Graphs with Large Girth and Large Chromatic Number
Andreas Klappenecker

We denote by $\chi(G)$ the chromatic number of a graph $G$, by $\alpha(G)$ the independence number of $G$, and by $girth(G)$ the girth of $G$.

**Theorem 1** (Erdős). For all nonnegative integers $k$ and $\ell$ there exists a graph $G$ with $girth(G) > \ell$ and $\chi(G) > k$.

**Proof.** Let $G = G(n,p)$ be a random graph with $n$ vertices, where for each pair of vertices an edge is chosen with probability $p$ independently of other edges. Let us chose $\theta$ such that $\theta < 1/\ell$ and $p = n^{\theta-1}$.

We can specify a cycle with $i$ edges by selecting a sequence of $i$ vertices, namely the sequence $(v_1, v_2, \cdots, v_i)$ of $i$ vertices specifies the cycle consisting of the $i$ edges $(v_1, v_2), (v_2, v_3), \ldots (v_{i-1}, v_i), (v_i, v_1)$.

There are $n(n-1)\cdots(n-i+1)$ sequences of $i$ vertices in $G$, but not all of them specify distinct cycles. The $i$ cyclically rotated sequences 

$$(v_{1+a}, v_{2+a}, \cdots, v_i, v_1, \ldots, v_a)$$

with $a \in \{0, \cdots, i-1\}$, and the $i$ mirrored sequences 

$$(v_a, v_{a-1}, \ldots, v_1, v_i, \ldots, v_{1+a})$$

all specify the same cycle of edges. Therefore, there are $n(n-1)\cdots(n-i+1)/2i$ sequences of vertices that specify different cycles.

Let $X$ denote the number of cycles in $G$ of size at most $\ell$. Then

$$E[X] = \sum_{i=3}^{\ell} \frac{n(n-1)\cdots(n-i+1)}{2i} p^i.$$ 

Since $(n-i+1)p < \cdots < np < n^{\theta}$, we get

$$E[X] \leq \sum_{i=3}^{\ell} n^{\theta i} = o(n).$$

It follows from Markov’s inequality that

$$\Pr[X \geq n/2] = o(1).$$

Let us choose $x = \lceil \frac{3}{p} \ln n \rceil$. By a union bound,

$$\Pr[\alpha(G) \geq x] \leq \binom{n}{x} (1-p)^{(\lceil \frac{3}{p} \ln n \rceil)}.$$ 

Notice that $\binom{n}{x} \leq n^x$. Furthermore, using the inequality $1 - y \leq e^{-y}$, we get $(1-p)^{(\lceil \frac{3}{p} \ln n \rceil)} \leq e^{-p(\lceil \frac{3}{p} \ln n \rceil)}$. Therefore,

$$\Pr[\alpha(G) \geq x] \leq n^x e^{-p(\lceil \frac{3}{p} \ln n \rceil)} = \left(n e^{-p(x-1)/2}\right)^x = o(1).$$
Let \( n \) be sufficiently large such that \( \Pr[X \geq n/2] < 1/2 \) and \( \Pr[\alpha(G) \geq x] < 1/2 \). Then there exists a \( G \) with less than \( n/2 \) cycles of length at most \( \ell \) and with \( \alpha(G) < 3n^{1-\theta} \ln n \). Remove from \( G \) a vertex from each cycle of length at most \( \ell \). This gives a graph \( G^* \) with at least \( n/2 \) vertices.

Since an independent set in \( G^* \) is an independent set in \( G \), we have \( \alpha(G^*) \leq \alpha(G) \).

If \( h = \alpha(G^*) \), then no class of colors in \( G^* \) can have more than \( h \) vertices. Therefore, \( \chi(G^*) \geq |G^*|/\alpha(G^*) \). It follows that

\[
\chi(G) \geq \chi(G^*) \geq \frac{|G^*|}{\alpha(G^*)} \geq \frac{n/2}{3n^{1-\theta} \ln n} = \frac{n^\theta}{6 \ln n}.
\]

Now simply choose \( n \) sufficiently large such that \( \frac{n^\theta}{6 \ln n} \geq k \).