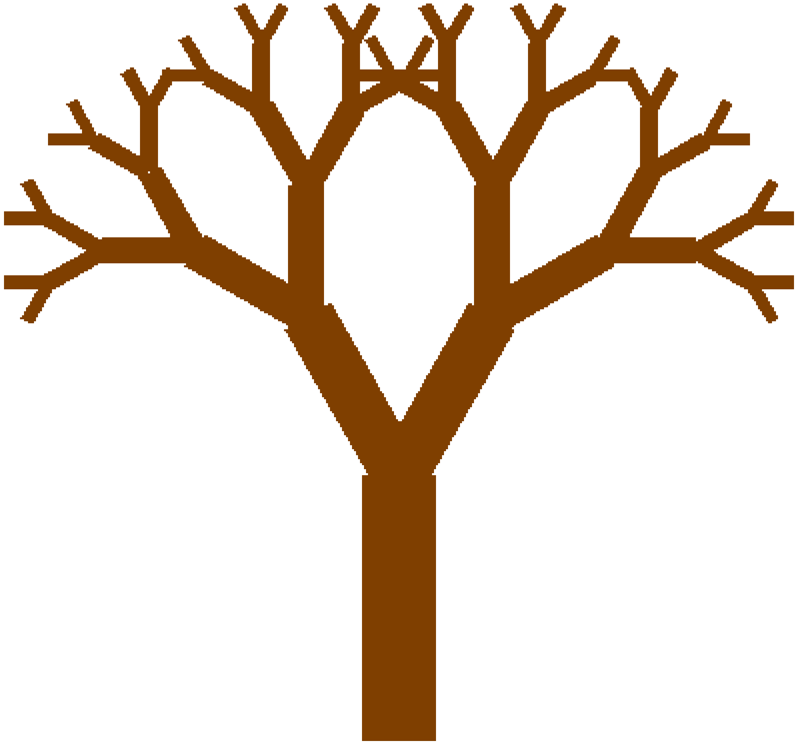
Condensation Sets – Example



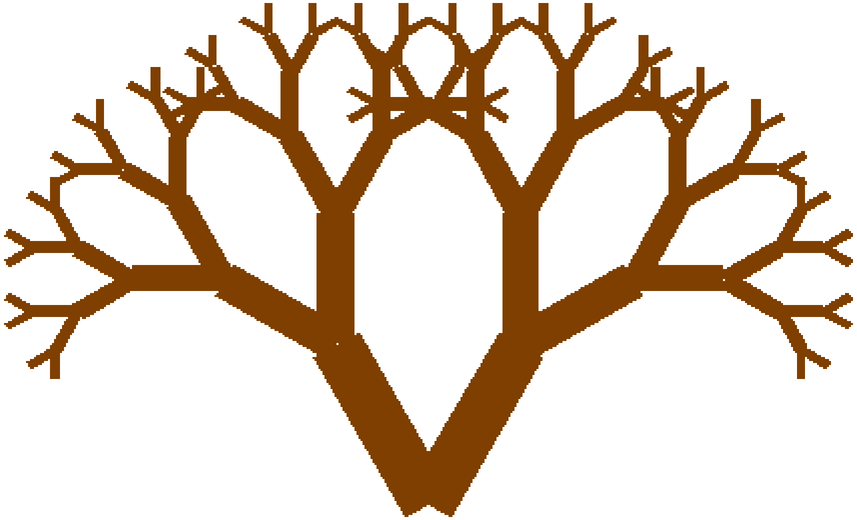
Condensation Sets – Example



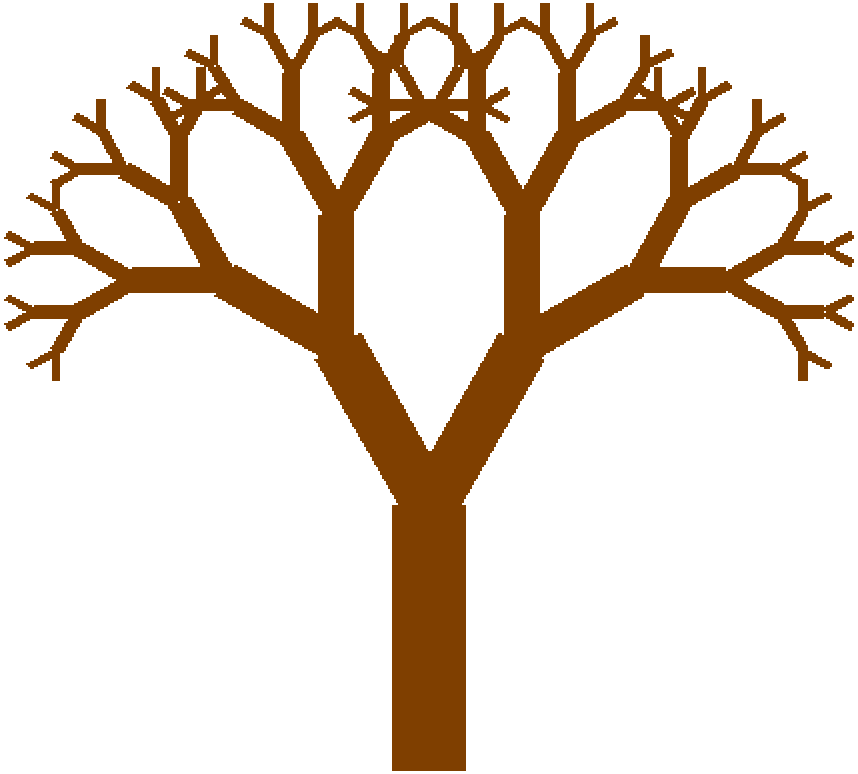
Condensation Sets – Example



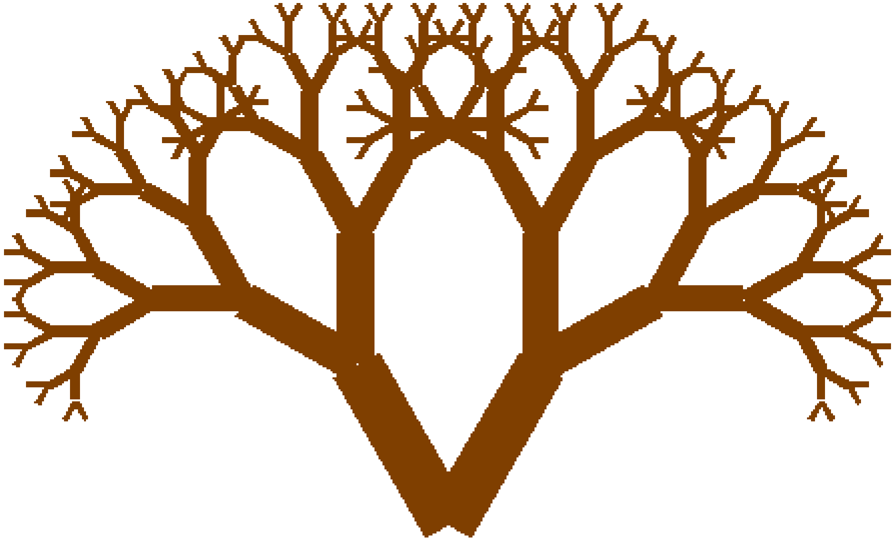
Condensation Sets – Example



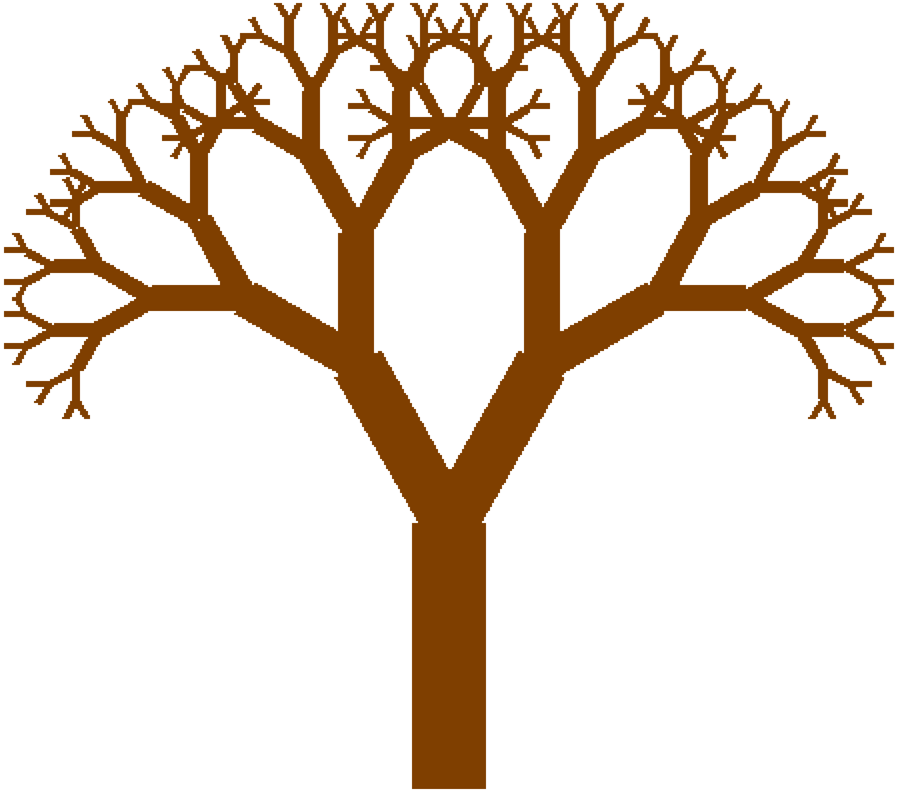
Condensation Sets – Example



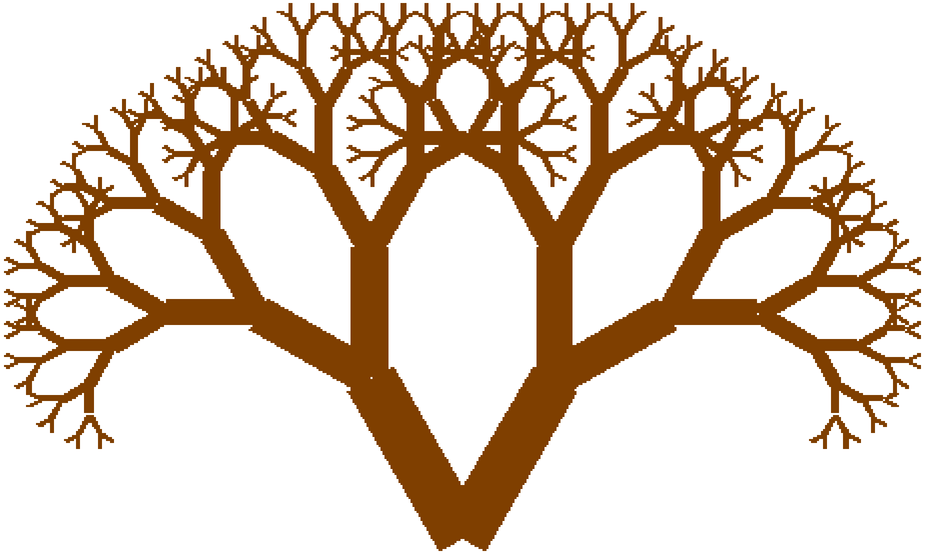
Condensation Sets – Example



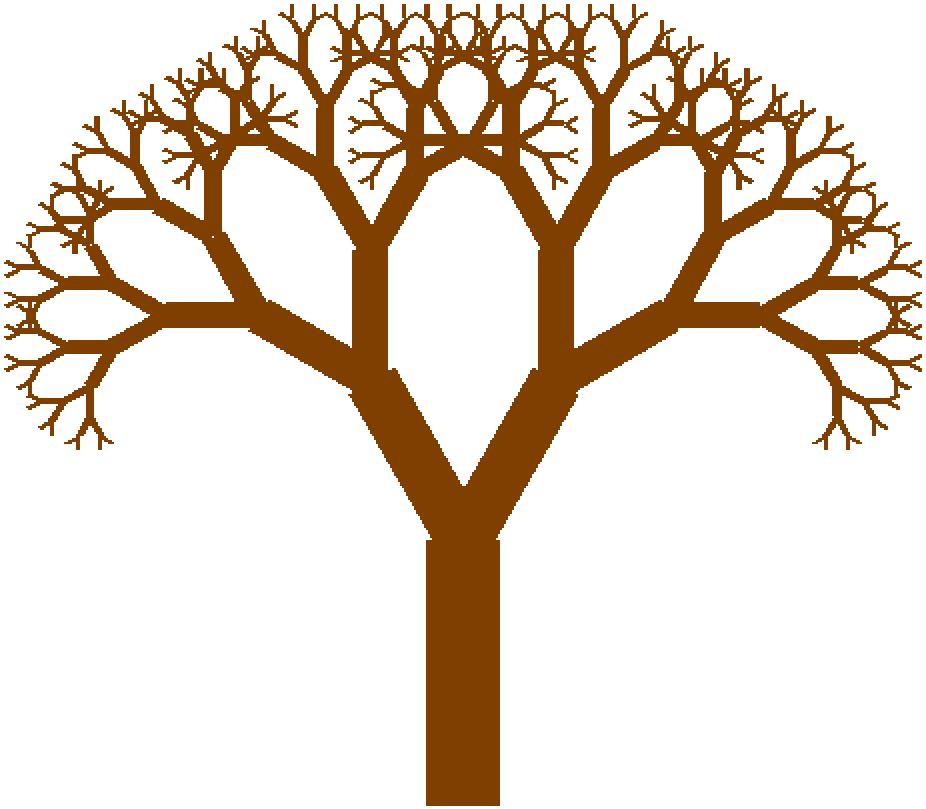
Condensation Sets – Example



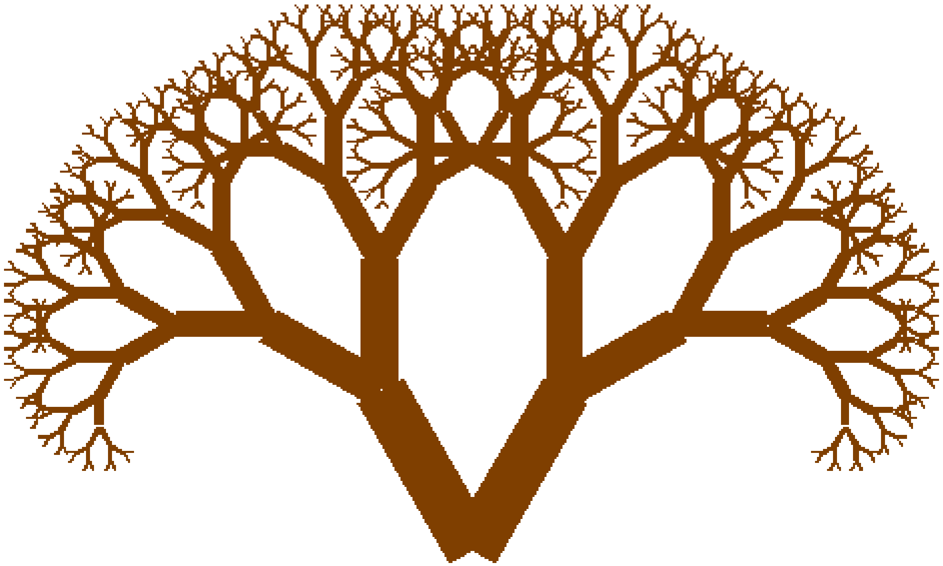
Condensation Sets – Example



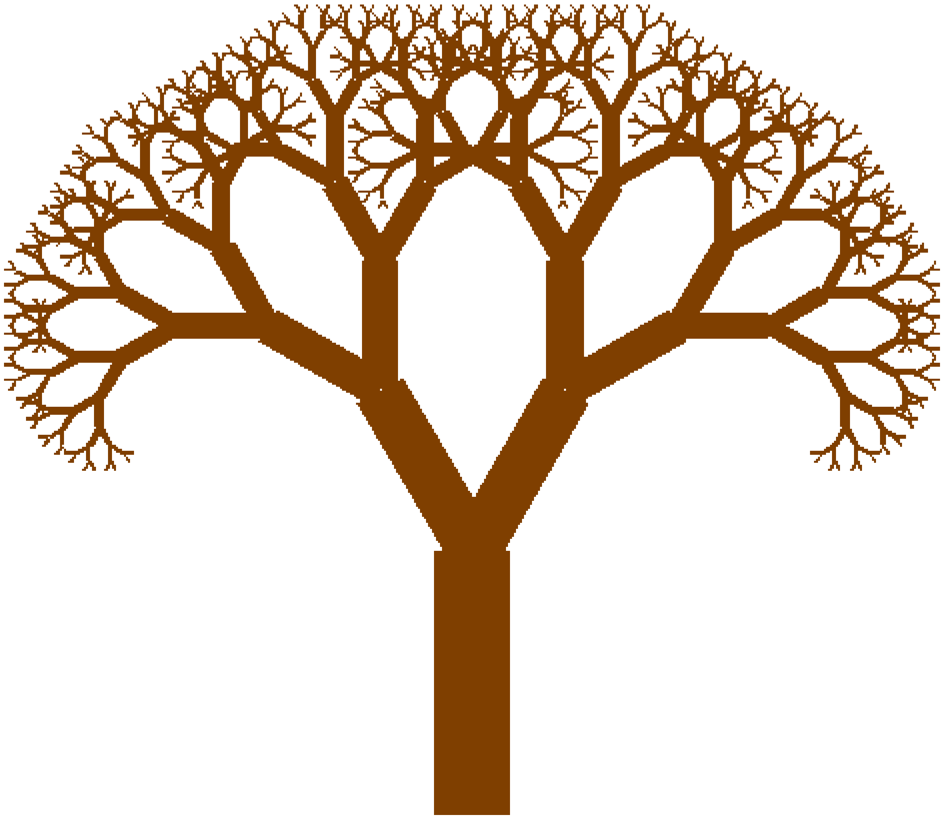
Condensation Sets – Example



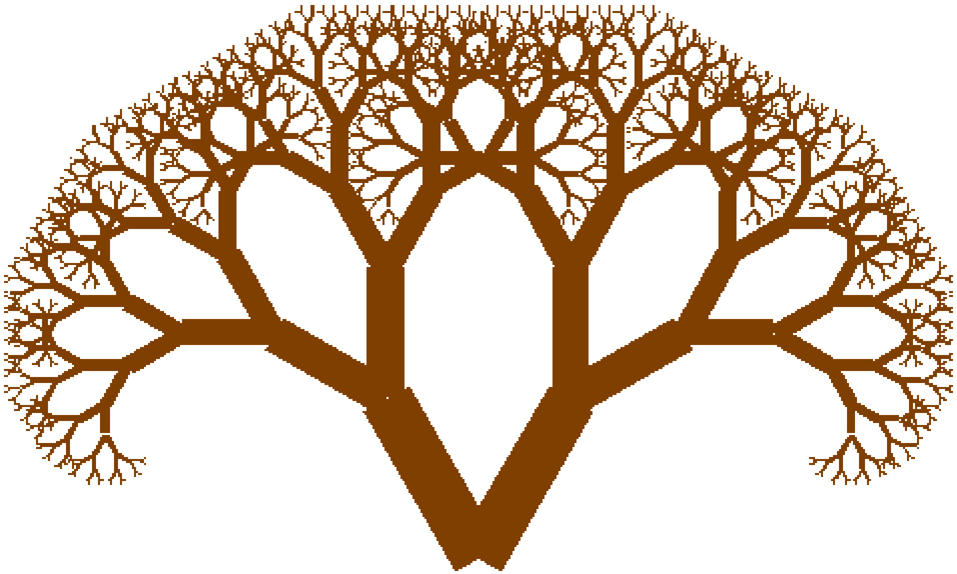
Condensation Sets – Example



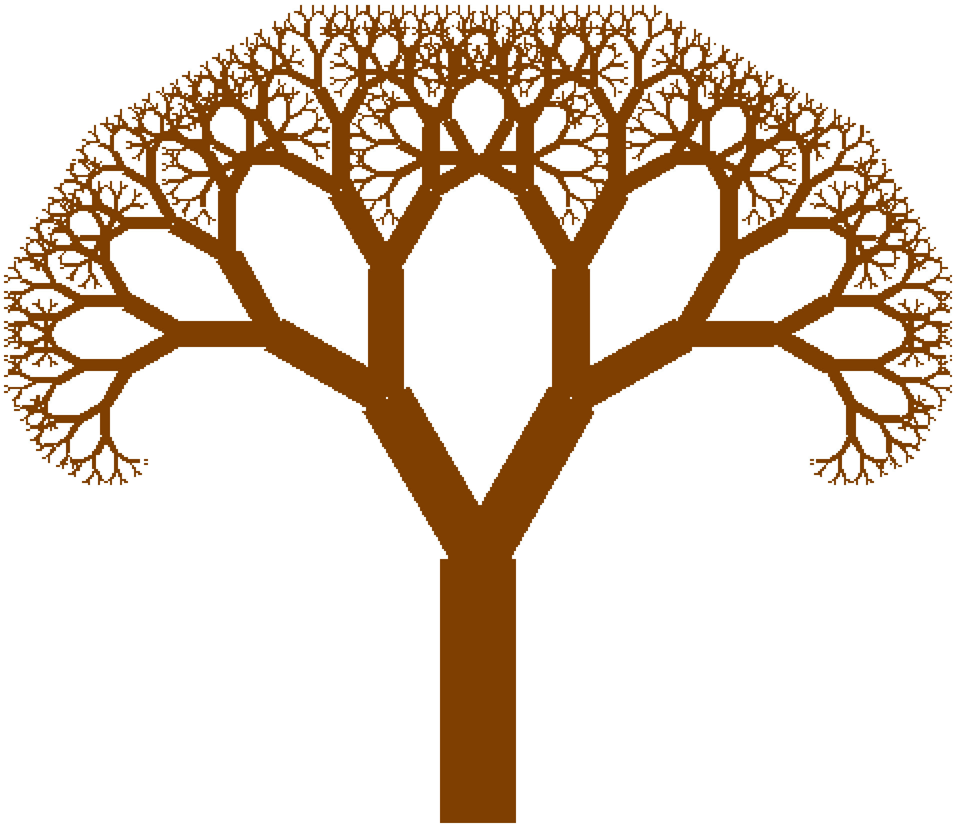
Condensation Sets – Example



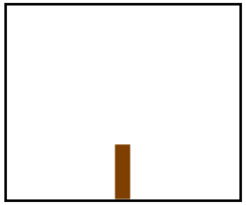
Condensation Sets – Example



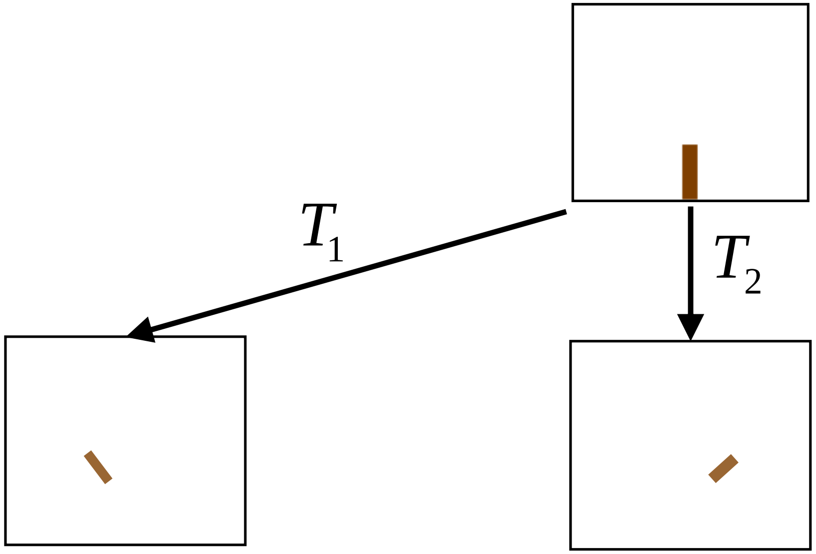
Condensation Sets – Example



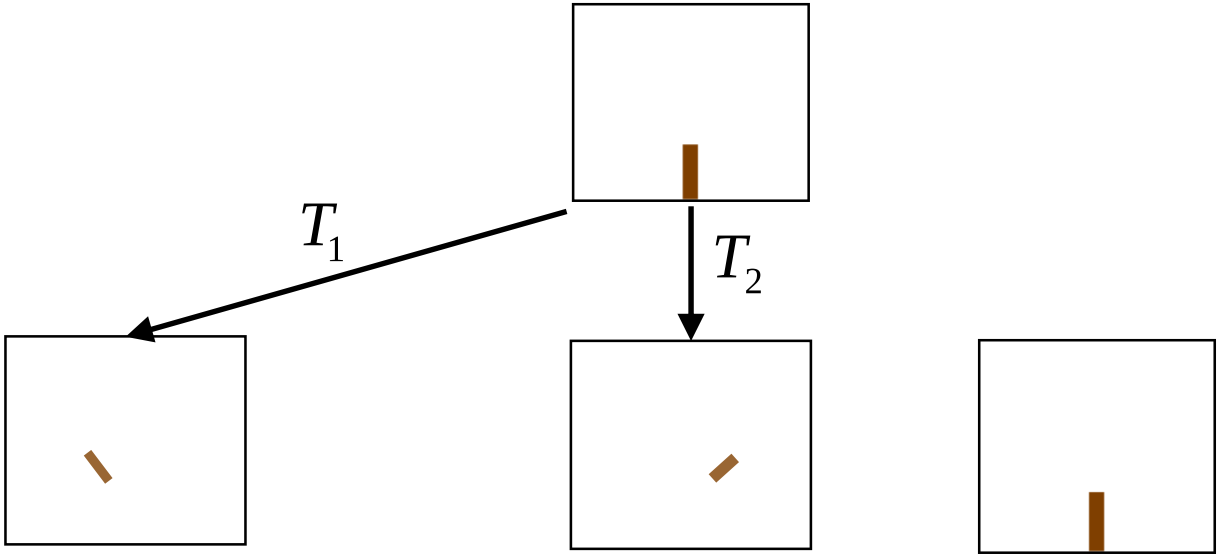
Rendering Fractals



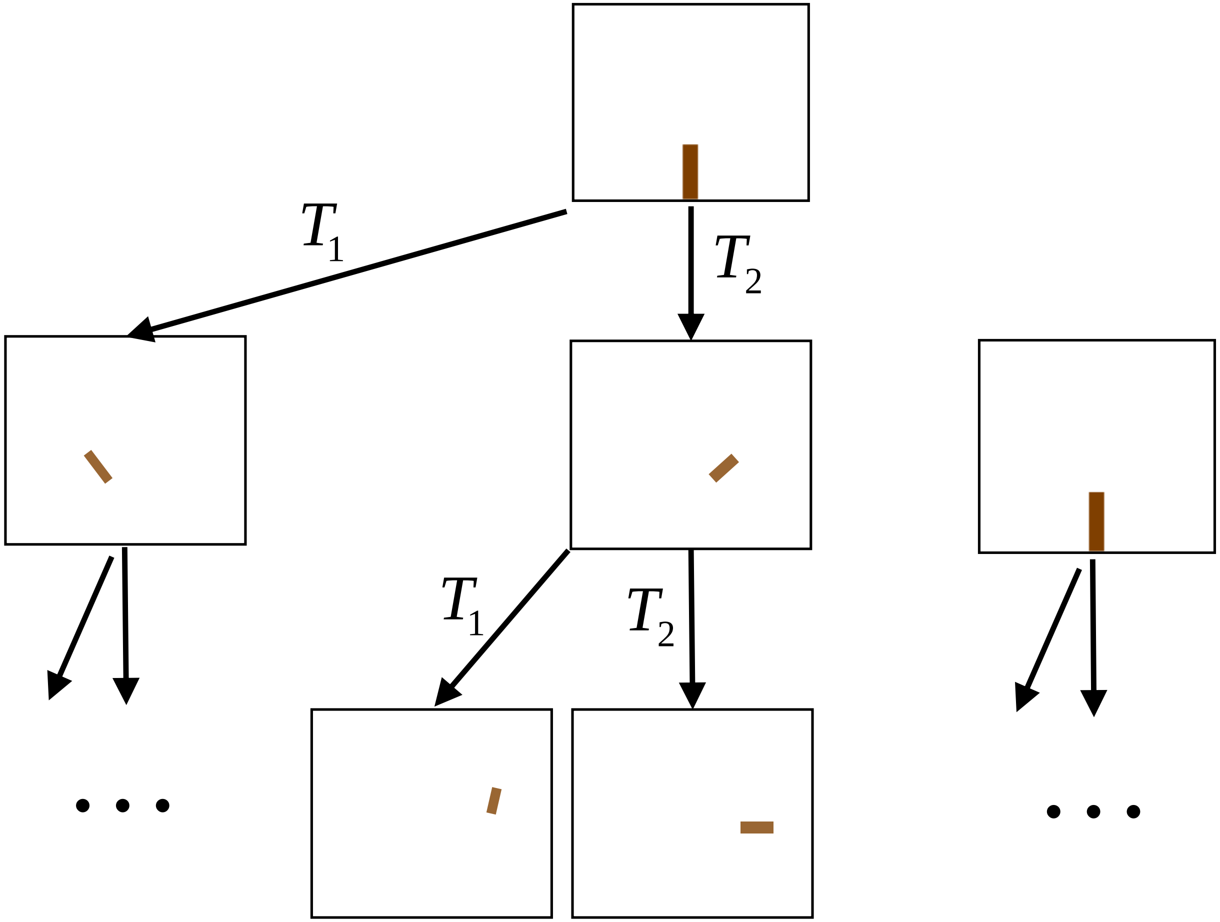
