Challenge Problem 2
CPSC 489/689 Quantum Algorithms
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The following quantum circuit represents a half-adder; it calculates the sum $a + b \mod 2$, and the carry $ab$ of the inputs $a$ and $b$:

```
  c
   ┌─┐
  ┤⊙├─┐ c + ab \mod 2
  └─┘

  b
   ┌─┐
  ┤⊙├─┐ a + b \mod 2
  └─┘

  a
```

The circuit implements a unitary matrix $U_{\text{add}}$, which is determined by

$$
U_{\text{add}}|000\rangle = |000\rangle, \quad U_{\text{add}}|100\rangle = |100\rangle,
$$

$$
U_{\text{add}}|001\rangle = |011\rangle, \quad U_{\text{add}}|101\rangle = |111\rangle,
$$

$$
U_{\text{add}}|010\rangle = |010\rangle, \quad U_{\text{add}}|110\rangle = |110\rangle,
$$

$$
U_{\text{add}}|011\rangle = |101\rangle, \quad U_{\text{add}}|111\rangle = |001\rangle.
$$

Let $m(U)$ denote the minimal number of controlled-not and single qubit gates, which are needed to realize $U \in U(2^n)$. The challenge is to determine $m(U_{\text{add}})$. In other words, how many controlled-not gates and single qubits gates are needed in an optimal implementation of $U_{\text{add}}$? You need to prove your result.

Remark. Let $T$ denote the unitary matrix corresponding to the Toffoli gate. Notice that $|m(T) - m(U_{\text{add}})| \leq 1$.

I offer a Challenges in Quantum Computing Award, worth US$ 100, for the first correct solution to this problem.

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