Rendering Fractals



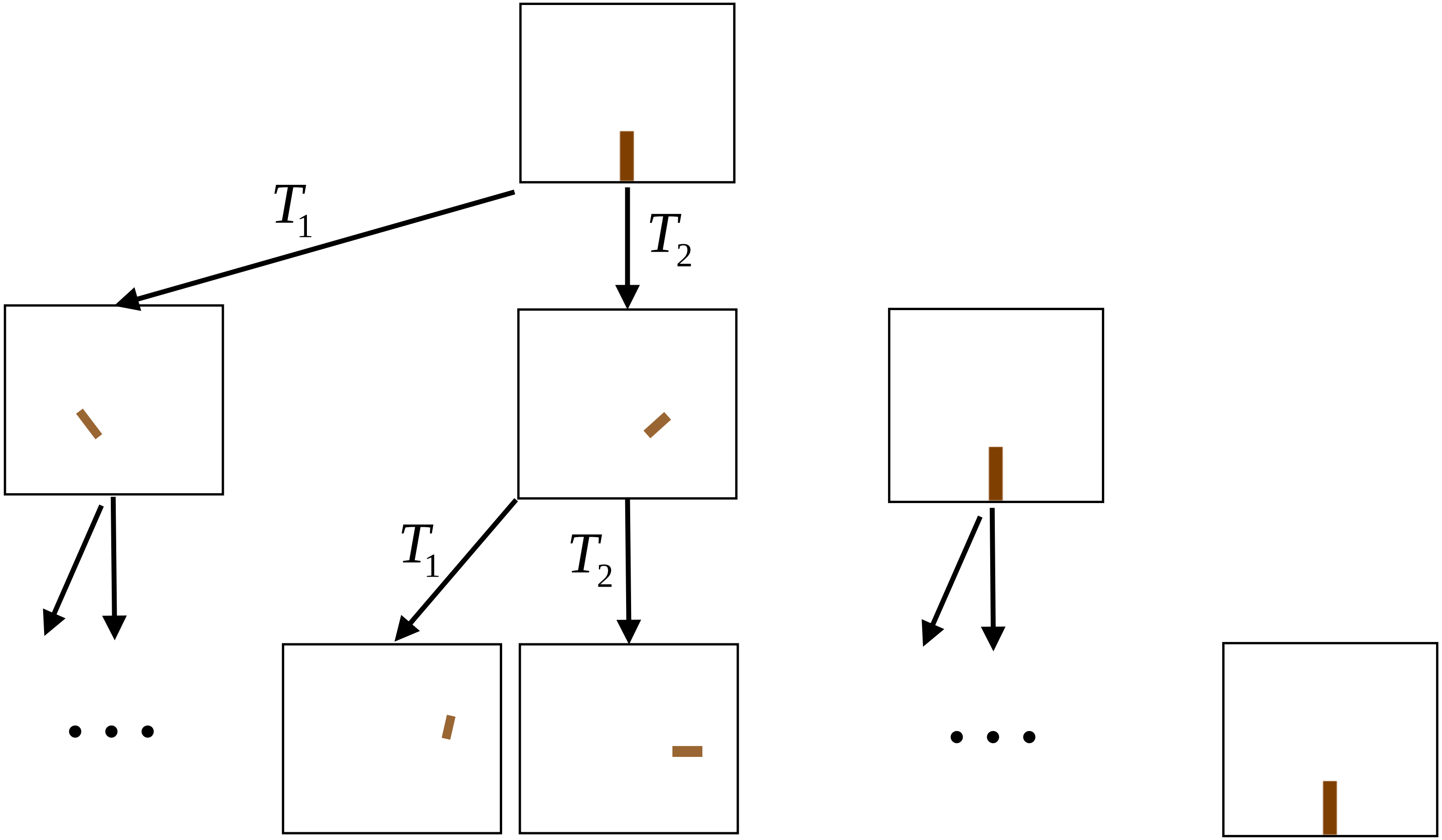
Rendering Fractals



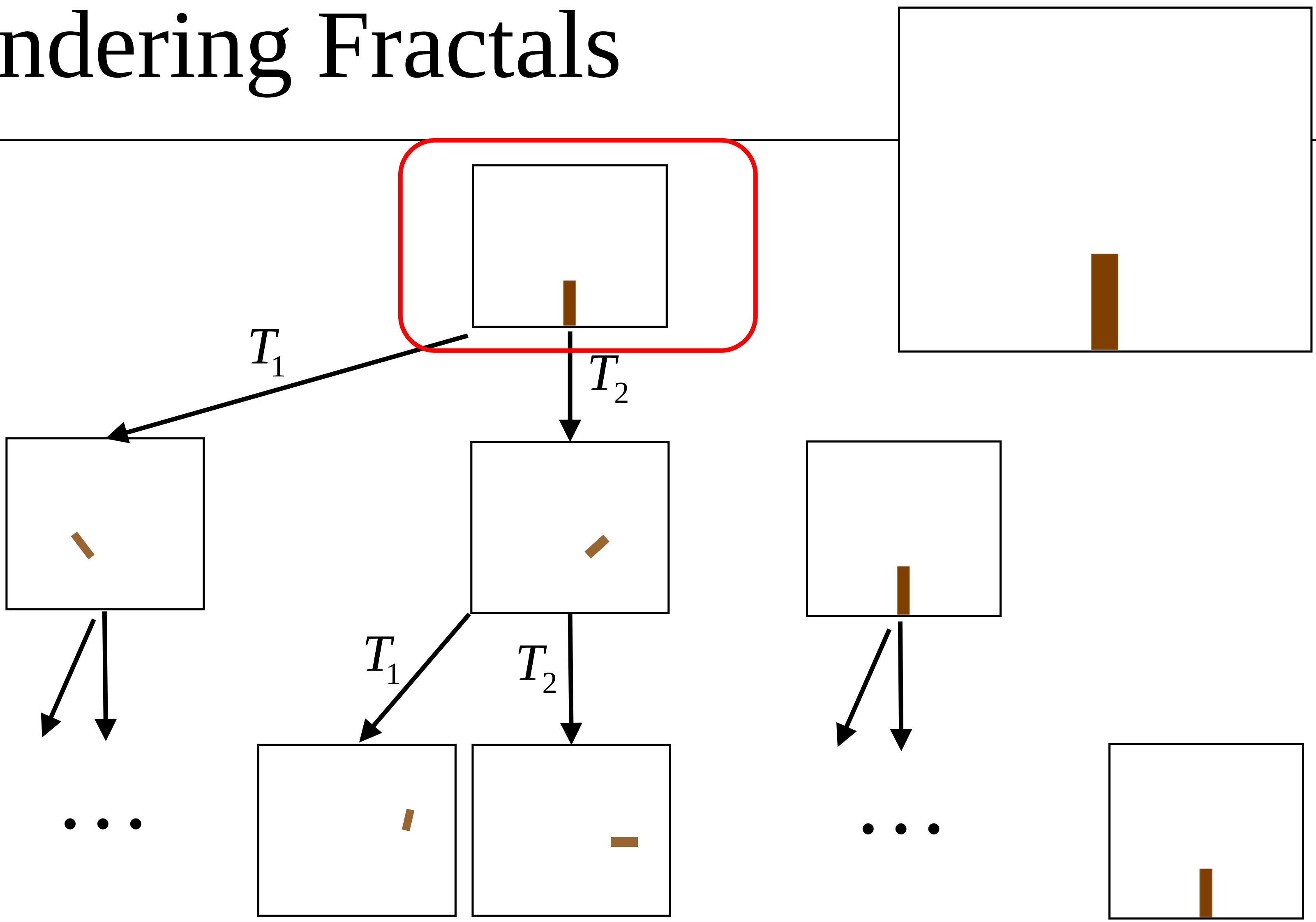
Rendering Fractals



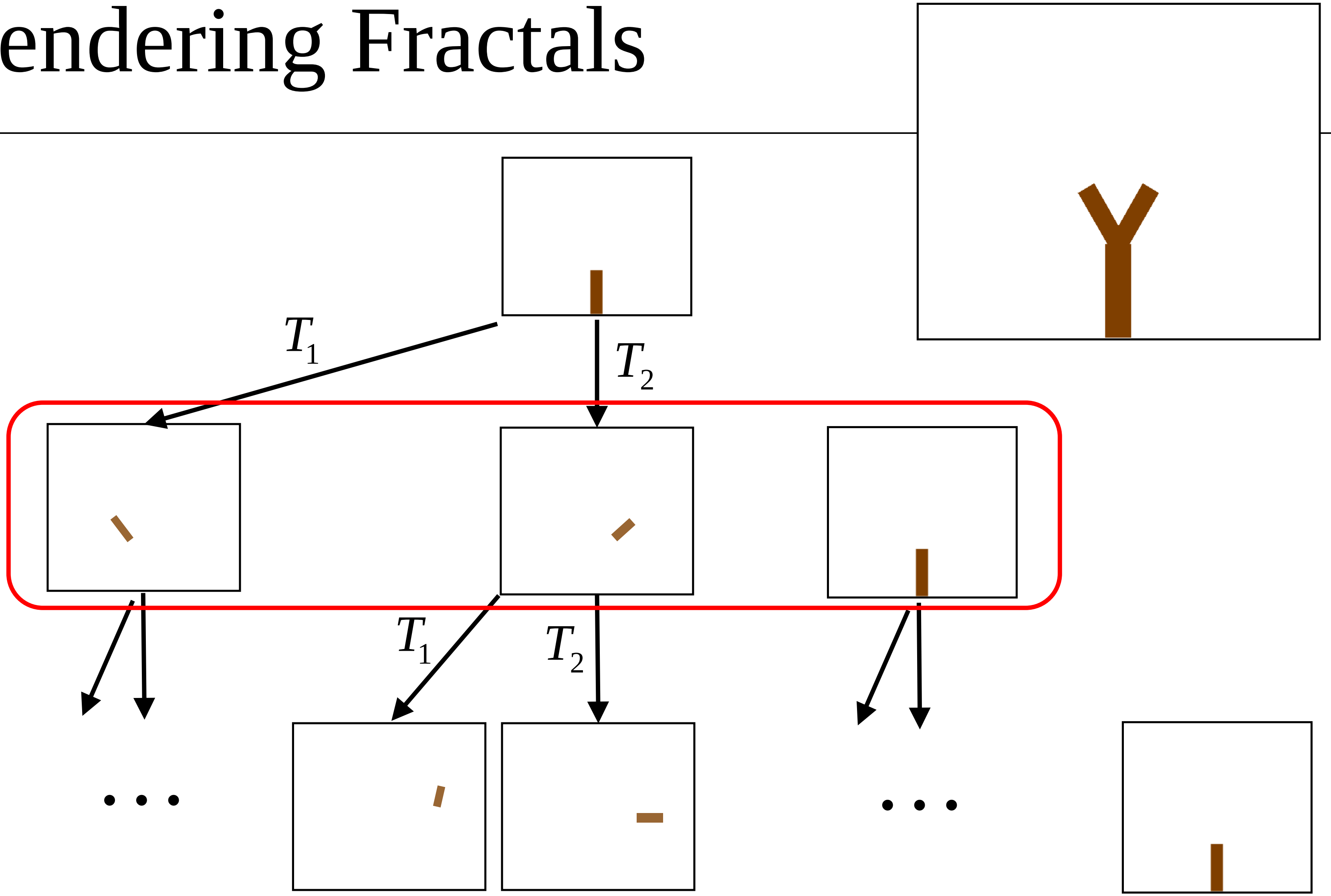
Rendering Fractals



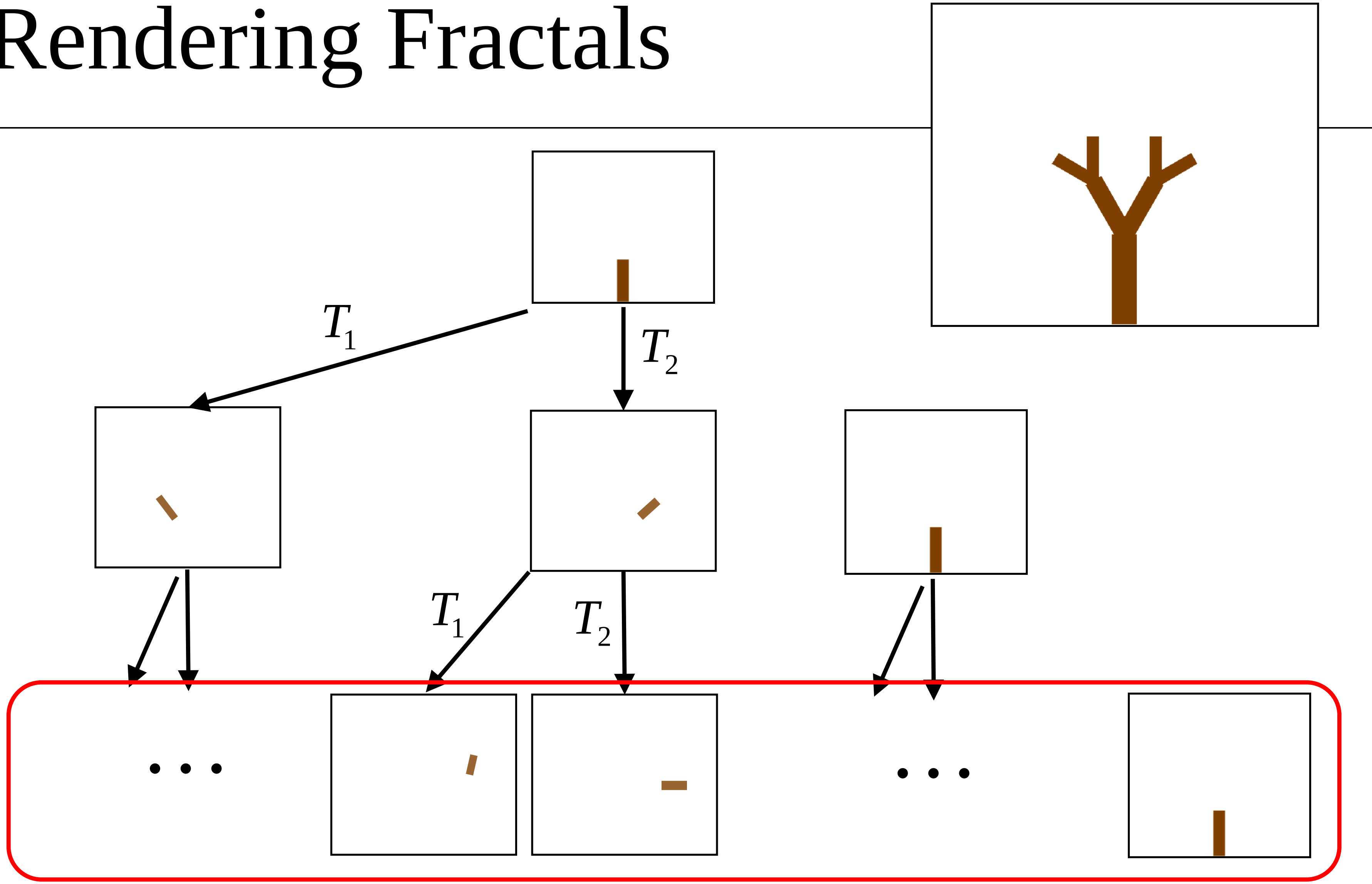
Rendering Fractals



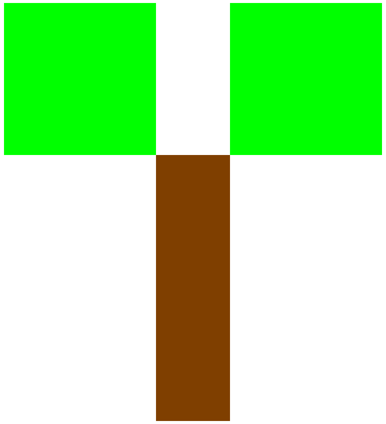
Rendering Fractals



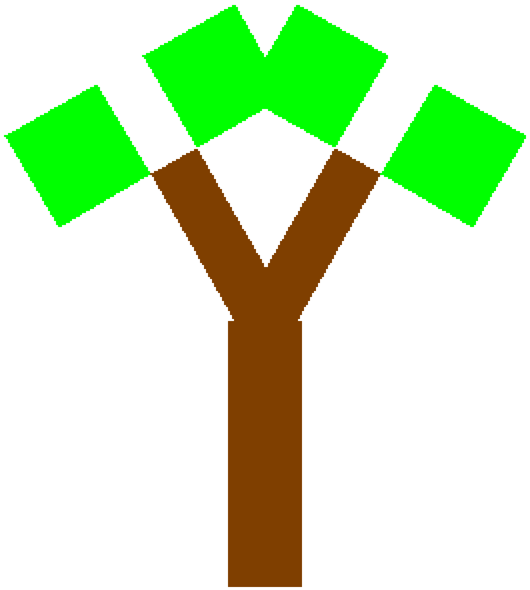
Rendering Fractals



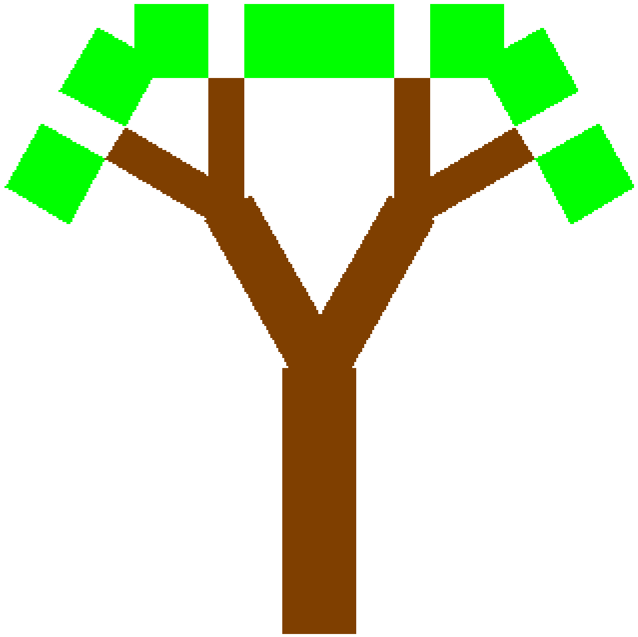
Condensation Set Example



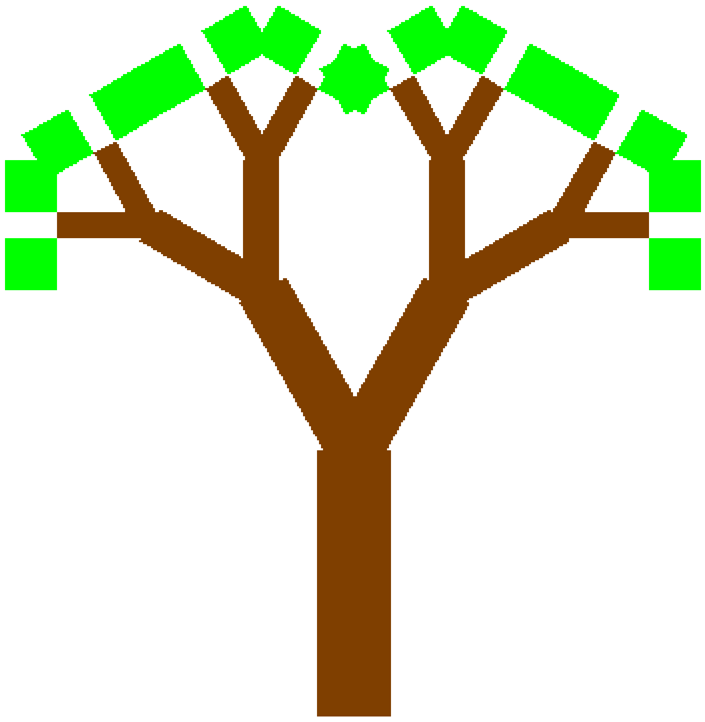
Condensation Set Example



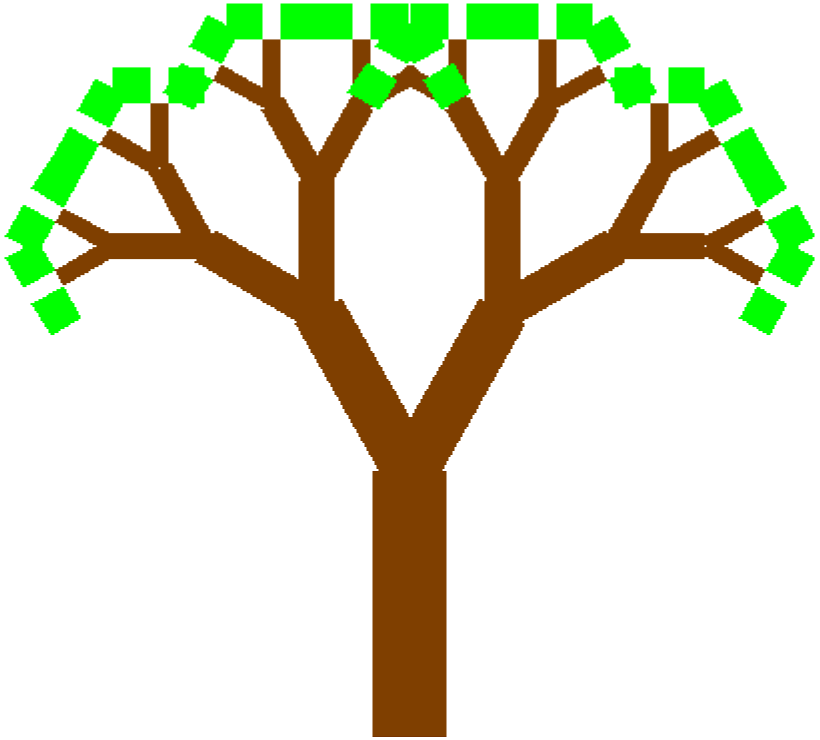
Condensation Set Example



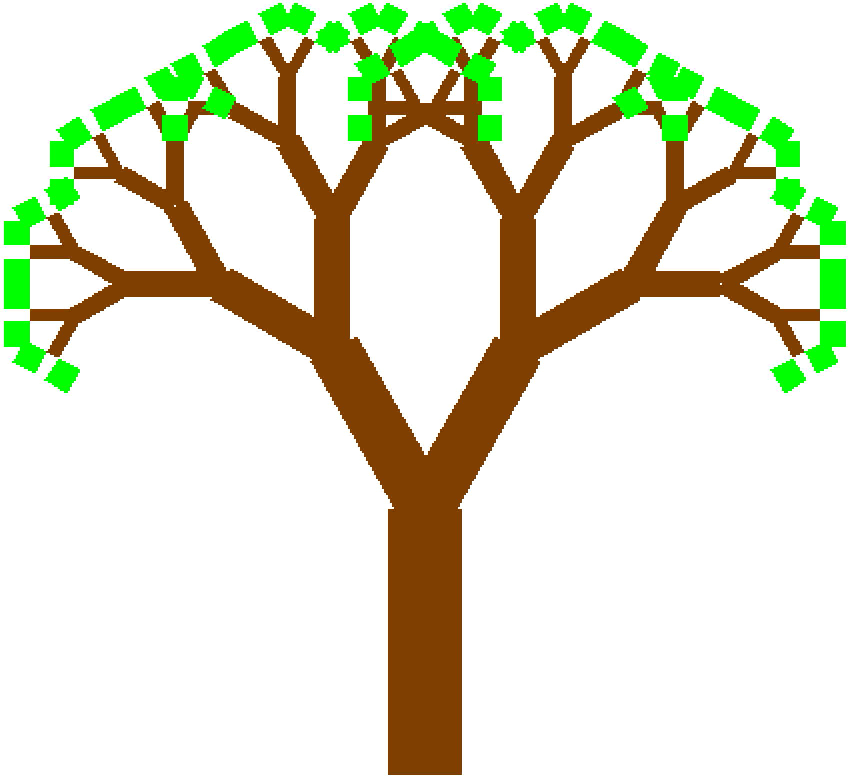
Condensation Set Example



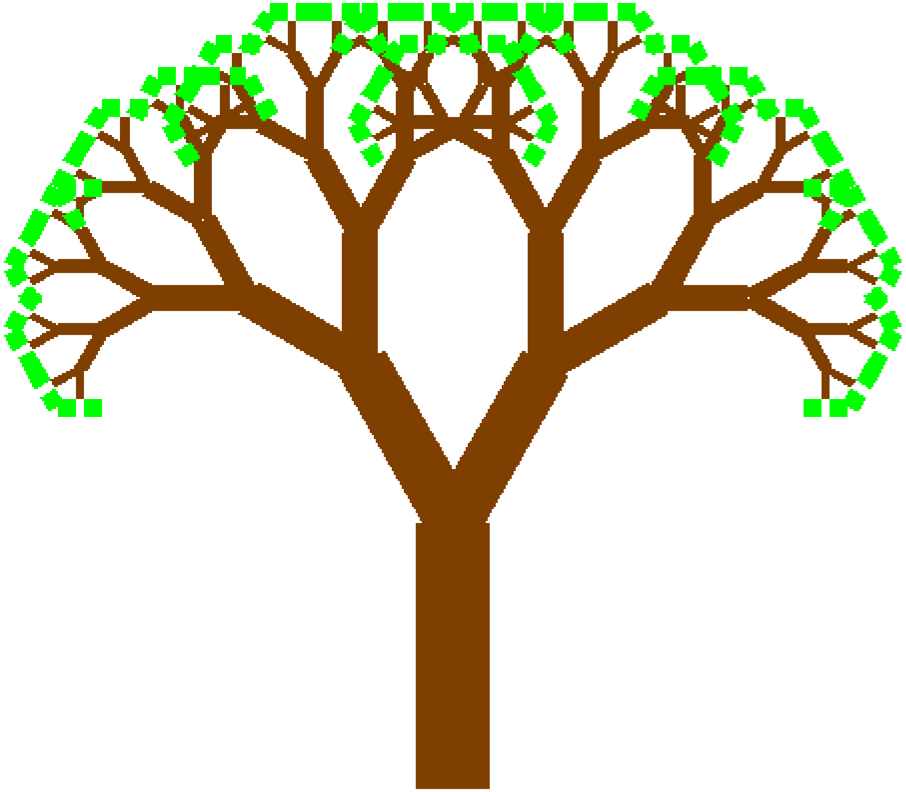
Condensation Set Example



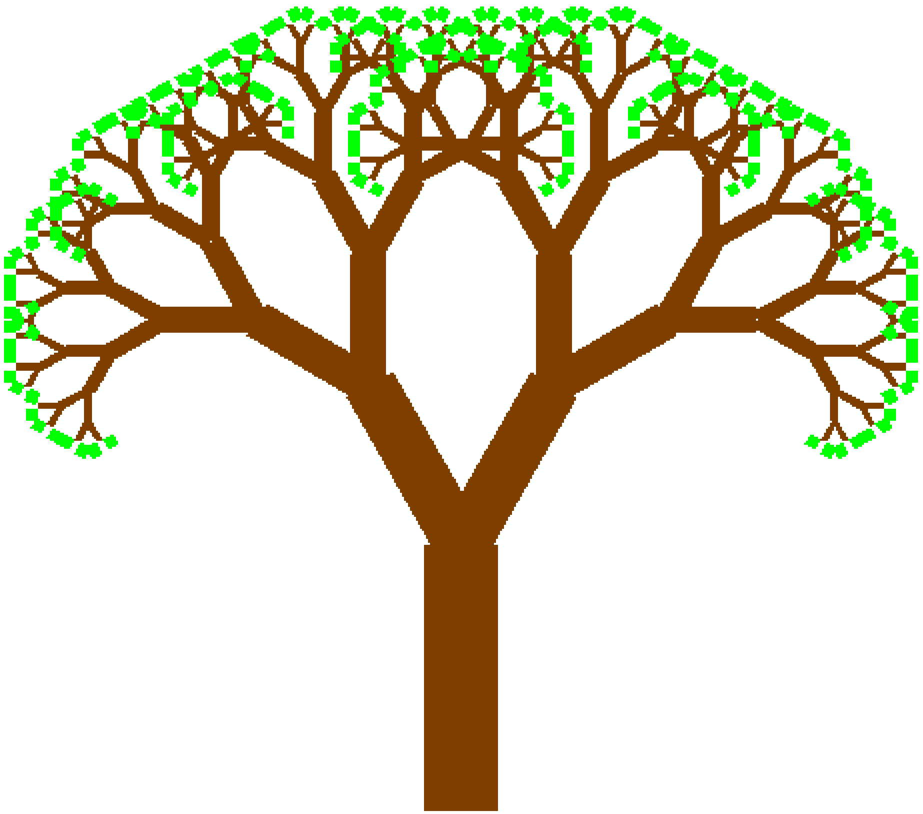
Condensation Set Example



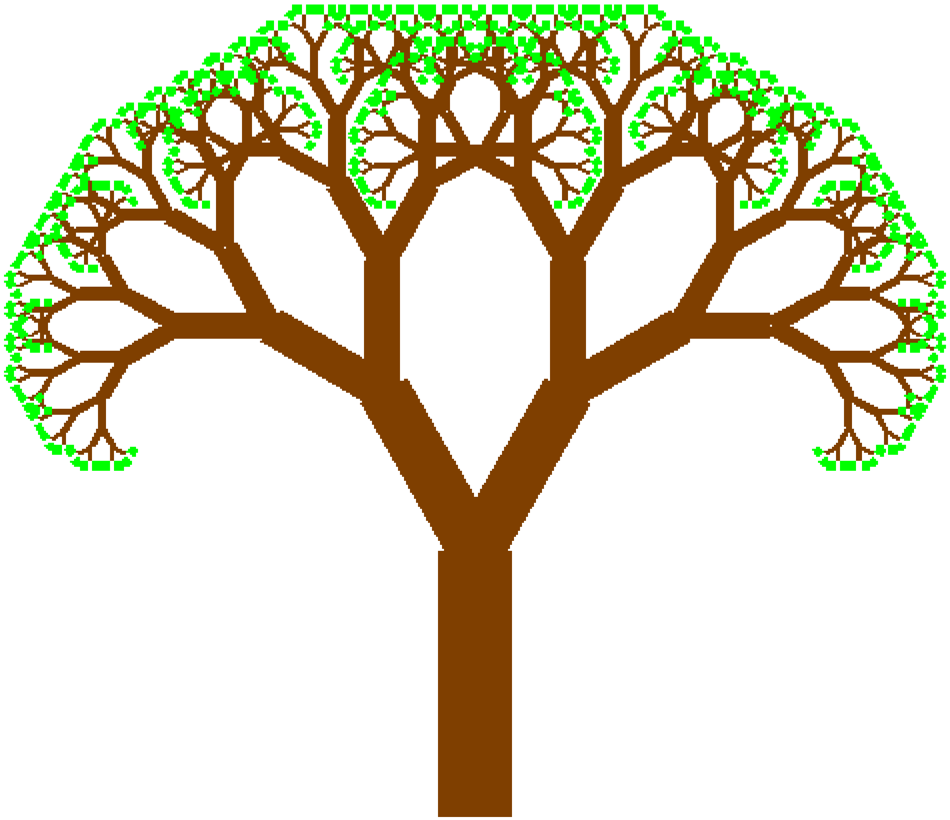
Condensation Set Example



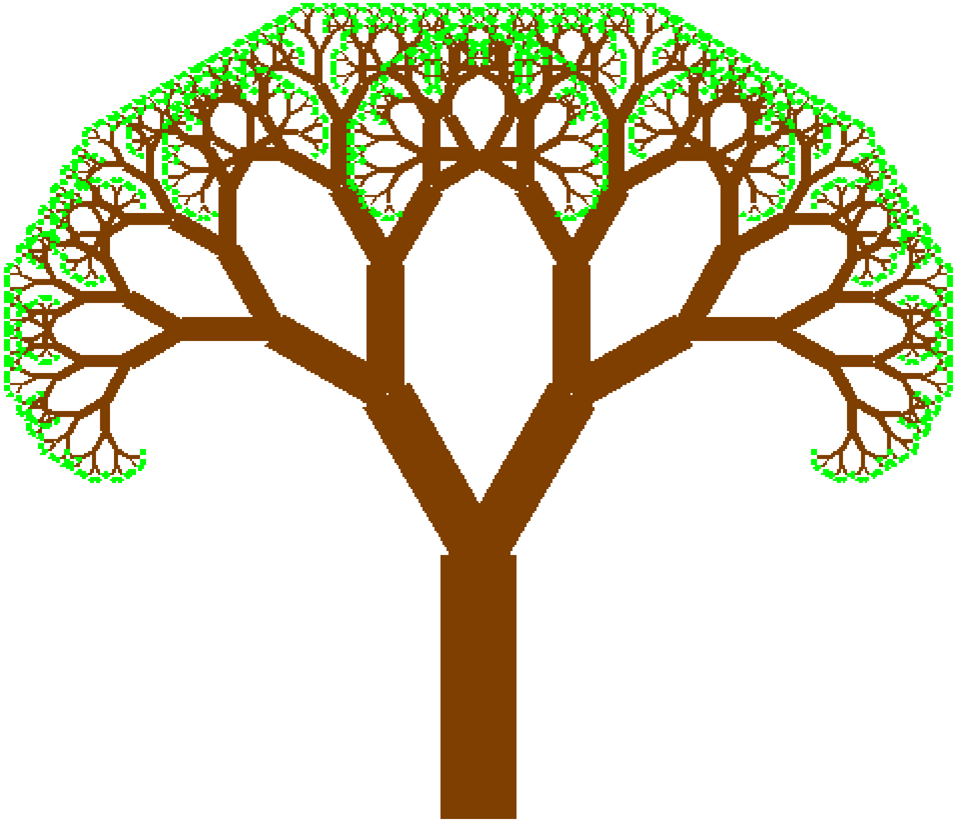
Condensation Set Example



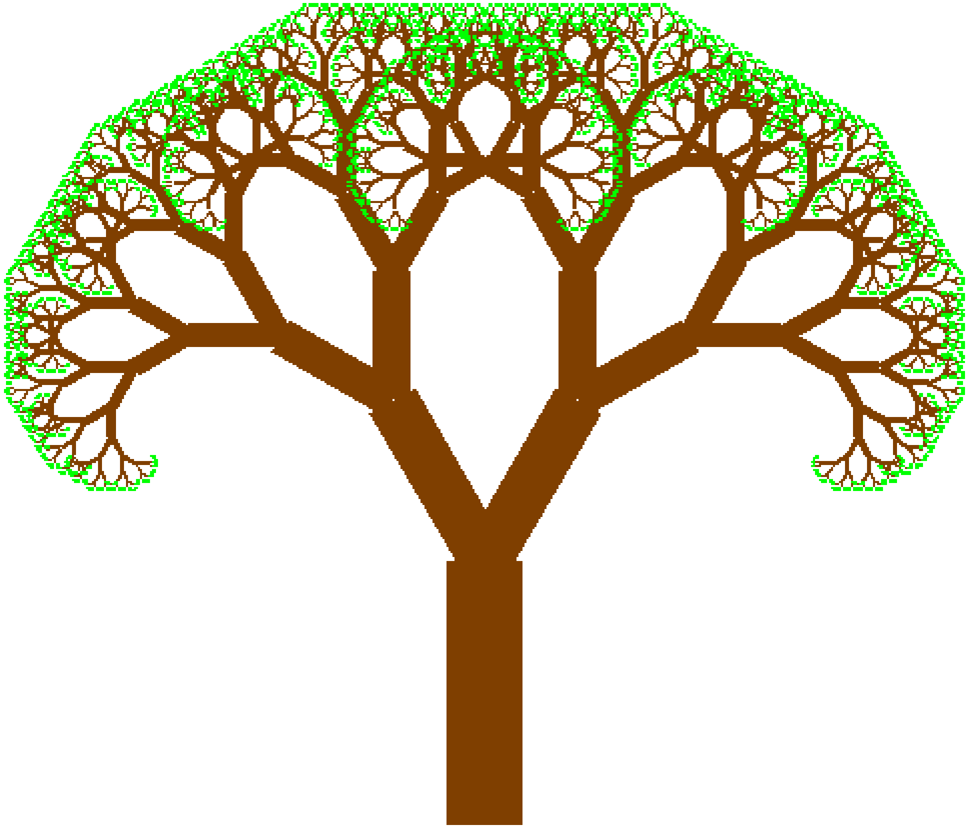
Condensation Set Example



Condensation Set Example



Condensation Set Example

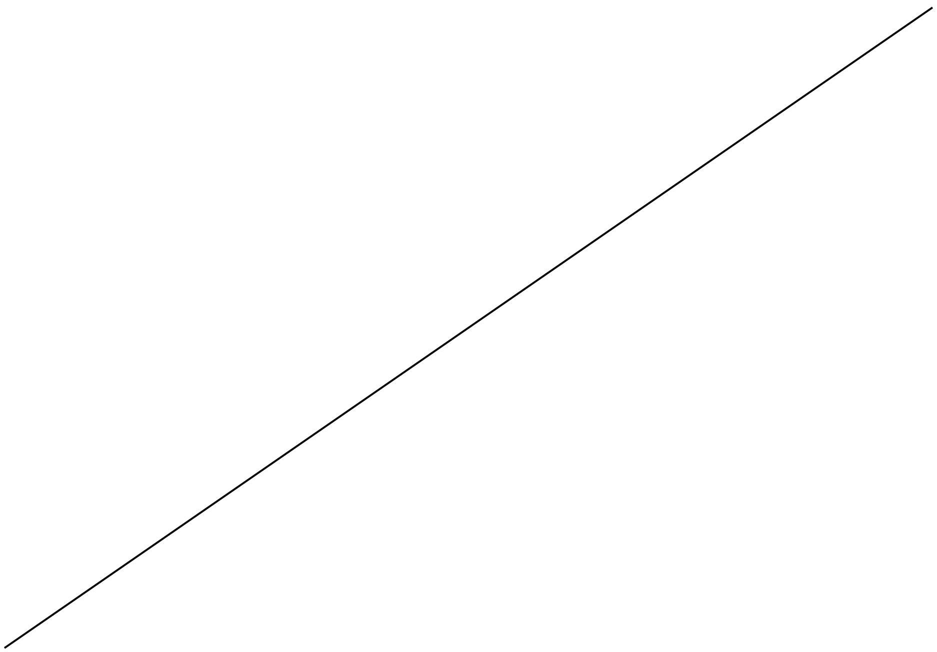


Fractal Dimension

- Fractal dimension = $-\frac{\log(\#transformations)}{\log(scale\ factor)}$

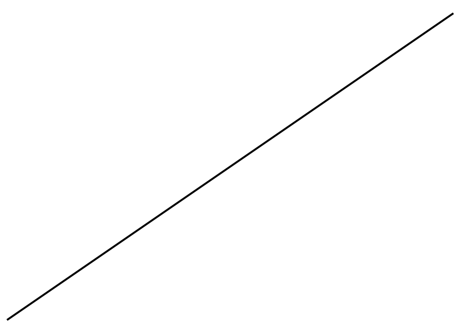
Fractal Dimension

- Fractal dimension = $-\dfrac{\log(\#\mathit{transformations})}{\log(\mathit{scale\ factor})}$



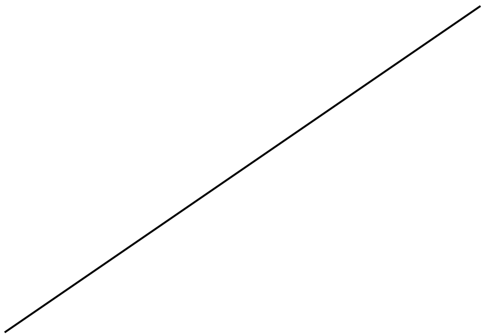
Fractal Dimension

- Fractal dimension = $-\dfrac{\log(\#\mathit{transformations})}{\log(\mathit{scale\ factor})}$



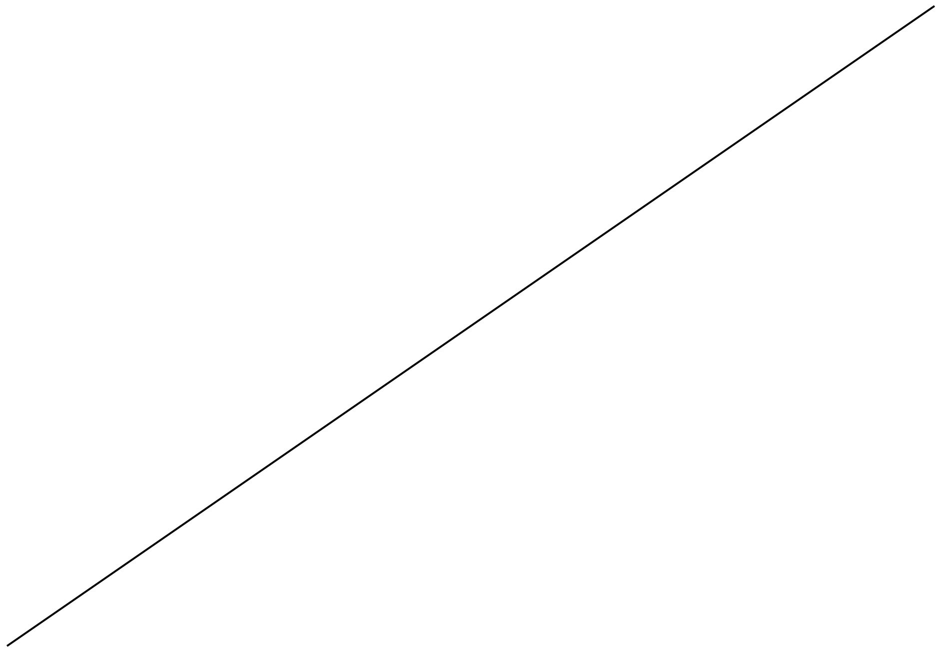
Fractal Dimension

- Fractal dimension = $-\dfrac{\log(\#\mathit{transformations})}{\log(\mathit{scale\ factor})}$



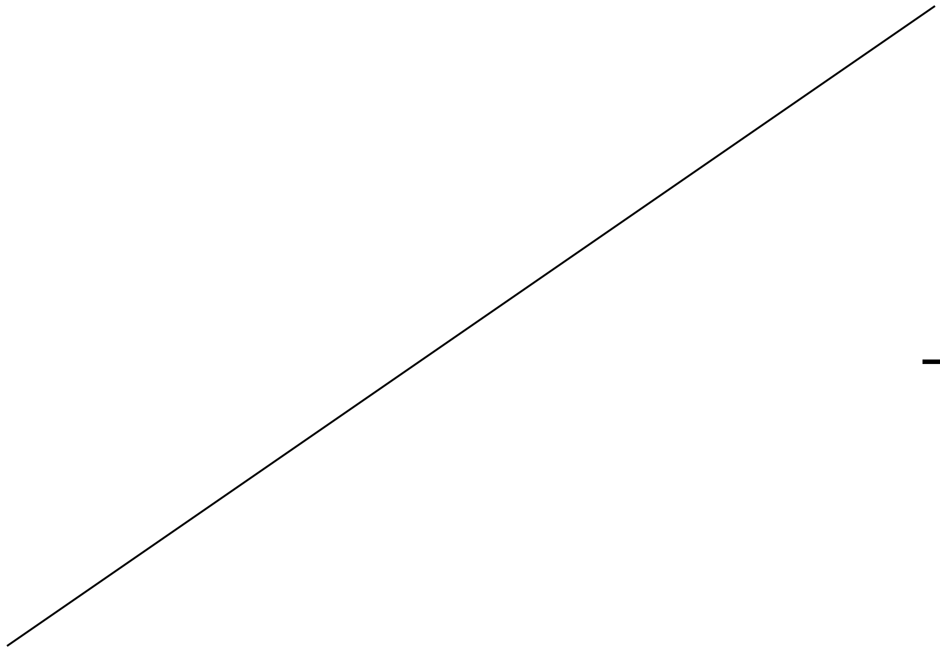
Fractal Dimension

- Fractal dimension = $-\dfrac{\log(\#\mathit{transformations})}{\log(\mathit{scale\ factor})}$



Fractal Dimension

- Fractal dimension = $-\dfrac{\log(\#\,transformations)}{\log(scale\ factor)}$



$$-\frac{\log(2)}{\log(1/2)} = 1$$

Fractal Dimension

- Fractal dimension = $-\dfrac{\log(\#\mathit{transformations})}{\log(\mathit{scale\ factor})}$

Fractal Dimension

- Fractal dimension = $-\dfrac{\log(\#\mathit{transformations})}{\log(\mathit{scale\ factor})}$

Fractal Dimension

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Fractal Dimension

- Fractal dimension = $-\dfrac{\log(\#transformations)}{\log(scale\ factor)}$

Fractal Dimension

- Fractal dimension = $-\dfrac{\log(\#\mathit{transformations})}{\log(\mathit{scale\ factor})}$

Fractal Dimension

- Fractal dimension = $-\dfrac{\log(\#transformations)}{\log(scale\ factor)}$

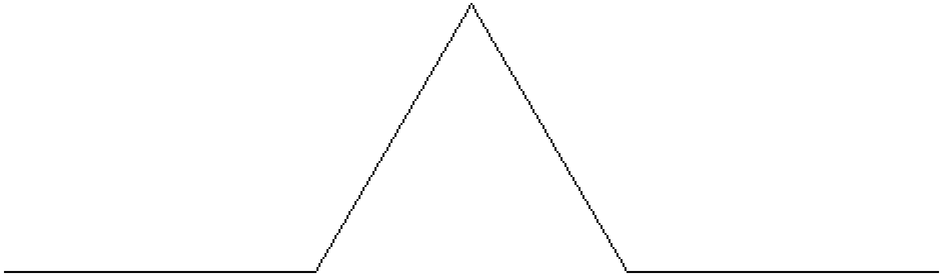
Fractal Dimension

- Fractal dimension = $-\dfrac{\log(\#\mathit{transformations})}{\log(\mathit{scale\ factor})}$

$$-\frac{\log(4)}{\log(\frac{1}{2})} = 2$$

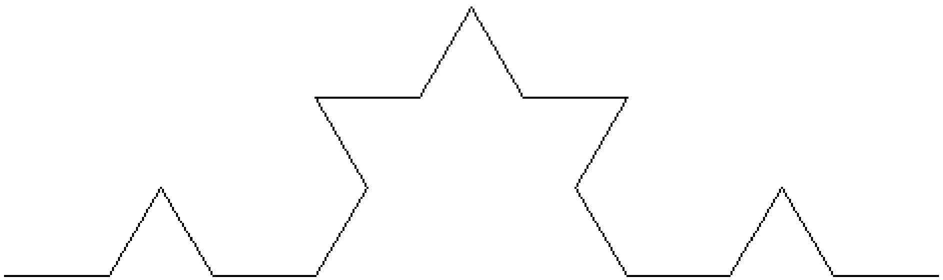
Fractal Dimension

- Fractal dimension = $-\dfrac{\log(\#transformations)}{\log(scale\ factor)}$



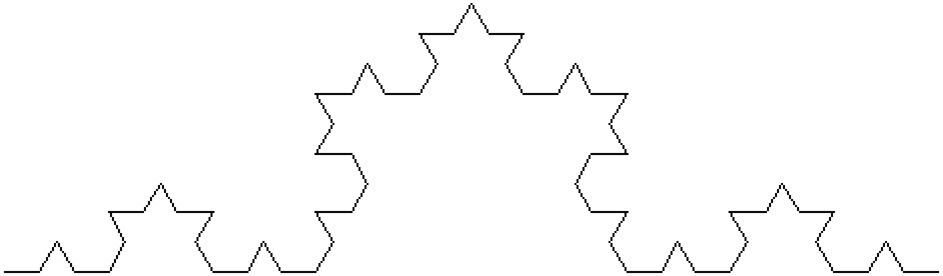
Fractal Dimension

- Fractal dimension = $-\dfrac{\log(\#\mathit{transformations})}{\log(\mathit{scale\ factor})}$



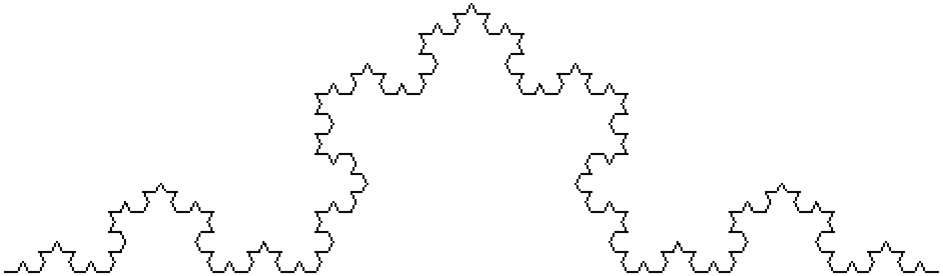
Fractal Dimension

- Fractal dimension = $-\dfrac{\log(\#\mathit{transformations})}{\log(\mathit{scale\ factor})}$



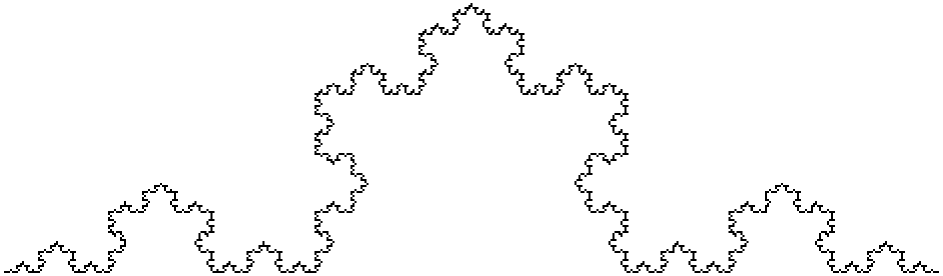
Fractal Dimension

- Fractal dimension = $-\dfrac{\log(\#transformations)}{\log(scale\ factor)}$



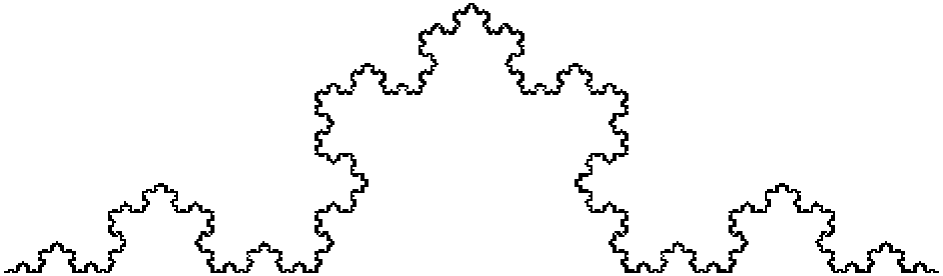
Fractal Dimension

- Fractal dimension = $-\dfrac{\log(\#\mathit{transformations})}{\log(\mathit{scale\ factor})}$



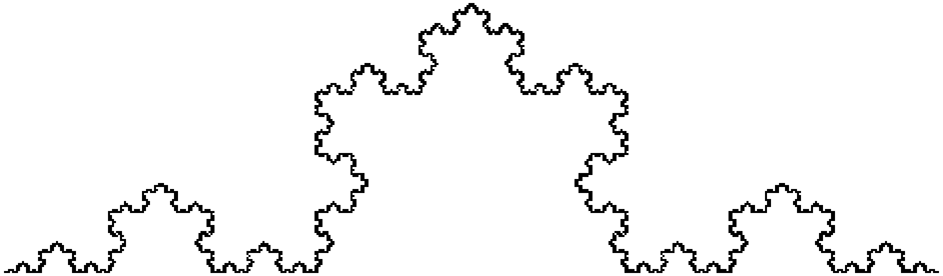
Fractal Dimension

- Fractal dimension = $-\dfrac{\log(\#transformations)}{\log(scale\ factor)}$



Fractal Dimension

- Fractal dimension = $-\dfrac{\log(\#\mathit{transformations})}{\log(\mathit{scale\ factor})}$



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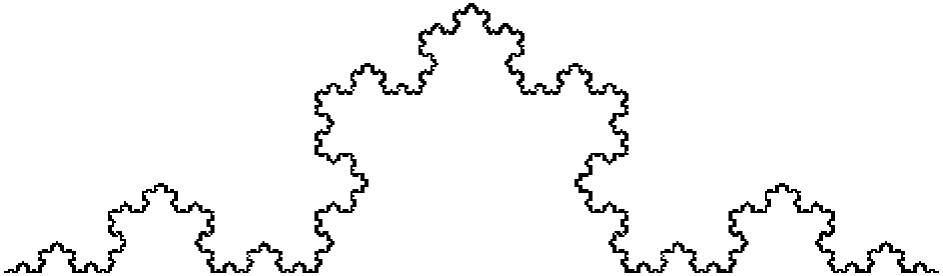

$$-\frac{\log(4)}{\log(1/3)} \approx 1.26$$

Fractal Dimension

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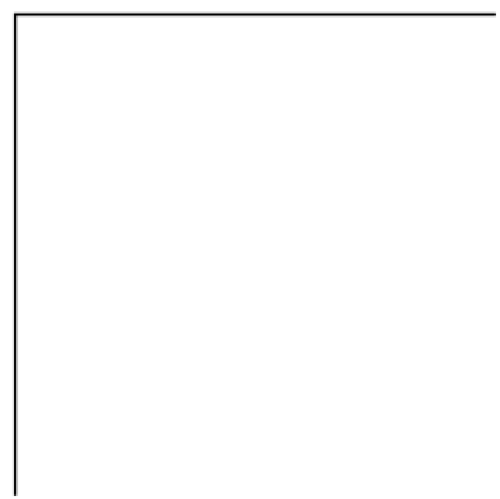

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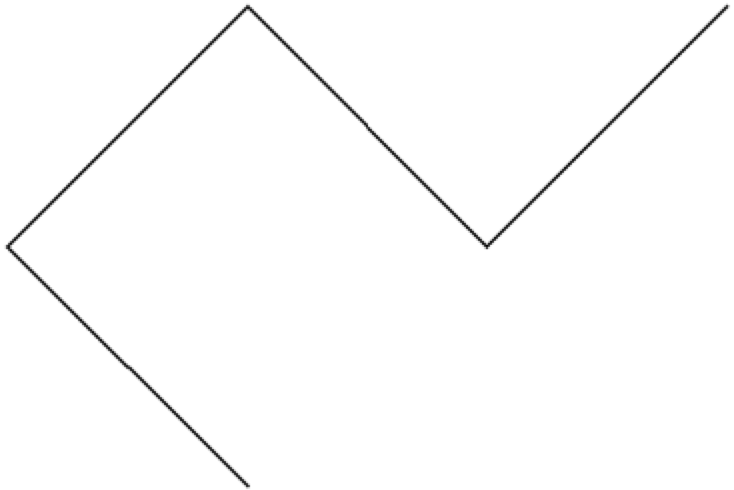

Fractal Dimension

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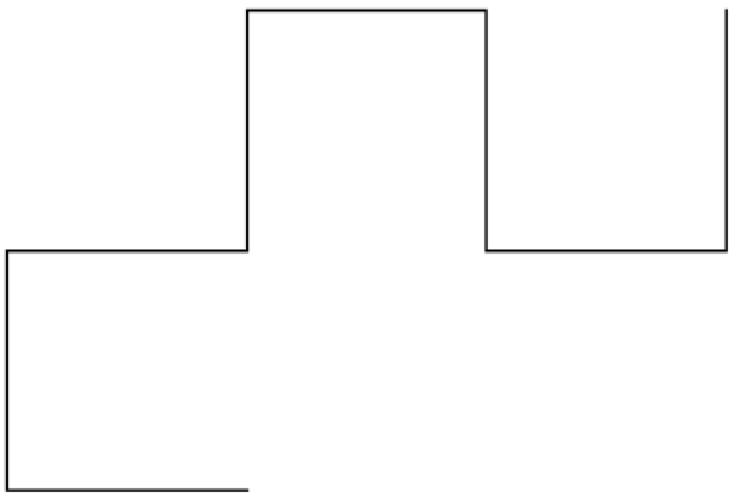

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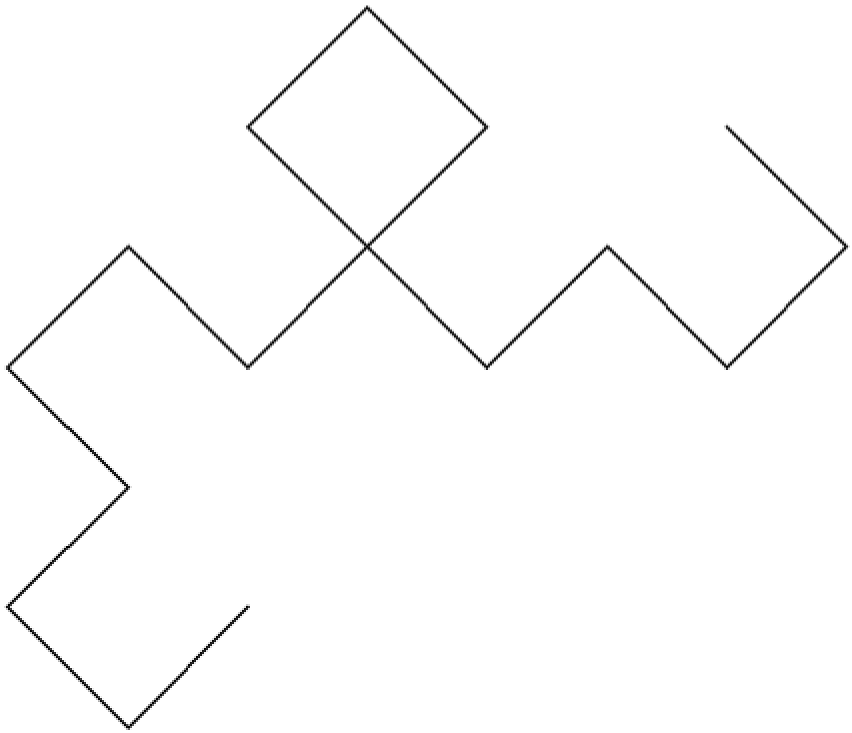

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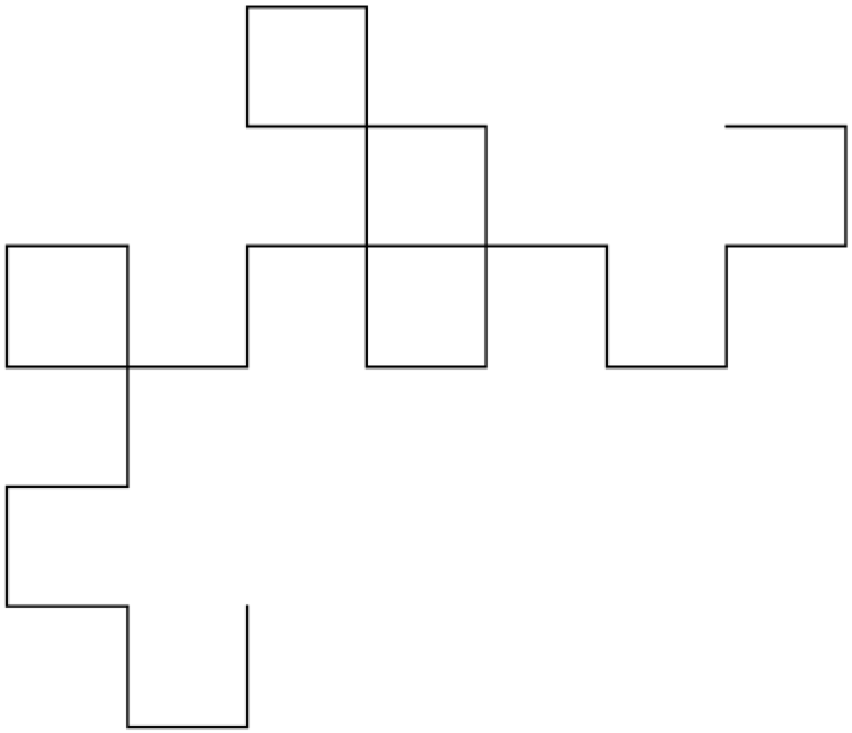

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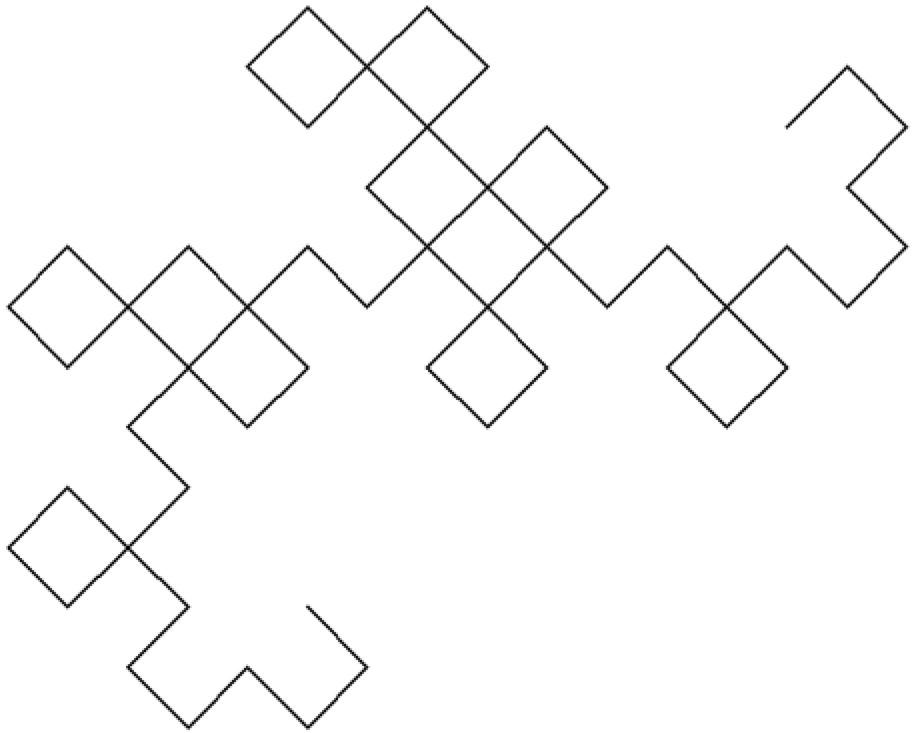

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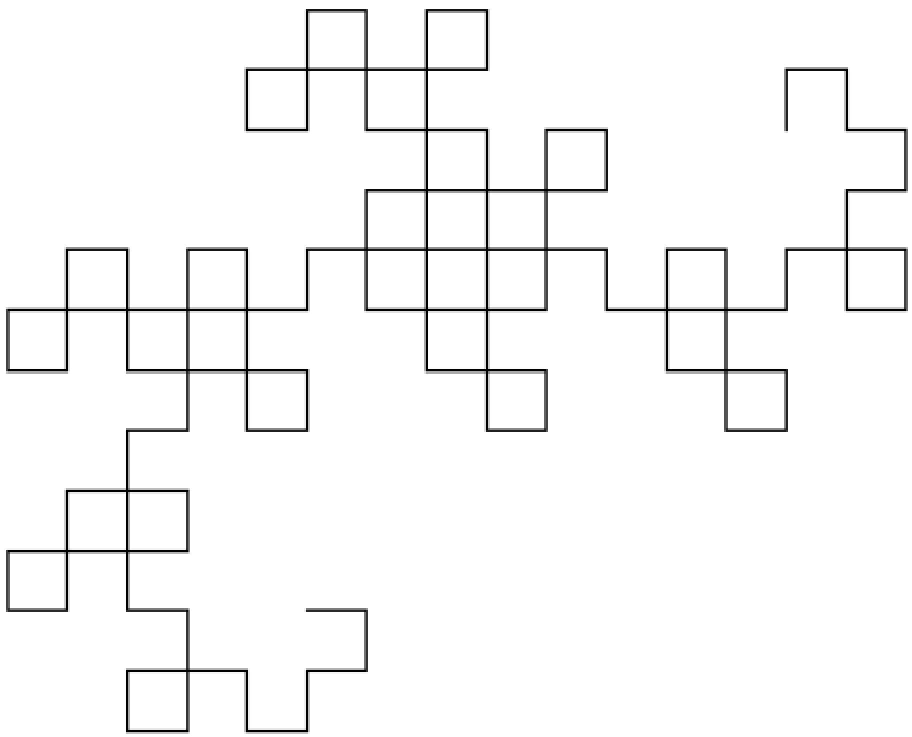

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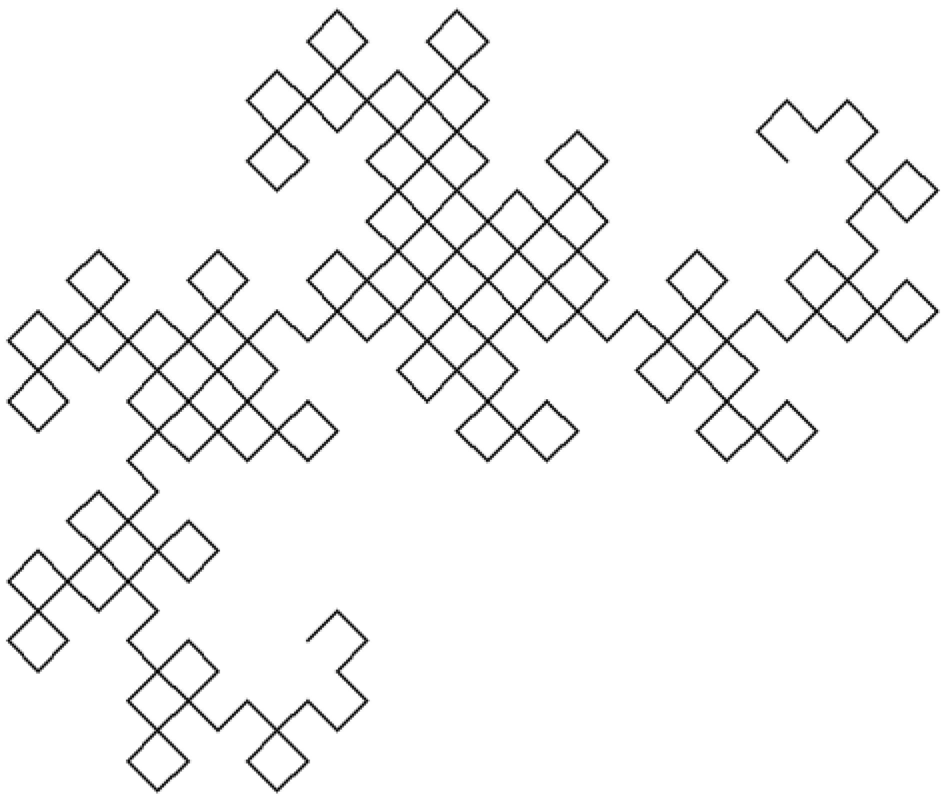

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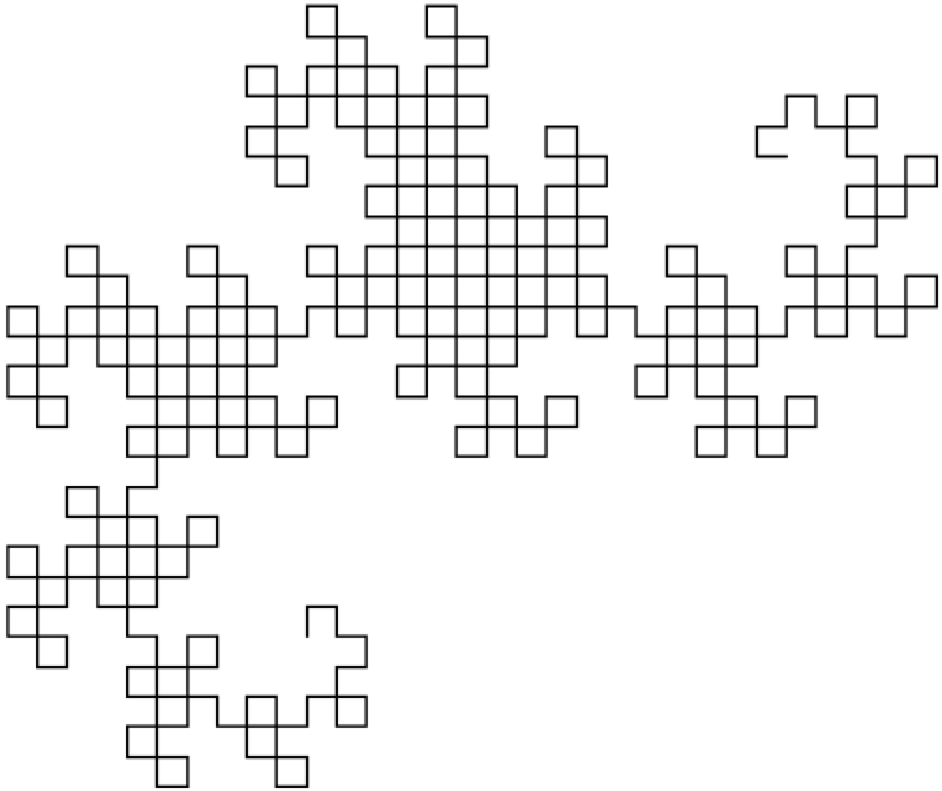

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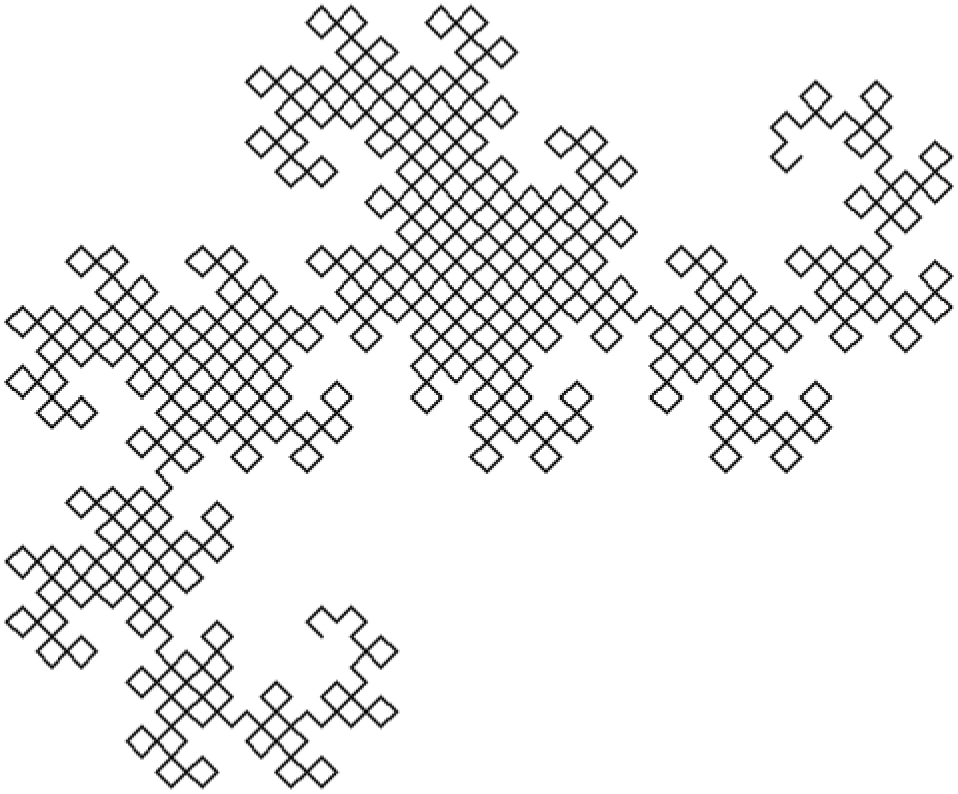

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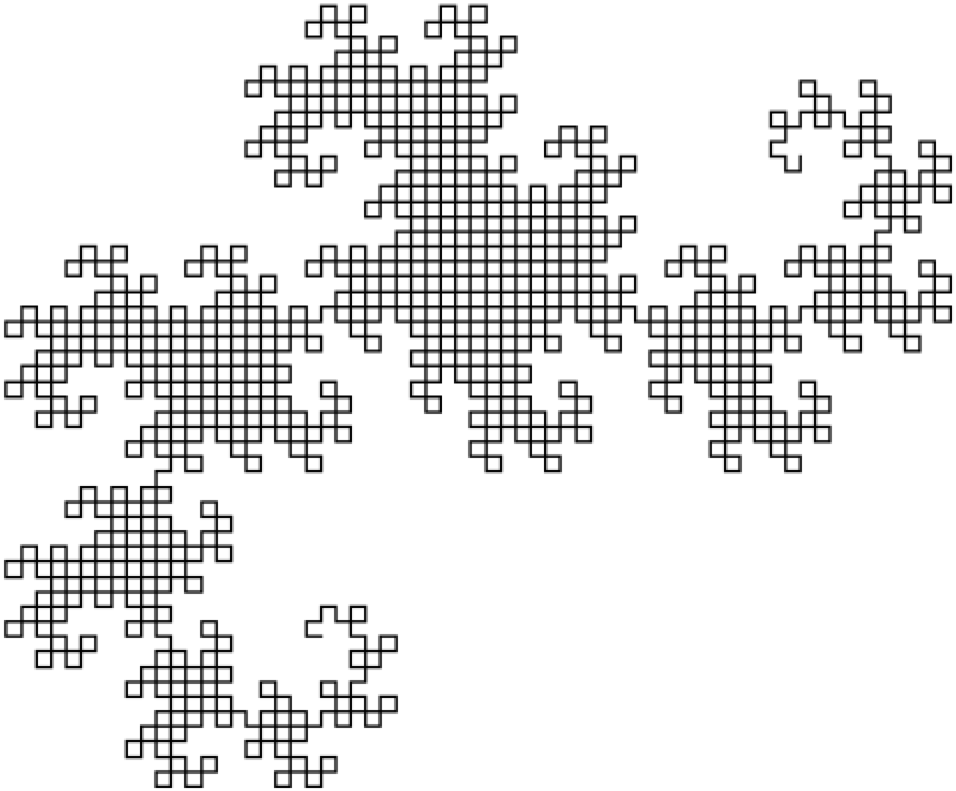

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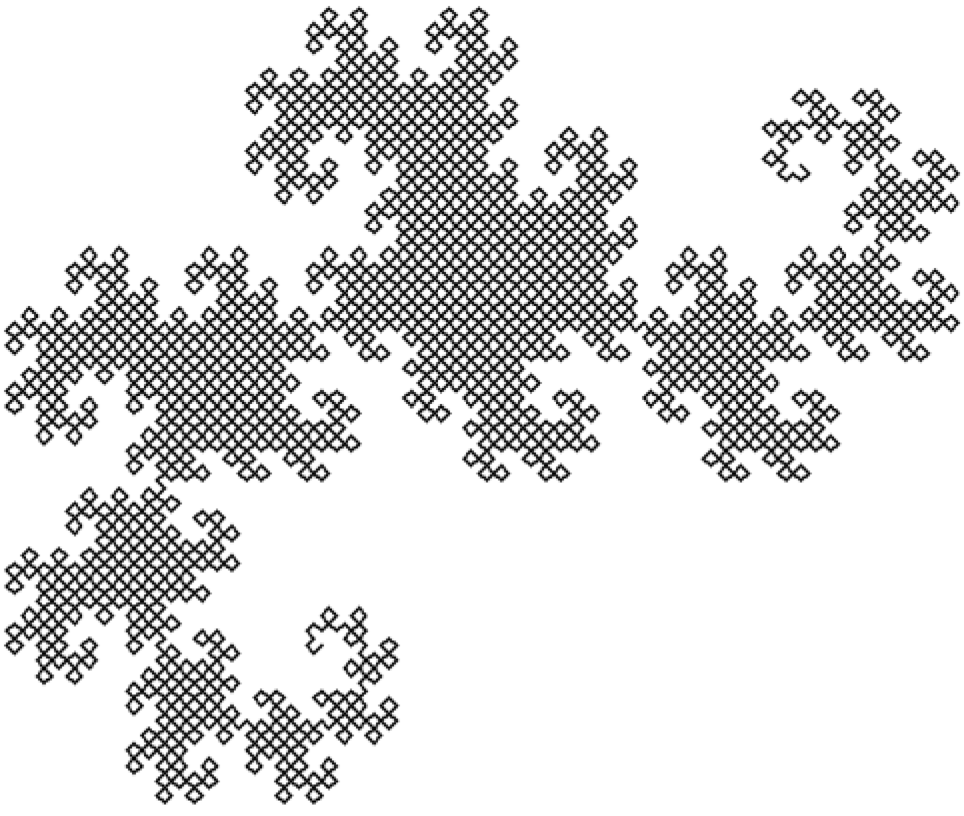

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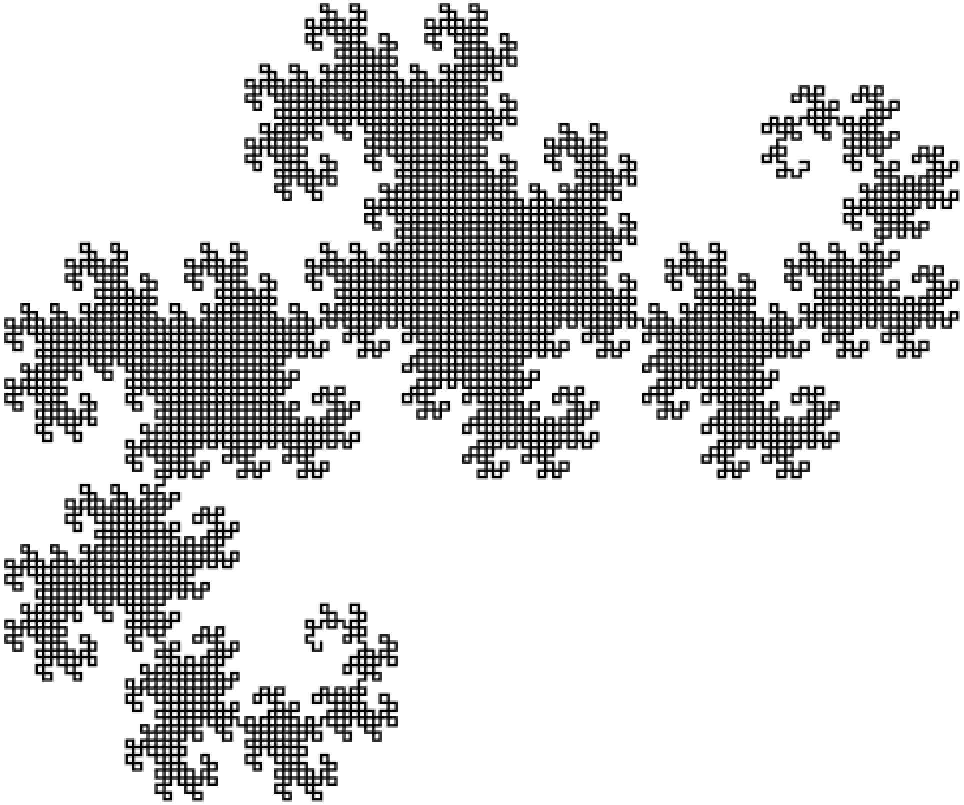

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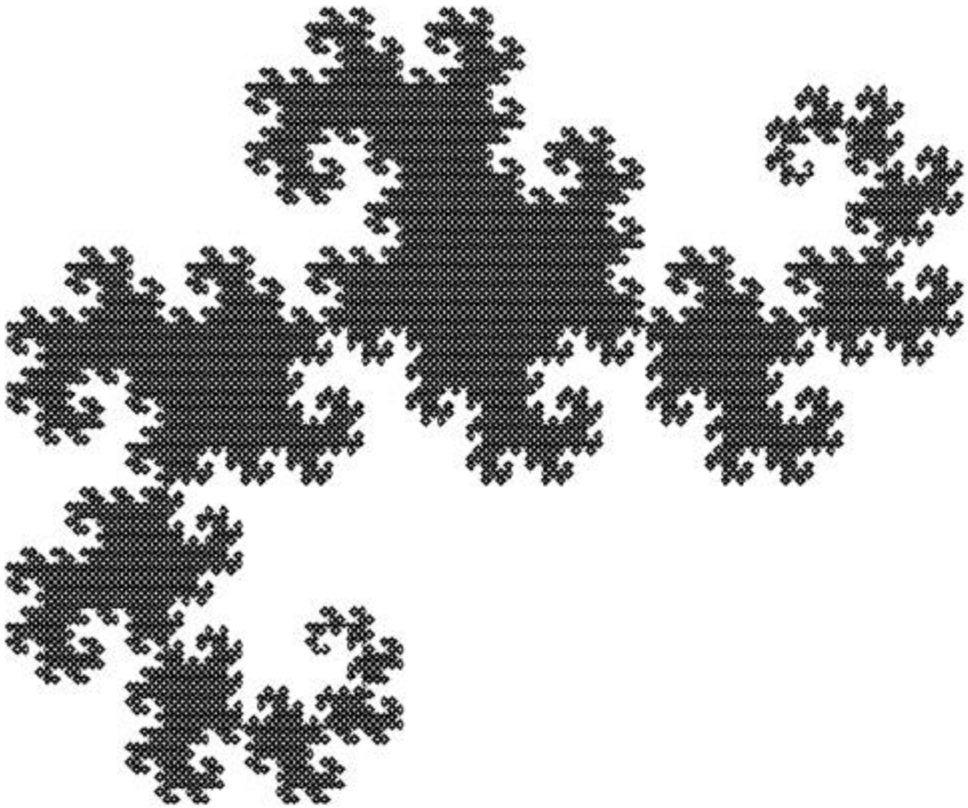

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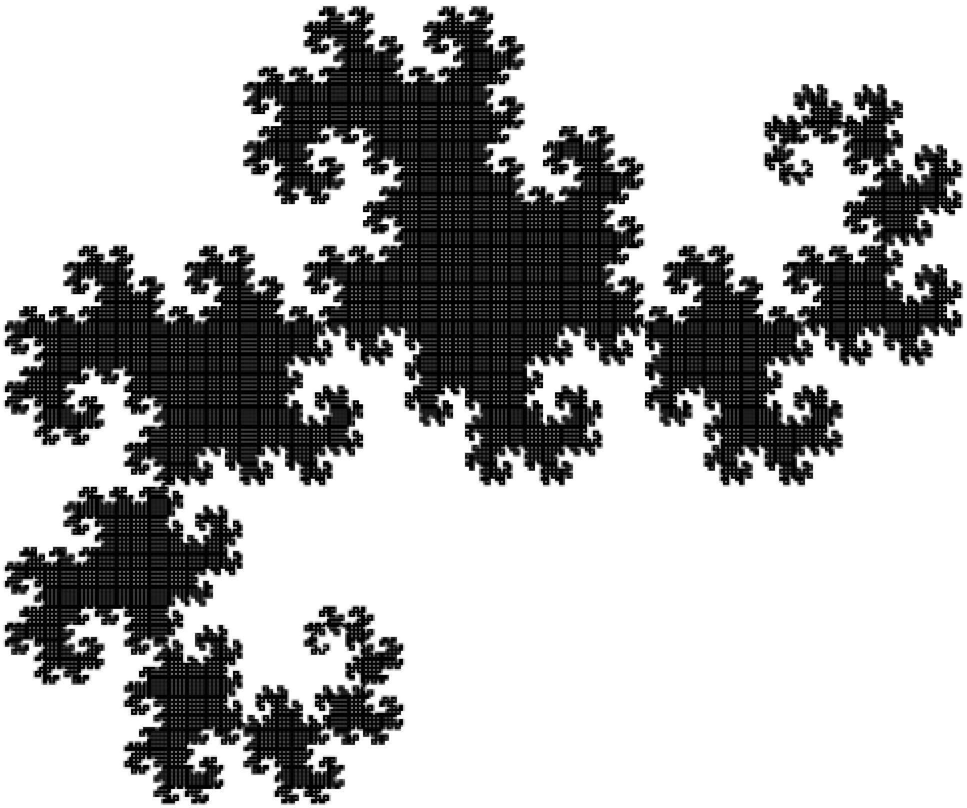

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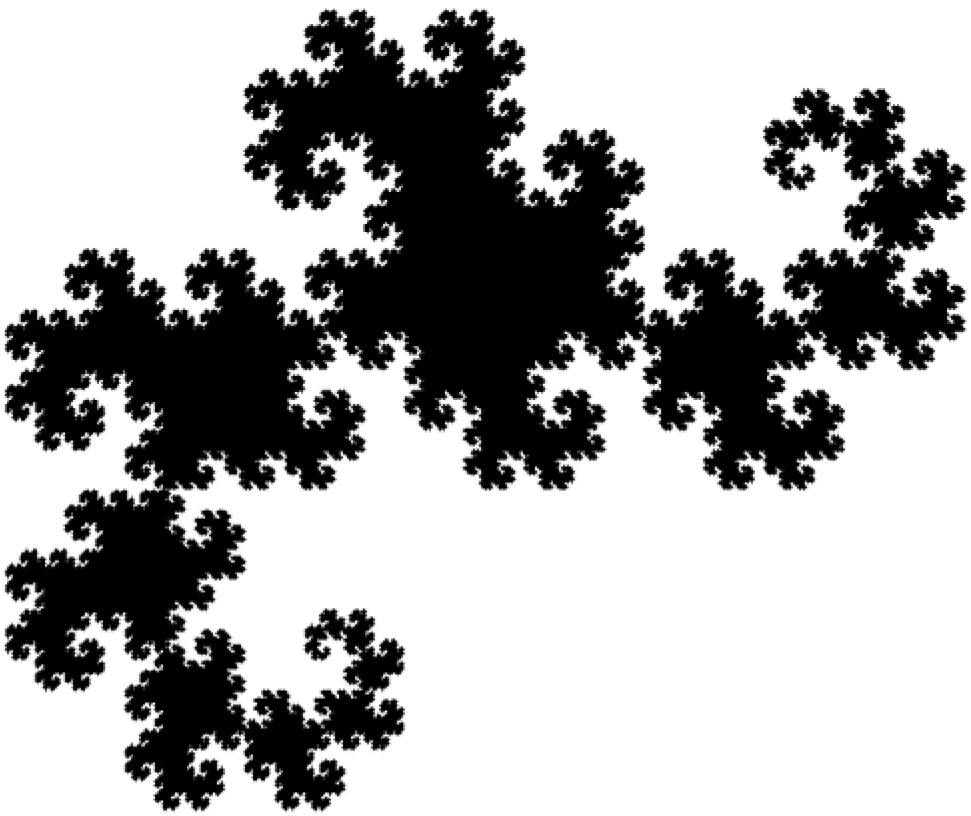

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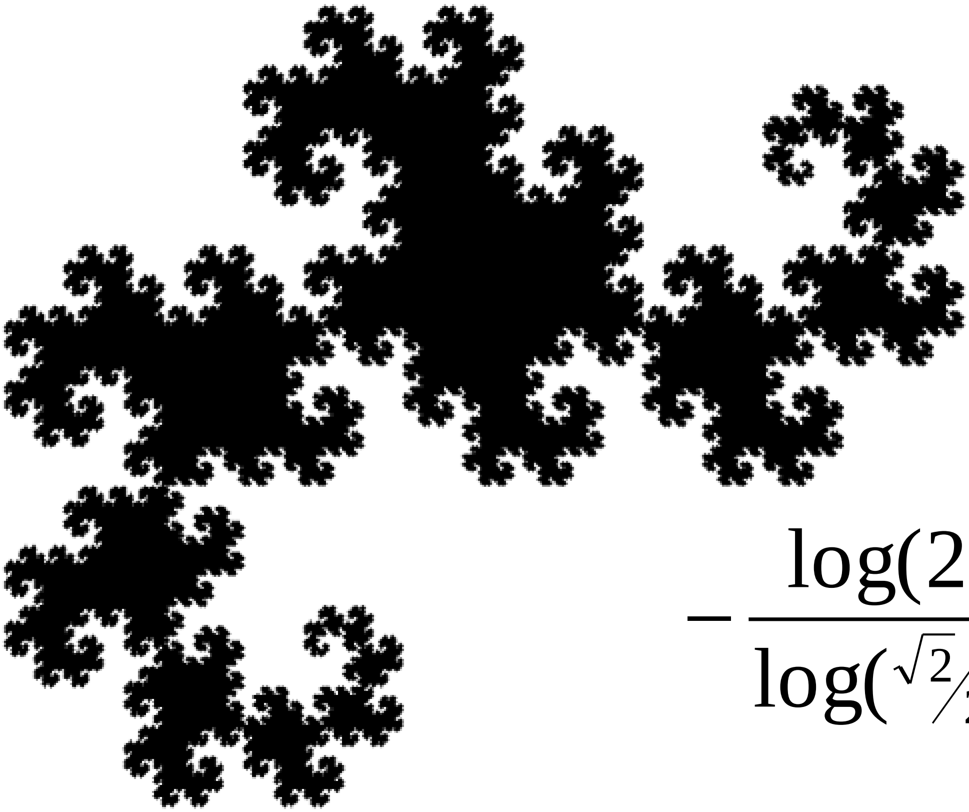
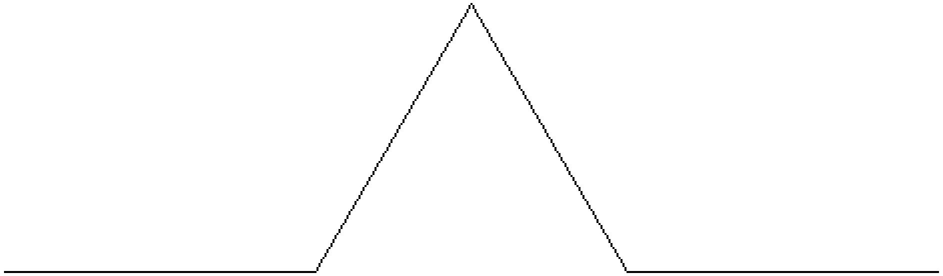


$$-\frac{\log(2)}{\log(\sqrt{2}/2)} = 2$$

Weird Fractal Properties

- Fractal curves can have infinite length!!!



$$len_0 = 4$$

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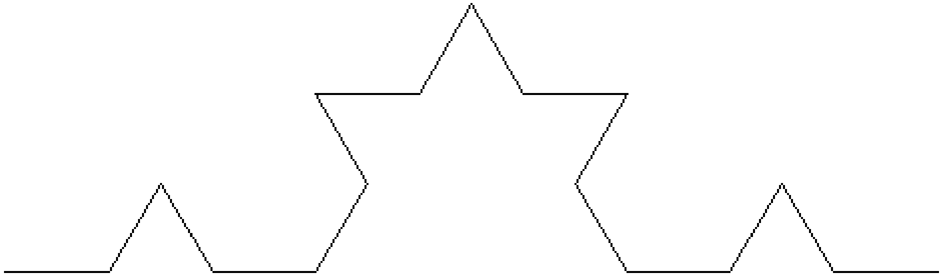

$$len_1 = ?$$

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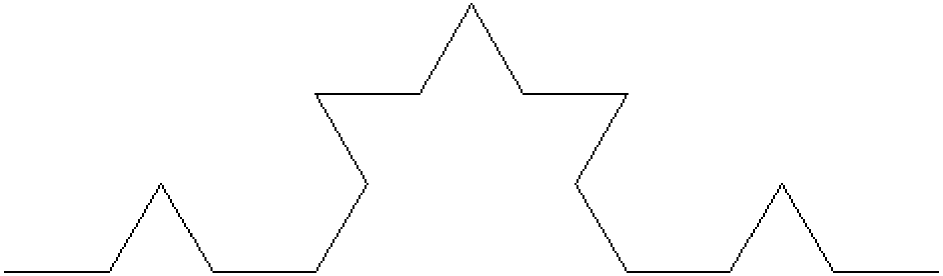

$$len_1 = \frac{4}{3} 4$$

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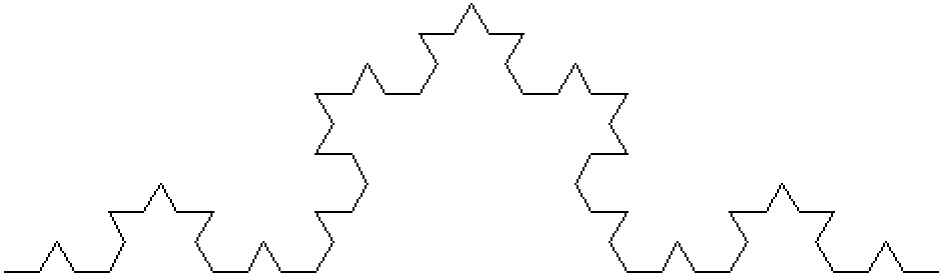

$$len_2 = \left(\frac{4}{3}\right)^2 4$$

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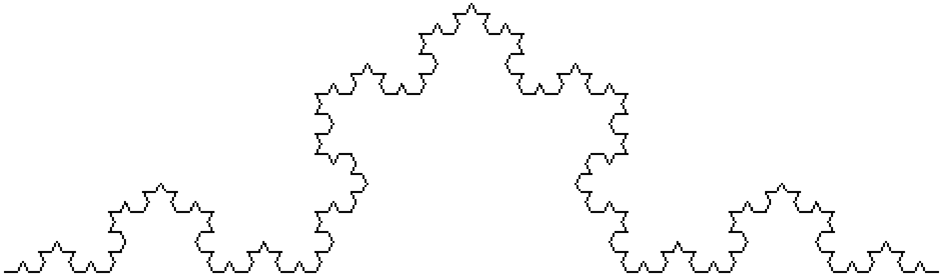

$$len_3 = \left(\frac{4}{3}\right)^3 4$$

Weird Fractal Properties

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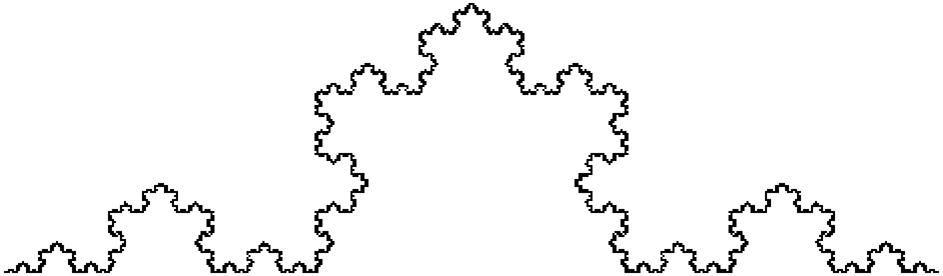
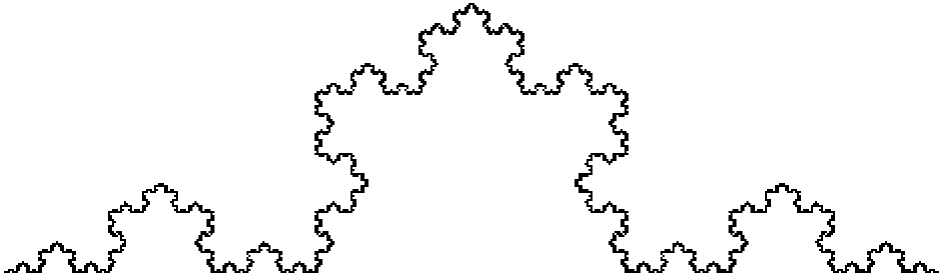


$$len_i = \left(\frac{4}{3}\right)^i 4$$

Weird Fractal Properties

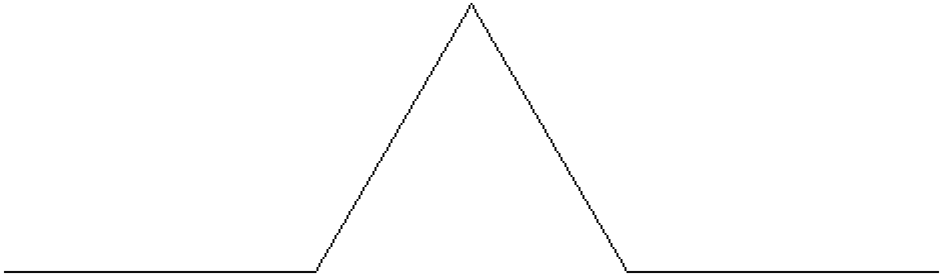
- Fractal curves can have infinite length!!!



$$len_{\infty} = \lim_{i \to \infty} \left(\frac{4}{3} \right)^{i} 4 = \infty$$

Weird Fractal Properties

- Fractal curves can have infinite length!!!
- But only enclose finite area?



$$area_0 = \frac{\sqrt{3}}{4}$$

Weird Fractal Properties

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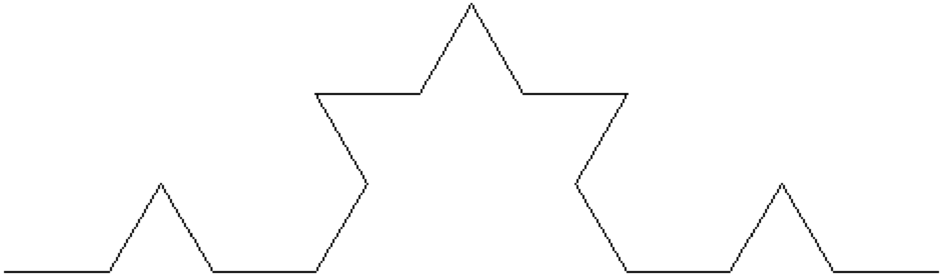

$$area_1 = ?$$

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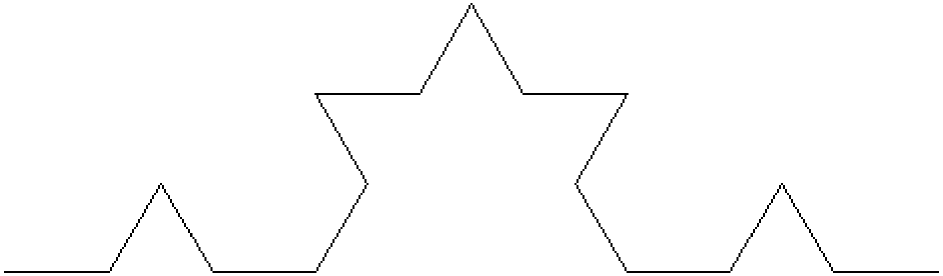

$$area_1 = \left(1 + \frac{4}{9}\right)\frac{\sqrt{3}}{4}$$

Weird Fractal Properties

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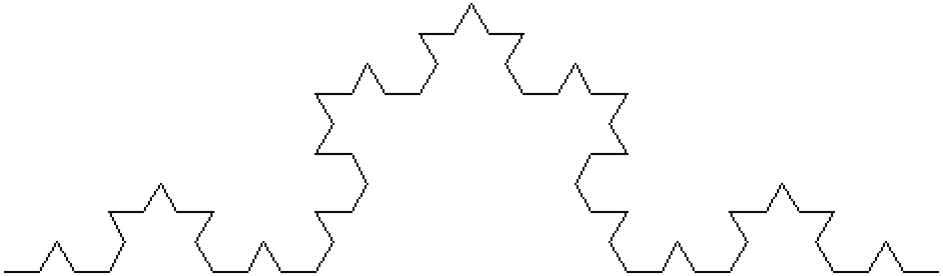

$$area_2 = \left(1 + \frac{4}{9} + \frac{16}{81}\right)\frac{\sqrt{3}}{4}$$

Weird Fractal Properties

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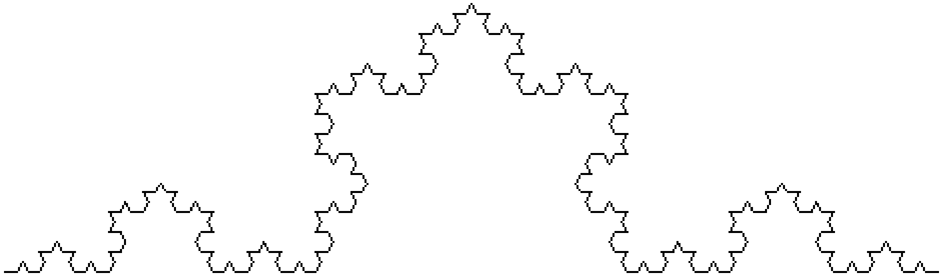

$$area_3 = \left(1 + \frac{4}{9} + \frac{16}{81} + \frac{64}{729}\right)\frac{\sqrt{3}}{4}$$

Weird Fractal Properties

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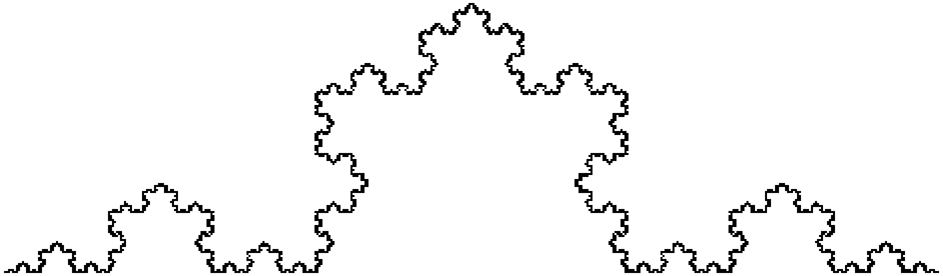

$$area_i = \frac{\sqrt{3}}{4}\sum_{j=0}^{i}\left(\frac{4}{9}\right)^j$$

Weird Fractal Properties

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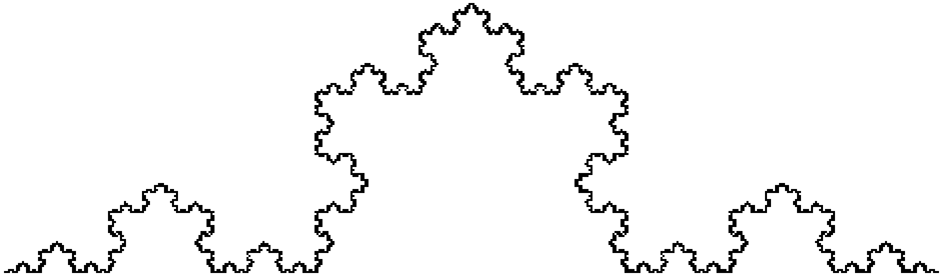

$$area_{\infty} = \lim_{i \to \infty} \frac{\sqrt{3}}{4} \sum_{j=0}^{i} \left(\frac{4}{9} \right)^{j} = \frac{\sqrt{3}}{4} \frac{1}{1 - \frac{4}{9}} \approx .78$$

Weird Fractal Properties